

Bifurcation reordering programs snap-through symmetry in folded elastic ribbons

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Abstract

Snap-through in slender elastic structures is often viewed as a sudden transition between stable configurations, yet the pathway taken during this transition can differ fundamentally. A structure may snap while preserving symmetry, or first lose symmetry and pass through an asymmetric state before reaching its final configuration. What selects between these pathways remains less well understood, especially when the geometry and loading are themselves symmetric. Here, we show that snap-through symmetry can be programmed by reordering competing bifurcations in folded elastic ribbons. We study an elastic ribbon with two localized folds placed symmetrically about the midpoint and show that varying the fold position changes the relative order of two instabilities: a symmetry-breaking pitchfork bifurcation and a saddle-node bifurcation on the symmetry-preserving branch. When the pitchfork bifurcation occurs first, the ribbon loses symmetry before snapping and follows an asymmetric pathway. Conversely, when the saddle-node bifurcation occurs first, the ribbon loses stability while remaining on the symmetric branch, resulting in a symmetry-preserving transition. Combining experiments, discrete differential geometry simulations and numerical continuation, we map this exchange in bifurcation ordering and construct a phase diagram that predicts the switch between asymmetric and symmetric snap-through regimes. A reduced-order double-mass von Mises truss model captures the same mechanism as a generic competition between symmetry-breaking and symmetry-preserving instabilities. These results establish bifurcation reordering as a geometric mechanism for programming snap-through pathways in slender elastic structures, offering a design principle for multistable systems, morphing structures and instability-based mechanical devices.

Keywords: Elastic ribbon, Fold, Snap-through instability, Bifurcation reordering, Symmetry breaking, Discrete simulation, Reduced-order toy model

1. Introduction

Snap-through instabilities in slender elastic structures provide a powerful mechanism for rapid and reversible shape transformation (Holmes and Crosby, 2007; Yang et al., 2023). By exploiting geometric nonlinearity and elastic energy storage, beams, ribbons, plates, and shells can transition abruptly between distant equilibrium configurations while remaining within the elastic regime (Stoll, 1994; Plaut, 2015; Huang et al., 2024a,c, 2025; Lu et al., 2026a). Such transitions are typically governed by a nonlinear energy landscape containing multiple stable and unstable equilibrium states, so that small changes in geometry, loading or boundary conditions can produce large-amplitude configurational changes (Liu et al., 2021; Radisson and Kanso, 2023; Yang et al., 2025). This property has been widely harnessed in morphing structures (Liu et al., 2023; Rahman et al., 2024), fluidic devices (Gomez et al., 2017b; Jiao and Liu, 2020), deployable systems (Lalisani and Royer, 2026), mechanical metamaterials (Zhang et al., 2021; Yue et al., 2026), and soft robotics (Wang et al., 2023; Park et al., 2025), where fast actuation, multistable switching and large deformation are desirable. In these systems, snap-through is no longer viewed merely as a failure mode, but as a functional pathway for encoding programmable mechanical responses (Yan et al., 2025; Lu et al., 2026b).

For functional applications, however, it is not sufficient to determine only the onset or threshold of snap-through. It is also essential to understand how the transition proceeds, since the pathway governs the symmetry, dynamics and final configuration of the snapped state (Lu et al., 2023; de Bono et al., 2024; Feng et al., 2026). In geometrically nonlinear structures, multiple equilibrium branches and instability modes may coexist, allowing a structure to reach snap-through either along a symmetry-preserving branch or after first undergoing symmetry-breaking (Radisson and Kanso, 2023). The distinction between these pathways is important for mechanical functionality. For example, a symmetry-preserving transition may provide a repeatable and predictable morphing route, whereas a symmetry-breaking transition can select one of several asymmetric states and thereby introduce path dependence or mode selection (Pontecorvo et al., 2013; Wang et al., 2024). Understanding and controlling the symmetry of snap-through therefore remains a central challenge for designing reliable and deterministic morphing systems.

The symmetry of a snap-through transition is closely tied to the bifurcation structure of the underlying equilibrium landscape (Radisson and Kanso, 2023). In systems with an underlying geometric symmetry, a pitchfork bifurcation provides a canonical local signature of symmetry-breaking, whereby a symmetric equilibrium branch loses stability and a pair of symmetry-related asymmetric branches emerges (Gomez et al., 2017a; Yu and Hanna, 2019; Huang et al., 2020). By contrast, a saddle-node bifurcation corresponds to the disappearance of a stable equilibrium branch through a limit point, triggering an abrupt loss of stability and often providing a direct route to snap-through (Yan et al., 2019; Simpkins et al., 2026; Bhattacharyya and Rangarajan, 2026). These two bifurcations therefore represent distinct routes by which a structure may leave a stable equilibrium: one through symmetry-breaking and the other through loss of stability of an existing branch. However, how the competition and relative ordering of these bifurcations determine the symmetry of snap-through remains largely unexplored (Zhang et al., 2025).

Localized folds, creases and curvature discontinuities provide an appealing route to controlling such bifurcation landscapes. In origami-inspired structures, creases are commonly used to prescribe folding kinematics, introduce hinge-like compliance and enable programmable shape transformation in thin sheets and shell-like systems (Dias and Audoly, 2014; Misseroni et al., 2024). Beyond classical origami settings, folded and creased ribbons offer a particularly simple yet rich platform in which localized geometric features interact strongly with global bending, twisting and buckling modes (Jules et al., 2019; Marzin et al., 2025). For example, creased annuli have revealed tunable bistable and looping behaviours (Yu et al., 2023), while the integration of kinks and creases in meta-ribbons has enabled programmable folding sequences and reconfigurable shape transformations (Huang et al., 2024b). These studies highlight that localized folds are not merely passive kinematic features, but spatially localized design elements that can regulate global elastic responses. More recently, localized folds have been shown to act as geometric tuners that reshape the stability landscape of elastic ribbons, allowing pitchfork bifurcations to be biased, shifted or suppressed, thereby programming symmetry-breaking or symmetry-preserving deformation under shear (Huang et al., 2026). Yet an important question remains open: can fold placement be used not only to tune individual bifurcation points, but to reorder multiple competing bifurcations and thereby determine the symmetry of snap-through?

Here, we address this question by investigating pre-compressed elastic ribbons containing a pair of symmetrically placed localized folds, as shown in Fig. 1. We show that fold position can reorder two competing instabilities: a symmetry-breaking pitchfork bifurcation and a symmetry-preserving saddle-node bifurcation. This reordering determines whether snap-through occurs after or before symmetry-breaking, thereby selecting between asymmetric and symmetric transition pathways. Combining experiments, discrete simulations and reduced-order modeling, we establish bifurcation reordering as a geometric mechanism for programming snap-through symmetry in slender elastic structures. The remainder of this article is organized as follows. We first formulate the problem and describe the experimental setup in Section 2. The discrete model used to simulate elastic ribbons with localized folds is then introduced in Section 3. In Section 4, we combine numerical continuation, stability analysis and experiments to reveal the bifurcation reordering mechanism and the resulting transition between asymmetric and symmetric snap-through pathways. To clarify the underlying mechanics, Section 5 develops a reduced-order toy model that captures the competition between pitchfork and saddle-node bifurcations. Finally, concluding remarks and potential applications of geometry-programmed snap-through symmetry are discussed in Section 6.

2. Problem setup and experimental platform

We consider a thin, foldable elastic ribbon subjected to prescribed boundary displacements, as illustrated in Fig. 1. The ribbon has initial length l , width w , and thickness b , satisfying the slenderness condition $l \gg w \gg b$. A pair of localized folds is introduced symmetrically about the ribbon midpoint, with the folds located at $s = \gamma$ and $s = l - \gamma$ and characterized by a prescribed fold angle ϕ . The fold position and fold angle provide geometric design parameters for programming the stability landscape of the ribbon.

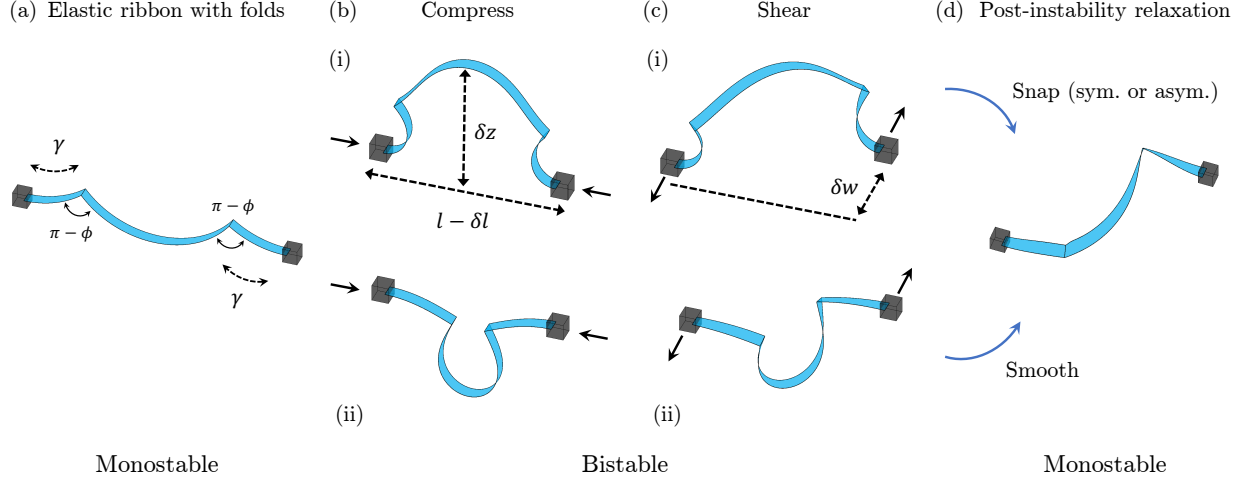


Figure 1: Evolution of static configurations and snap-through behavior of folded elastic ribbons. (a) Initial uncompressed folded ribbon containing a pair of symmetrically placed localized folds, characterized by the fold angle ϕ and fold position γ . (b) Longitudinal compression deforms the folded ribbon into pre-compressed arched configurations with end-shortening δl , clamped-end distance $l - \delta l$, and characteristic height δz . Two stable pre-compressed configurations coexist at this stage: (i) an upper state with positive midpoint height ($\Delta Z > 0$) and (ii) a lower state with negative midpoint height ($\Delta Z < 0$). (c) A subsequent transverse shear displacement δw is applied while the end-shortening is maintained. During shearing, the two stable branches persist until one approaches a stability limit. (d) Beyond a critical shear displacement, the bistable system becomes monostable: the lower branch evolves smoothly toward the remaining stable configuration, whereas the upper one loses stability and undergoes snap-through. This work focuses on the transition pathway of the snap-through event, in particular, whether it proceeds through a symmetric or asymmetric mode.

The mechanical response is studied through a three-stage loading protocol. Initially, the ribbon is in an uncompressed folded configuration containing the prescribed localized folds, as shown in Fig. 1(a). The ribbon is then subjected to longitudinal compression, reducing the distance between the two clamped ends from l to $l - \delta l$. This compression deforms the folded ribbon into pre-compressed arched configurations with characteristic midpoint height δz . At this stage, two stable equilibrium configurations coexist: an upper pre-compressed state, shown in Fig. 1(b)(i), and a lower pre-compressed state, shown in Fig. 1(b)(ii), distinguished by the sign of the midpoint height. Subsequently, a transverse shear displacement δw is applied at one end while the end-to-end distance $l - \delta l$ is maintained. The transverse shear displacement serves as the primary control parameter in the stability analysis. For small shear, both stable branches persist, as shown in Fig. 1(c). For the loading path considered here, beyond a critical shear displacement, bistability is lost: the lower branch evolves smoothly toward the remaining stable configuration, whereas the upper branch loses stability and undergoes snap-through, as illustrated in Fig. 1(d). The central question of this work is whether this snap-through transition is symmetric or asymmetric, corresponding respectively to a symmetry-preserving or symmetry-breaking pathway.

The experiments are conducted on a custom-designed platform for imposing prescribed boundary displacements, as illustrated in Fig. 2(a). The system consists of two programmable actuators mounted on an electric linear rail, allowing accurate control of the longitudinal shortening δl and transverse shear displacement δw . The setup is placed on an optical table to ensure mechanical stability, and a camera is used to

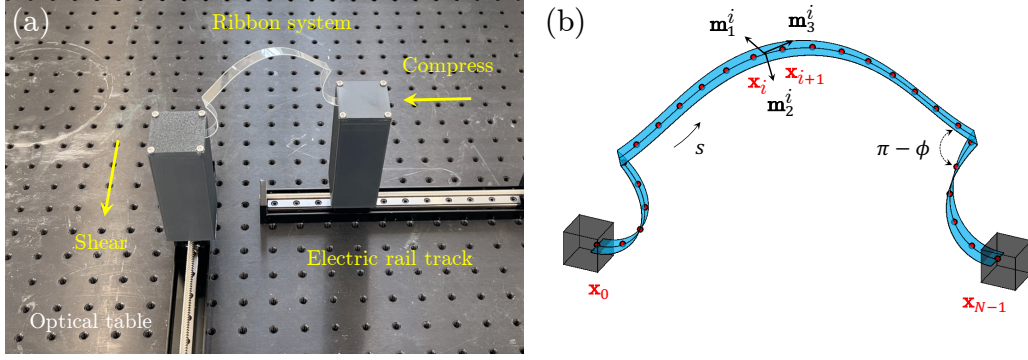


Figure 2: Experimental platform and discrete modeling framework for folded elastic ribbons. (a) Experimental setup. A folded elastic ribbon is mounted between two programmable actuators on an optical table. The electric linear rail controls the longitudinal compression δl and transverse shear displacement δw , enabling systematic exploration of the ribbon’s stability and snap-through response. (b) Discrete differential geometry (DDG) model representation. In the numerical model, the ribbon centerline is discretized into a sequence of vertices $\mathbf{x}_i$, connected by elastic segments. A discrete material frame $\{\mathbf{m}_1^i, \mathbf{m}_2^i, \mathbf{m}_3^i\}$ is assigned to each edge to evaluate bending and twisting deformation, while the localized fold is incorporated through a prescribed curvature discontinuity.

record the evolution of the ribbon configurations during deformation. The ribbon specimens are fabricated from elastic polyethylene terephthalate (PET) sheets with length $l = 200$ mm, width $w = 5.0$ mm, and thickness $b = 0.25$ mm. Localized folds are introduced manually at prescribed symmetric positions, $s = \gamma$ and $s = l - \gamma$, with care taken to achieve the target fold angle ϕ . These folds act as localized elastic hinges or curvature discontinuities, providing geometric constraints that couple to the ribbon’s large deformation and snap-through response.

3. Discrete model and stability analysis

We build on an established discrete differential geometry (DDG) framework to model the nonlinear deformation and stability of slender folded ribbons with localized out-of-plane creases (Huang et al., 2025). The ribbon is represented as an anisotropic slender body capable of undergoing large rotations and geometrically nonlinear deformation, while the prescribed folds are incorporated through localized curvature discontinuities along the centerline. Building on previous implementations for folded ribbons (Huang et al., 2024b, 2026), the present framework combines numerical continuation, eigenvalue-based stability analysis and dynamic simulation to track the competing pitchfork and saddle-node bifurcations in a two-fold geometry and resolve the associated post-instability response.

The ribbon is modeled as a slender body of length l , width w , and thickness b , satisfying $l \gg w \gg b$. Under this geometric separation of scales, an anisotropic rod idealization provides an efficient description of the nonlinear mechanics of narrow strips and folded ribbons (Yu and Hanna, 2019; Huang et al., 2020, 2024c, 2025). The cross-sectional area is $A = wb$, the principal second moments of area are $I_1 = wb^3/12$ and $I_2 = bw^3/12$, and the torsional constant is approximated as $J = wb^3/3$. The ribbon is made of an elastic material with Young’s modulus E , shear modulus G , and density ρ .

Kinematics. Referring to Fig. 2(b), the ribbon centerline is discretized into N nodes,

$$[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}], \quad (1)$$

which define $N - 1$ edge vectors

$$[\mathbf{e}^0, \mathbf{e}^1, \dots, \mathbf{e}^{N-2}], \quad (2)$$

with

$$\mathbf{e}^i = \mathbf{x}_{i+1} - \mathbf{x}_i. \quad (3)$$

To describe the orientation of each edge, we associate with edge i an orthonormal reference frame

$$\{\mathbf{d}_1^i, \mathbf{d}_2^i, \mathbf{d}_3^i\}, \quad (4)$$

and a material frame

$$\{\mathbf{m}_1^i, \mathbf{m}_2^i, \mathbf{m}_3^i\}. \quad (5)$$

Both frames are adapted to the centerline, so that their third director coincides with the unit tangent,

$$\mathbf{d}_3^i \equiv \mathbf{m}_3^i = \frac{\mathbf{e}^i}{\|\mathbf{e}^i\|}. \quad (6)$$

For a ribbon, the first material director $\mathbf{m}_1^i$ is chosen as the local surface normal, and the second director $\mathbf{m}_2^i$ spans the width direction,

$$\mathbf{m}_2^i = \mathbf{m}_3^i \times \mathbf{m}_1^i. \quad (7)$$

The reference frame is updated by parallel transport along the centerline (Bergou et al., 2008, 2010), while the material frame is obtained from the reference frame through a scalar twist angle θ^i on each edge. The discrete degrees of freedom are therefore collected into

$$\mathbf{q} = [\mathbf{x}_0, \theta^0, \mathbf{x}_1, \theta^1, \dots, \mathbf{x}_{N-2}, \theta^{N-2}, \mathbf{x}_{N-1}]^T, \quad (8)$$

which contains $3N + (N - 1)$ unknowns.

Elastic energy. The elastic deformation of the ribbon consists of stretching, bending, and twisting (Huang et al., 2025). The axial strain associated with edge i is defined by

$$\varepsilon^i = \frac{\|\mathbf{e}^i\|}{\|\hat{\mathbf{e}}^i\|} - 1, \quad (9)$$

where a hat denotes a quantity evaluated in the stress-free reference configuration, and $\|\cdot\|$ is the Euclidean norm. Bending is measured at the internal node $\mathbf{x}_i$, $i = 1, \dots, N - 2$, through the curvature binormal

$$(\boldsymbol{\kappa}\mathbf{b})_i = \frac{2\mathbf{e}^{i-1} \times \mathbf{e}^i}{\|\mathbf{e}^{i-1}\| \|\mathbf{e}^i\| + \mathbf{e}^{i-1} \cdot \mathbf{e}^i}. \quad (10)$$

The two material curvatures are then obtained by projecting the curvature binormal onto the averaged material frame,

$$\kappa_{1,i} = \frac{1}{2} \frac{(\mathbf{m}_2^{i-1} + \mathbf{m}_2^i) \cdot (\boldsymbol{\kappa}\mathbf{b})_i}{\Delta l_i}, \quad (11)$$

$$\kappa_{2,i} = -\frac{1}{2} \frac{(\mathbf{m}_1^{i-1} + \mathbf{m}_1^i) \cdot (\boldsymbol{\kappa}\mathbf{b})_i}{\Delta l_i}, \quad (12)$$

where

$$\Delta l_i = \frac{\|\mathbf{e}^{i-1}\| + \|\mathbf{e}^i\|}{2} \quad (13)$$

is the Voronoi length associated with node i . Here, $\kappa_{1,i}$ and $\kappa_{2,i}$ denote the curvatures about the $\mathbf{m}_1$ - and $\mathbf{m}_2$ -axes, respectively. The discrete twist strain at node i is defined as (Bergou et al., 2008, 2010; Huang et al., 2025),

$$\kappa_{3,i} = \frac{\theta^i - \theta^{i-1} + m_i^{\text{ref}}}{\Delta l_i}, \quad i = 1, \dots, N - 2, \quad (14)$$

where m_i^{ref} is the reference twist induced by parallel transport of the reference frame. The total elastic energy is written as

$$E^{\text{ela}} = E^{\text{s}} + E_1^{\text{b}} + E_2^{\text{b}} + E^{\text{t}}, \quad (15)$$

with

$$\begin{aligned}
E^s &= \sum_{i=0}^{N-2} \frac{1}{2} EA_i (\varepsilon^i)^2 \|\hat{\mathbf{e}}^i\|, \\
E_1^b &= \sum_{i=1}^{N-2} \frac{1}{2} EI_{1,i} (\kappa_{1,i} - \hat{\kappa}_{1,i})^2 \Delta \hat{l}_i, \\
E_2^b &= \sum_{i=1}^{N-2} \frac{1}{2} EI_{2,i} (\kappa_{2,i} - \hat{\kappa}_{2,i})^2 \Delta \hat{l}_i, \\
E^t &= \sum_{i=1}^{N-2} \frac{1}{2} GJ_i (\kappa_{3,i} - \hat{\kappa}_{3,i})^2 \Delta \hat{l}_i.
\end{aligned} \tag{16}$$

Here, EA_i is the axial stiffness, $EI_{1,i}$ and $EI_{2,i}$ are the two bending stiffnesses, and GJ_i is the twisting stiffness. For slender ribbons, the axial stiffness is typically much larger than the bending and twisting stiffnesses, so the stretching energy effectively acts as a penalty term that enforces near-inextensibility of the centerline (Huang et al., 2025).

Abrupt folds. A key advantage of the present discrete formulation is that geometric discontinuities can be introduced naturally at the nodal level. If a crease with prescribed turning angle ϕ_i is placed at node $\mathbf{x}_i$, its effect is represented through an imposed intrinsic curvature (Huang et al., 2024b)

$$\hat{\kappa}_{2,i} = 2 \frac{\tan(\phi_i/2)}{\Delta l_i}. \tag{17}$$

To model a sharp fold, the corresponding bending stiffness $EI_{2,i}$ is chosen sufficiently large, so that the constraint $\kappa_{2,i} = \hat{\kappa}_{2,i}$ is strongly enforced (Huang et al., 2026). In this way, the discrete model can accommodate both smooth elastic deformation and localized fold kinematics within a unified energy-based framework.

Boundary condition. To impose the clamped-clamped boundary condition, the first two nodes $\{\mathbf{x}_0, \mathbf{x}_1\}$, the last two nodes $\{\mathbf{x}_{N-2}, \mathbf{x}_{N-1}\}$, and the associated rotational degrees of freedom $\{\theta^0, \theta^{N-2}\}$ are prescribed and updated consistently with the boundary constraints.

Static arc-length continuation. The equilibrium path of folded ribbons generally contains limit points associated with snap-through and snap-back instabilities. In such cases, conventional load-controlled Newton iterations fail once the tangent stiffness matrix becomes singular. To trace the complete equilibrium branch, including unstable segments, we employ the modified generalized displacement control method (MGDCM), which is a robust arc-length-type continuation scheme (Leon et al., 2014; Liu and Paulino, 2017). The nonlinear equilibrium equation is

$$\mathbf{R}(\mathbf{q}, \lambda) = \lambda \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{ela}}(\mathbf{q}) = \mathbf{0}, \tag{18}$$

where $\mathbf{q}$ is the generalized coordinate vector, $\mathbf{F}^{\text{ela}}$ is the internal elastic force vector, $\mathbf{F}^{\text{ext}}$ is the reference external load vector, and λ is the load factor. Note that the internal elastic force vector is obtained from the variation of the total elastic energy,

$$\mathbf{F}^{\text{ela}} = -\frac{\partial E^{\text{ela}}}{\partial \mathbf{q}}. \tag{19}$$

At iteration j of load step k , the linearized equilibrium equation is

$$\mathbb{K}_{j-1}^k \Delta \mathbf{q}_j^k = \Delta \lambda_j^k \mathbf{F}^{\text{ext}} + \mathbf{R}_{j-1}^k, \tag{20}$$

where the tangent stiffness matrix is defined by

$$\mathbb{K} = \frac{\partial^2 E^{\text{ela}}}{\partial \mathbf{q}^2}, \tag{21}$$

and the residual vector is

$$\mathbf{R}_{j-1}^k = (\lambda_{j-1}^k) \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{ela}}(\mathbf{q}_{j-1}^k). \quad (22)$$

The solution is updated according to

$$\mathbf{q}_j^k = \mathbf{q}_{j-1}^k + \Delta \mathbf{q}_j^k, \quad \lambda_j^k = \lambda_{j-1}^k + \Delta \lambda_j^k. \quad (23)$$

Since both $\Delta \mathbf{q}_j^k$ and $\Delta \lambda_j^k$ are unknown, Eq. (20) is supplemented by a scalar constraint equation,

$$\mathbf{a}_j^k \cdot \Delta \mathbf{q}_j^k + b_j^k \Delta \lambda_j^k = c_j^k. \quad (24)$$

The displacement increment is decomposed as

$$\Delta \mathbf{q}_j^k = (\Delta \mathbf{q}^{\text{r}})_j^k + \Delta \lambda_j^k (\Delta \mathbf{q}^{\text{ext}})_j^k, \quad (25)$$

where $(\Delta \mathbf{q}^{\text{r}})_j^k$ and $(\Delta \mathbf{q}^{\text{ext}})_j^k$ satisfy

$$\mathbb{K}_{j-1}^k (\Delta \mathbf{q}^{\text{r}})_j^k = \mathbf{R}_{j-1}^k, \quad (26)$$

$$\mathbb{K}_{j-1}^k (\Delta \mathbf{q}^{\text{ext}})_j^k = \mathbf{F}^{\text{ext}}. \quad (27)$$

Substitution into the constraint equation gives

$$\Delta \lambda_j^k = \frac{c_j^k - \mathbf{a}_j^k \cdot (\Delta \mathbf{q}^{\text{r}})_j^k}{\mathbf{a}_j^k \cdot (\Delta \mathbf{q}^{\text{ext}})_j^k + b_j^k}. \quad (28)$$

Based on the above formulation, the load factor increment is updated as (Leon et al., 2014; Liu and Paulino, 2017),

$$\Delta \lambda_j^k = \begin{cases} \Delta \bar{\lambda}, & k = 1, j = 1, \\ -\frac{(\Delta \mathbf{q}^{\text{ext}})_1^1 \cdot (\Delta \mathbf{q}^{\text{r}})_j^1}{(\Delta \mathbf{q}^{\text{ext}})_1^1 \cdot (\Delta \mathbf{q}^{\text{ext}})_j^1}, & k = 1, j > 1, \\ \pm \Delta \bar{\lambda} \sqrt{\frac{(\Delta \mathbf{q}^{\text{ext}})_1^1 \cdot (\Delta \mathbf{q}^{\text{ext}})_1^1}{(\Delta \mathbf{q}^{\text{ext}})_1^k \cdot (\Delta \mathbf{q}^{\text{ext}})_1^k}}, & k > 1, j = 1, \\ -\frac{(\Delta \mathbf{q}^{\text{ext}})_1^k \cdot (\Delta \mathbf{q}^{\text{r}})_j^k}{(\Delta \mathbf{q}^{\text{ext}})_1^k \cdot (\Delta \mathbf{q}^{\text{ext}})_j^k}, & k > 1, j > 1. \end{cases} \quad (29)$$

Here, the sign of the load factor increment is determined according to the direction of the displacement increment between two consecutive load steps, i.e.,

$$\text{sgn}((\Delta \mathbf{q}^{\text{ext}})_1^{k-1} \cdot (\Delta \mathbf{q}^{\text{ext}})_1^k). \quad (30)$$

The parameter $\Delta \bar{\lambda}$ represents the prescribed initial load increment. Local stability is determined from the tangent stiffness matrix. An equilibrium configuration is stable when all eigenvalues remain positive, while a bifurcation point is signaled by the vanishing of one eigenvalue. The corresponding eigenvector supplies a natural perturbation direction for branch switching. This procedure allows the solver to systematically trace connected equilibrium branches and their bifurcation structure.

Dynamic simulation. In addition to static continuation, we employ dynamic relaxation to compute stable post-buckled configurations and to capture rapid transitions between equilibria (Huang et al., 2025). The governing equation is written as

$$\mathbb{M} \ddot{\mathbf{q}} + \mathbb{C} \dot{\mathbf{q}} + \mathbb{K} \mathbf{q} = \mathbf{F}^{\text{ext}} \quad (31)$$

where $\mathbb{M}$ is the diagonal mass matrix and $\mathbb{C} = \mu \mathbb{M}$ is a viscous damping matrix. Equation (31) is integrated in time using the implicit Euler scheme. Advancing from t_k to $t_{k+1} = t_k + \delta t$, we write

$$\mathbb{M} \ddot{\mathbf{q}}(t_{k+1}) + \mathbb{C} \dot{\mathbf{q}}(t_{k+1}) + \mathbb{K} \mathbf{q}(t_{k+1}) = \mathbf{F}^{\text{ext}}(t_{k+1}), \quad (32)$$

$$\mathbf{q}(t_{k+1}) = \mathbf{q}(t_k) + \delta t \dot{\mathbf{q}}(t_{k+1}), \quad (33)$$

$$\dot{\mathbf{q}}(t_{k+1}) = \dot{\mathbf{q}}(t_k) + \delta t \ddot{\mathbf{q}}(t_{k+1}). \quad (34)$$

The inertia and damping can capture the transient response and drive the system toward a stable equilibrium state. This strategy is particularly effective for transient problems involving severe nonlinearity, snap events, and abrupt changes in configuration.

4. Bifurcation reordering and snap-through pathways

In this section, we present combined numerical and experimental results to characterize the configurational evolution and snap-through pathways of folded elastic ribbons under shear loading. We consider a ribbon with two symmetrically distributed folds located at $\{\Gamma, 1 - \Gamma\} = \{\gamma/l, 1 - \gamma/l\}$. Our main objective is to determine whether the shear-induced snap-through transition proceeds through a symmetric or asymmetric deformation pathway. Throughout this section, the axial compression between the two clamped ends is normalized as $\Delta L = \delta l/l$, and the transverse shear displacement is defined as $\Delta W = \delta w/l$. Unless otherwise specified, the compression is fixed at $\Delta L = 0.5$, and ΔW is taken as the primary bifurcation parameter. The ribbon deformation is characterized by the normalized midpoint height $\Delta Z = \delta z/l$. We focus on the upper pre-compressed equilibrium branch shown in Fig. 1(b)(i), track its evolution under transverse shear as illustrated in Fig. 1(c)(i), and examine how it loses stability and relaxes toward the remaining stable configuration. To quantify the degree of symmetry during deformation, we define the asymmetry measure $\mathcal{A}$ as

$$\mathcal{A} = \int_0^{1/2} \kappa_2(s) ds - \int_{1/2}^1 \kappa_2(s) ds. \quad (35)$$

A value of $\mathcal{A} = 0$ indicates a symmetric configuration, whereas $\mathcal{A} \neq 0$ indicates symmetry-breaking.

In the numerical simulations, the ribbon is taken to have length $l = 1.0$ m, width $w = 0.2$ m, and thickness $b = 1$ mm. The material is characterized by Young's modulus $E = 1.0$ GPa, shear modulus $G = E/3$, and density $\rho = 1000$ kg/m³. For the static continuation, an adaptive step size is used. In the dynamic simulations, the transverse shear is applied at a constant rate $\delta \dot{w} = 0.01$ m/s, which is sufficiently small to ensure a quasi-static loading process. Unless otherwise specified, the ribbon is discretized with $N = 201$ nodes, and a time step of $\delta t = 1$ ms is used, as justified by a convergence study.

4.1. Pitchfork-first ordering and asymmetric snap-through

We first examine a representative case with fold angle $\phi = 90^\circ$ and fold position $\Gamma = 0.13$. The equilibrium branches are obtained by static numerical continuation, and the resulting bifurcation diagram is shown in Fig. 3(a). For clarity, we denote the equilibrium branch with positive midpoint height $\Delta Z > 0$ as the *plus* state $\mathcal{C}^P$, and the equilibrium branch with negative midpoint height $\Delta Z < 0$ as the *minus* state $\mathcal{C}^M$. These two branches represent the two competing stable equilibrium modes of the pre-compressed folded ribbon in the small-shear regime.

As the transverse shear ΔW increases, the stability of the plus branch $\mathcal{C}^P$ changes. The bifurcation diagram shows that this stable branch first loses stability through a subcritical pitchfork bifurcation. Beyond this point, the system departs from the symmetry-preserving branch and evolves toward an *asymmetric* configuration, denoted by $\mathcal{C}^A$. Meanwhile, the unstable continuation of the $\mathcal{C}^P$ branch bends backward and connects to a *M-shaped unstable* configuration, labeled as $\mathcal{C}^U$. This branch structure shows that the loss of stability of the plus state is governed by symmetry breaking rather than by a direct saddle-node instability. Because the subcritical pitchfork bifurcation is encountered before the saddle-node point as the shear increases, the ribbon loses symmetry before snap-through. As a result, the subsequent transition occurs through an asymmetric pathway toward the competing stable configuration. Representative equilibrium configurations on the different branches, including $\mathcal{C}^P$, $\mathcal{C}^A$, $\mathcal{C}^U$ and $\mathcal{C}^M$, for both $\Delta W = 0$ and $\Delta W = 0.2$, are shown in Fig. 3(b), providing a physical interpretation of the bifurcation diagram.

To further elucidate the transient and nonequilibrium dynamics associated with this instability, we perform dynamic simulations under the same loading conditions, as shown in Fig. 3(c). A sufficiently slow shear

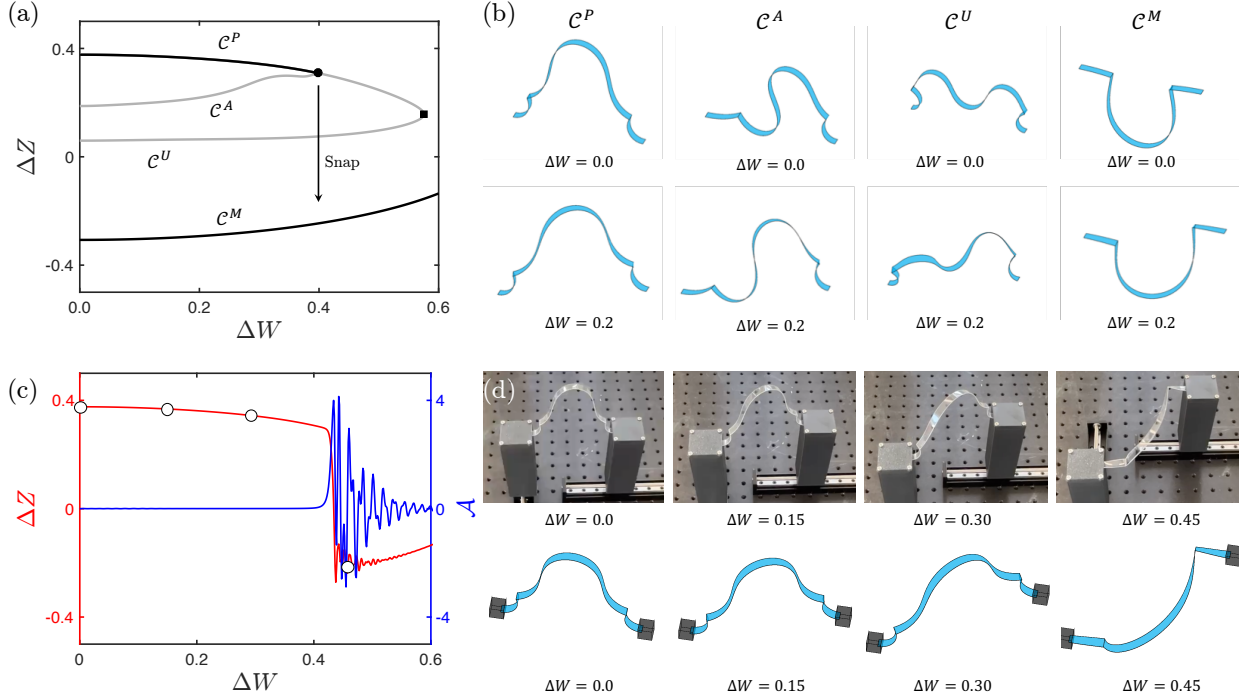


Figure 3: Asymmetric snap-through induced by pitchfork-first bifurcation ordering in a folded elastic ribbon with fold position $\Gamma = 0.13$. (a) Static bifurcation diagram showing the normalized midpoint height ΔZ as a function of the transverse shear ΔW . Black and grey curves denote stable and unstable equilibrium branches, respectively. Circular markers indicate pitchfork bifurcation points, while square markers indicate saddle-node bifurcation points. For this fold position, the pitchfork bifurcation is encountered before the saddle-node bifurcation, so that the ribbon loses symmetry before snap-through. (b) Representative equilibrium configurations associated with different deformation branches: the plus state $\mathcal{C}^P$ with $\Delta Z > 0$, asymmetric state $\mathcal{C}^A$, unstable state $\mathcal{C}^U$, and minus state $\mathcal{C}^M$ with $\Delta Z < 0$. (c) Dynamic simulation showing the normalized midpoint height ΔZ and asymmetry measure $\mathcal{A}$ as functions of the transverse shear ΔW . The sharp growth of $\mathcal{A}$ near snap-through indicates the emergence of an asymmetric transition pathway. (d) Experimental snapshots and corresponding numerical configurations showing the deformation sequence of the folded ribbon under increasing transverse shear.

rate is imposed so that the loading process remains close to quasi-static while still allowing the transient snap-through dynamics to be resolved. The dynamic response shows that the normalized midpoint height ΔZ varies smoothly at first, but then exhibits a sudden jump after the pitchfork instability is encountered. This abrupt change signals the loss of stability of the plus state $\mathcal{C}^P$ and the onset of snap-through. At the same time, the asymmetry measure $\mathcal{A}$ departs markedly from zero during the transition. Since $\mathcal{A} = 0$ corresponds to a symmetric deformation state, the emergence of a finite value of $\mathcal{A}$ demonstrates that the ribbon passes through an asymmetric deformation pathway during snap-through.

The dynamic results therefore indicate that the transition is not symmetry-preserving, but instead proceeds through a pronounced symmetry-breaking process. The corresponding experimental snapshots are presented in Fig. 3(d). The experiments show excellent qualitative agreement with the numerical predictions, capturing both the sequence of configurational changes and the asymmetric character of the snap-through event. In particular, the observed deformation pathway confirms that the ribbon does not transform directly between two symmetry-preserving configurations, but instead passes through an intermediate asymmetric shape before reaching the final stable state. Additional dynamic renderings of this transient process are provided in Supplementary Movie S1.

4.2. Saddle-node-first ordering and symmetric snap-through

Next, we consider another representative case with fold angle $\phi = 90^\circ$ and fold position $\Gamma = 0.21$. The corresponding equilibrium bifurcation structure obtained by static numerical continuation is shown in

Fig. 4(a). As before, the branch with $\Delta Z > 0$ is denoted by $\mathcal{C}^P$ and referred to as the *plus* state, while the branch with $\Delta Z < 0$ is denoted by $\mathcal{C}^M$ and referred to as the *minus* state. These two branches correspond to the competing stable equilibrium states of the pre-compressed ribbon at small shear.

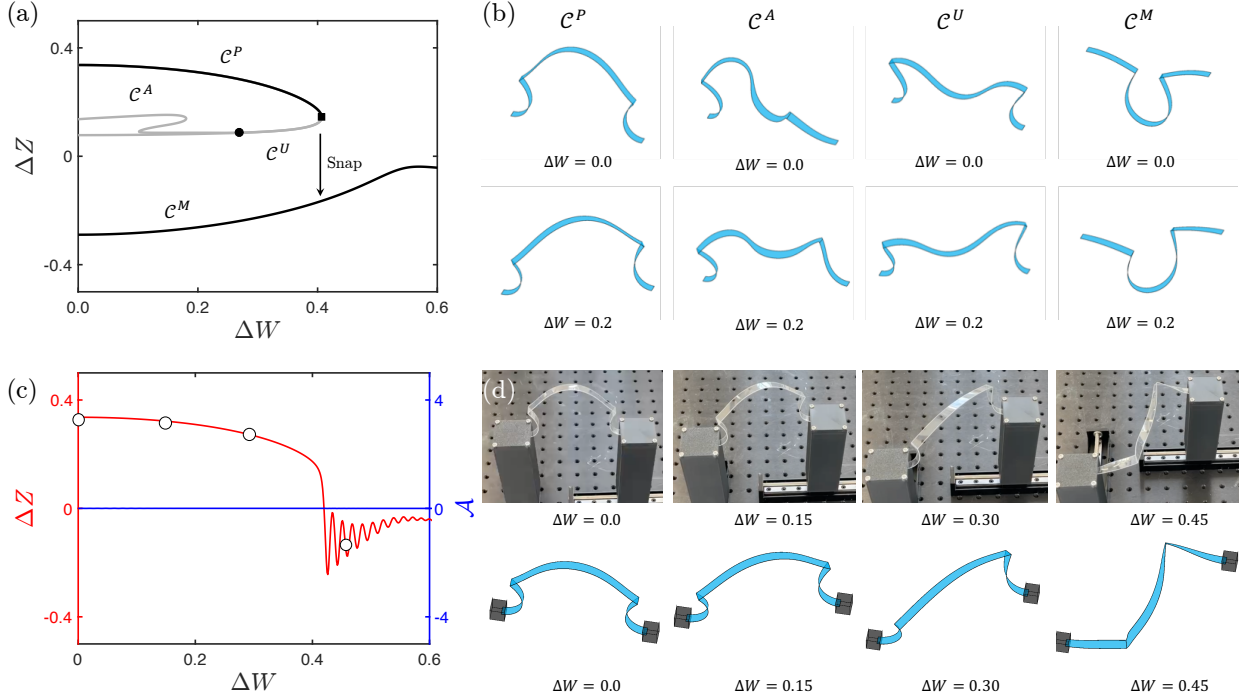


Figure 4: Symmetric snap-through induced by saddle-node-first bifurcation ordering in a folded elastic ribbon with fold position $\Gamma = 0.21$. (a) Static bifurcation diagram showing the normalized midpoint height ΔZ as a function of transverse shear ΔW . Black and grey curves denote stable and unstable equilibrium branches, respectively. Circular markers indicate pitchfork bifurcation points, while square markers indicate saddle-node bifurcation points. For this fold position, the saddle-node bifurcation is encountered before the pitchfork bifurcation, so that the ribbon snaps while maintaining configurational symmetry. (b) Representative equilibrium configurations associated with different deformation branches: the plus state $\mathcal{C}^P$, asymmetric state $\mathcal{C}^A$, unstable state $\mathcal{C}^U$, and minus state $\mathcal{C}^M$. (c) Dynamic simulation showing the normalized midpoint height ΔZ and asymmetry measure $\mathcal{A}$ as functions of transverse shear ΔW . The asymmetry measure remains close to zero during snap-through, confirming a symmetry-preserving transition pathway. (d) Experimental snapshots and corresponding numerical configurations showing the deformation sequence of the folded ribbon under increasing transverse shear.

With increasing transverse shear ΔW , the plus branch $\mathcal{C}^P$ loses stability through a saddle-node bifurcation. Beyond this limit point, the branch turns backward and connects to an *unstable* symmetric M-shaped configuration, labeled as $\mathcal{C}^U$. A secondary bifurcation is also observed on this unstable branch, from which an asymmetric branch $\mathcal{C}^A$ emerges. However, this symmetry-breaking bifurcation is reached only after the saddle-node point. Therefore, the primary instability of the plus state is governed by the saddle-node bifurcation rather than by the pitchfork bifurcation. Since the ribbon loses stability while still remaining on the symmetry-preserving branch, the resulting snap-through transition is triggered before symmetry-breaking can occur. Representative equilibrium configurations associated with $\mathcal{C}^P$, $\mathcal{C}^A$, $\mathcal{C}^U$, and $\mathcal{C}^M$ for both $\Delta W = 0$ and $\Delta W = 0.2$ are displayed in Fig. 4(b).

The corresponding dynamic simulations are shown in Fig. 4(c). Under a sufficiently slow loading rate, the normalized midpoint height ΔZ exhibits a sudden jump when the shear reaches the critical value associated with the saddle-node bifurcation. This jump signifies the onset of snap-through from the plus state toward the competing stable state. Unlike the pitchfork-first case discussed previously, the asymmetry measure $\mathcal{A}$ remains close to zero throughout the transition. This indicates that the snap-through process follows a symmetry-preserving pathway, with no transient symmetry breaking during the dynamic response. The corresponding experimental observations are shown in Fig. 4(d). The experiments are in good qualitative

agreement with the numerical results and confirm that the transition proceeds symmetrically. Additional visualizations of the transient dynamics are provided in Supplementary Movie S2.

4.3. Phase diagram of bifurcation-order-controlled snap-through

We next perform a systematic parameter sweep to identify how fold geometry controls the transition between asymmetric and symmetric snap-through. We first vary the fold position Γ while fixing the fold angle at $\phi = 90^\circ$. The resulting equilibrium surface, shown in Fig. 5(a), reveals a continuous evolution of the bifurcation structure as the two folds move along the ribbon. To more clearly visualize this transition, the corresponding bifurcation points are projected onto the $\{\Delta W, \Gamma\}$ plane in Fig. 5(b), where circular markers denote pitchfork bifurcation points and square markers denote saddle-node bifurcation points.

For small Γ , the pitchfork bifurcation is encountered before the saddle-node bifurcation as the transverse shear ΔW increases. In this regime, the plus branch first loses stability through symmetry breaking, and the ribbon departs from the symmetry-preserving branch before snap-through occurs. The resulting transition therefore follows an asymmetric pathway, consistent with the representative case shown in Fig. 3. As Γ increases, the pitchfork and saddle-node points shift relative to each other, and their ordering eventually reverses. Beyond the transition value of Γ , the saddle-node bifurcation is encountered before the pitchfork bifurcation. The plus branch then remains symmetry-preserving until it reaches the limit point, so the ribbon snaps before any symmetry-breaking instability is triggered. This produces a symmetric snap-through transition, as illustrated by the representative case in Fig. 4.

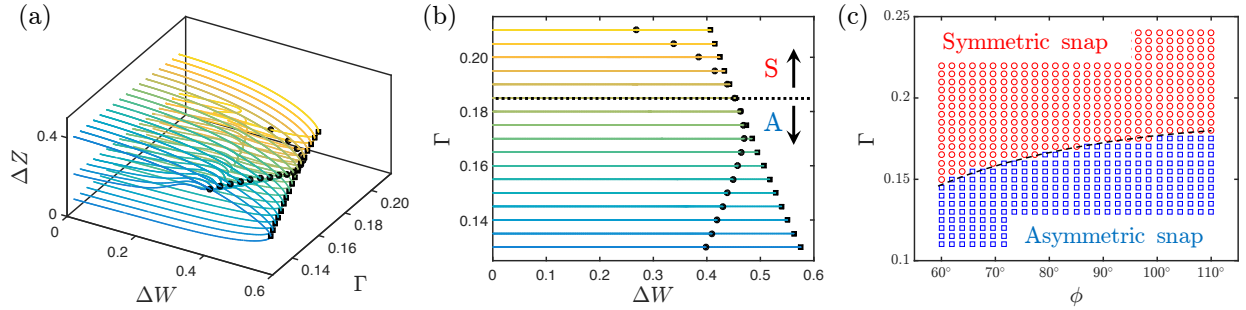


Figure 5: Global bifurcation landscape and phase diagram for folded elastic ribbons. (a) Three-dimensional equilibrium surface showing the normalized midpoint height ΔZ as a function of transverse shear ΔW and fold position Γ for a ribbon with fixed fold angle $\phi = 90^\circ$. The two folds are placed symmetrically at $\{\Gamma, 1 - \Gamma\}$. (b) Projection of the bifurcation structure in (a) onto the $\{\Delta W, \Gamma\}$ plane. Circular markers denote pitchfork bifurcation points, while square markers denote saddle-node bifurcation points. The dashed line marks the transition where the ordering of the two bifurcations changes, separating asymmetric-snap and symmetric-snap regimes. (c) Phase diagram in the ϕ, Γ space. Blue squares and red circles denote asymmetric and symmetric snap-through regimes, respectively. The dashed black line represents the phase boundary, corresponding to the exchange in the ordering of the pitchfork and saddle-node bifurcations.

The dashed line in Fig. 5(b) marks the condition at which the pitchfork and saddle-node bifurcations occur at the same shear displacement. This condition defines the exchange in bifurcation ordering and separates the asymmetric-snap and symmetric-snap regimes. Thus, the transition between the two snap-through modes is not governed simply by the presence of localized folds, but by whether the symmetry-breaking pitchfork bifurcation or the limit-point instability is encountered first during loading.

To generalize this design criterion, we further vary both the fold angle ϕ and fold position Γ , and classify the resulting response according to the observed snap-through pathway. The phase diagram in Fig. 5(c) shows two distinct regimes. In the asymmetric-snap regime, the pitchfork bifurcation precedes the saddle-node bifurcation, so symmetry-breaking occurs before snap-through. In the symmetric-snap regime, the saddle-node bifurcation occurs first, so the ribbon loses stability while still remaining on the symmetry-preserving branch. The phase boundary corresponds to the exchange in ordering between these two bifurcations, providing a geometric criterion for predicting the dominant transition pathway. This map demonstrates that snap-through symmetry can be programmed by tuning the fold position and fold angle, offering a practical design rule for controlling transition pathways in folded elastic ribbons.

5. Reduced-order toy model

To rationalize the bifurcation reordering mechanism observed in the folded ribbon, we adopt a reduced-order double-mass von Mises truss model (Wang et al., 2024; Giudici et al., 2025). The model is not intended to reproduce the full geometry of the ribbon, but to capture the minimal competition between a symmetry-breaking pitchfork bifurcation and a symmetry-preserving saddle-node bifurcation.

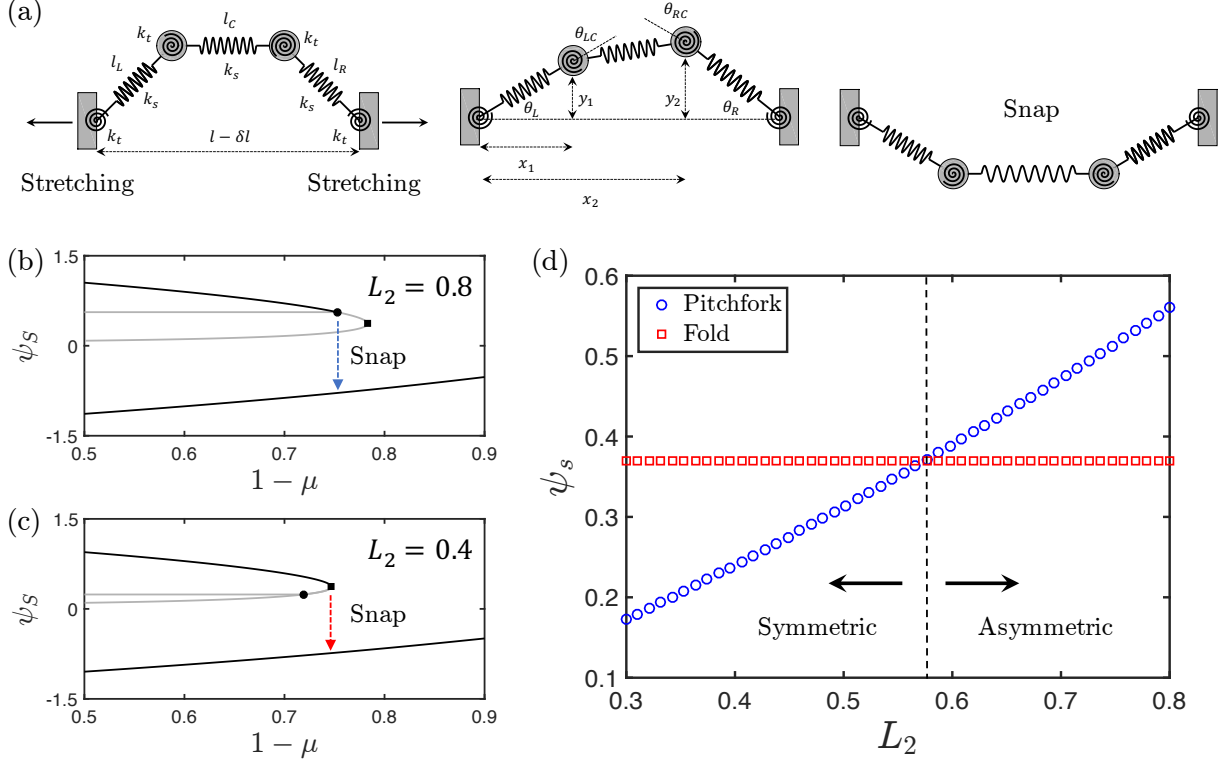


Figure 6: Reduced-order double-mass von Mises truss model for bifurcation reordering. (a) Schematic of the model. Two masses are connected by linear springs and torsional springs, providing a minimal system that captures the competition between symmetry-breaking and symmetry-preserving instabilities. The geometric parameter L_2 controls the relative position of the internal masses and plays a role analogous to the fold position in the ribbon system. (b,c) Static bifurcation diagrams for representative cases with (b) $L_2 = 0.8$ and (c) $L_2 = 0.4$. Black and grey curves denote stable and unstable equilibrium branches, respectively. Circular markers indicate pitchfork bifurcation points, while square markers indicate saddle-node bifurcation points. Changing L_2 reverses the ordering of the two bifurcations and switches the predicted snap-through pathway. (d) Evolution of the critical bifurcation points as a function of L_2 . The dashed line marks the transition where the pitchfork and saddle-node bifurcations coincide. The regime in which the saddle-node bifurcation occurs first corresponds to symmetric snap-through, whereas the regime in which the pitchfork bifurcation occurs first corresponds to asymmetric snap-through.

As shown in Fig. 6(a), the system consists of two point masses connected to the left and right supports by two linear springs of stiffness k_s and natural length l_1 . The instantaneous lengths of these two side springs are denoted by l_L and l_R , respectively. The two masses are further connected by a central spring with the same stiffness k_s and natural length l_2 , whose instantaneous length is denoted by l_C . Torsional springs of stiffness k_t are placed at the two supports and at the two internal joints. The support torsional springs have a prescribed rest angle α , whereas the internal torsional springs are relaxed when the adjacent springs are aligned. The geometric parameter l_2 controls the relative spacing between the two masses and plays a role analogous to the fold position in the ribbon system. Starting from the upper equilibrium branch, we gradually release the imposed compression and track the equilibrium paths and stability changes. This loading protocol allows us to determine whether the pitchfork or saddle-node bifurcation is encountered first, and hence whether the model predicts an asymmetric or symmetric snap-through.

Kinematics and energy formulation. The positions of the two nodes are employed to develop a model, $[x_1, y_1, x_2, y_2]$, and the total length of the system is

$$l = 2l_1 + l_2, \quad (36)$$

and the uniaxial compression is defined as δl . Based on the geometry, the instantaneous length of the left, middle, and right springs are given by,

$$\begin{aligned} l_L &= \sqrt{w_L^2 + y_1^2}, \\ l_R &= \sqrt{w_R^2 + y_2^2}, \\ l_C &= \sqrt{w_C^2 + y_C^2}, \end{aligned} \quad (37)$$

where

$$\begin{aligned} w_L &= x_1, \\ w_R &= l - \delta l - x_2, \\ w_C &= x_2 - x_1, \\ y_C &= y_1 - y_2. \end{aligned} \quad (38)$$

The angle between the two consecutive nodes is formulated as

$$\begin{aligned} \theta_L &= \arctan\left(\frac{y_1}{w_L}\right), \\ \theta_R &= \arctan\left(\frac{y_2}{w_R}\right), \\ \theta_{LC} &= \theta_L - \Delta\theta, \\ \theta_{RC} &= \theta_R + \Delta\theta, \end{aligned} \quad (39)$$

with

$$\Delta\theta = \arctan\left(\frac{y_C}{w_C}\right). \quad (40)$$

Following the previous study (Wang et al., 2024), we characterize the symmetric and asymmetric responses of the system using two parameters,

$$\begin{aligned} \theta_S &= \frac{1}{2}(\theta_L + \theta_R), \\ \theta_A &= \frac{1}{2}(\theta_L - \theta_R). \end{aligned} \quad (41)$$

The total energy is formulated the sum of the stretching and bending energies:

$$\mathcal{U} = \frac{1}{2}k_s \left[(l_L - l_1)^2 + (l_R - l_1)^2 + (l_C - l_2)^2 \right] + \frac{1}{2}k_t \left[(\theta_L + \alpha)^2 + (\theta_R + \alpha)^2 + \theta_{LC}^2 + \theta_{RC}^2 \right], \quad (42)$$

The dynamic equilibrium can be derived by solving the following equations,

$$\begin{aligned} m\ddot{x}_1 + \frac{\partial \mathcal{U}}{\partial x_1} &= 0, \\ m\ddot{y}_1 + \frac{\partial \mathcal{U}}{\partial y_1} &= 0, \\ m\ddot{x}_2 + \frac{\partial \mathcal{U}}{\partial x_2} &= 0, \\ m\ddot{y}_2 + \frac{\partial \mathcal{U}}{\partial y_2} &= 0. \end{aligned} \quad (43)$$

For a prescribed end displacement, the equilibrium configurations are obtained by solving the stationarity condition of the total elastic energy for the nodal coordinates $\mathbf{q}_{\text{toy}} = [x_1, y_1, x_2, y_2]$. Stability is then assessed from the local stiffness matrix, defined as the Hessian of the energy with respect to $\mathbf{q}_{\text{toy}}$: stable equilibria have positive eigenvalues, whereas bifurcation points are identified when the minimum eigenvalue approaches zero. The nodal coordinates are finally transformed into the symmetric and asymmetric coordinates, θ_S and θ_A , to distinguish symmetry-preserving and symmetry-breaking responses (Wang et al., 2024; Giudici et al., 2025).

Non-dimensionalization. Next, we rescale all the parameters and make the problem dimensionless. The normalized physical parameters are used,

$$\mu = \frac{\delta l}{\alpha^2 l}, \quad \psi_S = \frac{\theta_S}{\alpha}, \quad \psi_A = \frac{\theta_A}{\alpha}, \quad \Lambda = \frac{3k_t}{\alpha^2 k_s l_1^2}, \quad L_2 = \frac{l_2}{l_1}, \quad T = t \sqrt{\frac{k_s}{m}} \quad (44)$$

Static solution. By using the small-angle approximation, we can get the analytical solution of the equilibrium state (Wang et al., 2024). First of all, for the symmetric solution, $\psi_A = 0$, we can solve for ψ_S ,

$$\begin{aligned} \psi_S^N &= - \frac{\sqrt[3]{2} (a_3 \mu - a_2)}{\sqrt[3]{\sqrt{108 a_4^3 (a_2 - a_3 \mu)^3 + 729 a_1^2 a_4^4} + 27 a_1 a_4^2}} \\ &\quad - \frac{\sqrt[3]{\sqrt{108 a_4^3 (a_2 - a_3 \mu)^3 + 729 a_1^2 a_4^4} + 27 a_1 a_4^2}}{3 \sqrt[3]{2} a_4} \\ \psi_S^I &= - \frac{(1 + i\sqrt{3}) (a_2 - a_3 \mu)}{2^{2/3} \sqrt[3]{\sqrt{108 a_4^3 (a_2 - a_3 \mu)^3 + 729 a_1^2 a_4^4} + 27 a_1 a_4^2}} \\ &\quad + \frac{(1 - i\sqrt{3}) \sqrt[3]{\sqrt{108 a_4^3 (a_2 - a_3 \mu)^3 + 729 a_1^2 a_4^4} + 27 a_1 a_4^2}}{6 \sqrt[3]{2} a_4} \\ \psi_S^U &= - \frac{(1 - i\sqrt{3}) (a_2 - a_3 \mu)}{2^{2/3} \sqrt[3]{\sqrt{108 a_4^3 (a_2 - a_3 \mu)^3 + 729 a_1^2 a_4^4} + 27 a_1 a_4^2}} \\ &\quad + \frac{(1 + i\sqrt{3}) \sqrt[3]{\sqrt{108 a_4^3 (a_2 - a_3 \mu)^3 + 729 a_1^2 a_4^4} + 27 a_1 a_4^2}}{6 \sqrt[3]{2} a_4} \end{aligned} \quad (45)$$

where

$$a_1 = \frac{2}{3} \Lambda, \quad a_2 = \frac{4}{3} \Lambda, \quad a_3 = \frac{2}{3} (2 + L_2), \quad a_4 = \frac{2}{3}. \quad (46)$$

Here, ψ_S^I and ψ_S^N denote the inverted and natural states of the truss, respectively, which are analogous to the plus state $\mathcal{C}^P$ and minus state $\mathcal{C}^M$ in the folded-ribbon system. The intermediate branch ψ_S^U denotes the unstable state connecting these two symmetric configurations. The critical saddle-node bifurcation point is obtained from the condition $\psi_S^I = \psi_S^U$,

$$\psi_S^{\text{SN}} = \left(\frac{\Lambda}{2} \right)^{1/3}. \quad (47)$$

Secondly, for the asymmetric solution, $\psi_A \neq 0$, the solution is (Wang et al., 2024),

$$\psi_S^A = \frac{1}{4} L_2 (L_2 + 2), \quad (48)$$

and, thus, the critical pitchfork bifurcation occurs at:

$$\psi_S^{\text{PF}} = \frac{1}{4} L_2 (L_2 + 2). \quad (49)$$

On one side, when $\psi_S^{\text{PF}} > \psi_S^{\text{SN}}$, the pitchfork bifurcation is encountered before the saddle-node bifurcation during the release process. The system therefore loses symmetry before snap-through, leading to an asymmetric transition, as shown in Fig. 6(b). On the other side, when $\psi_S^{\text{SN}} > \psi_S^{\text{PF}}$, the saddle-node bifurcation is encountered first. In this case, the system loses stability while remaining on the symmetric branch, resulting in a symmetric snap-through transition, as shown in Fig. 6(c).

The critical value of L_2 separating these two bifurcation-ordering regimes is obtained by solving $\psi_S^{\text{SN}} = \psi_S^{\text{PF}}$, which gives

$$L_2^* = 2\sqrt{\left(\frac{\Lambda}{2}\right)^{1/3} + \frac{1}{4}} - 1. \quad (50)$$

This critical value is shown in Fig. 6(d). Note that in all plots in Fig. 6, we use $\alpha = 0.1$ and $\Lambda = 0.1$, yielding $L_2^* = 0.5728$. Thus, varying L_2 across this critical value reverses the ordering of the pitchfork and saddle-node bifurcations and switches the predicted snap-through pathway between asymmetric and symmetric modes.

Dynamic validation. Finally, we perform dynamic simulations of the reduced-order toy model to confirm the bifurcation-ordering prediction and to resolve the transient snap-through process. The bifurcation parameter is swept slowly by prescribing a small loading rate $\dot{\mu}$, such that $1 - \mu = \dot{\mu}T$, together with a small imperfection to trigger branch selection when symmetry-breaking occurs. In the simulations, we use $\dot{\mu} = 0.001$, which is sufficiently slow to approximate quasi-static loading while still capturing the dynamic transition between equilibrium states.

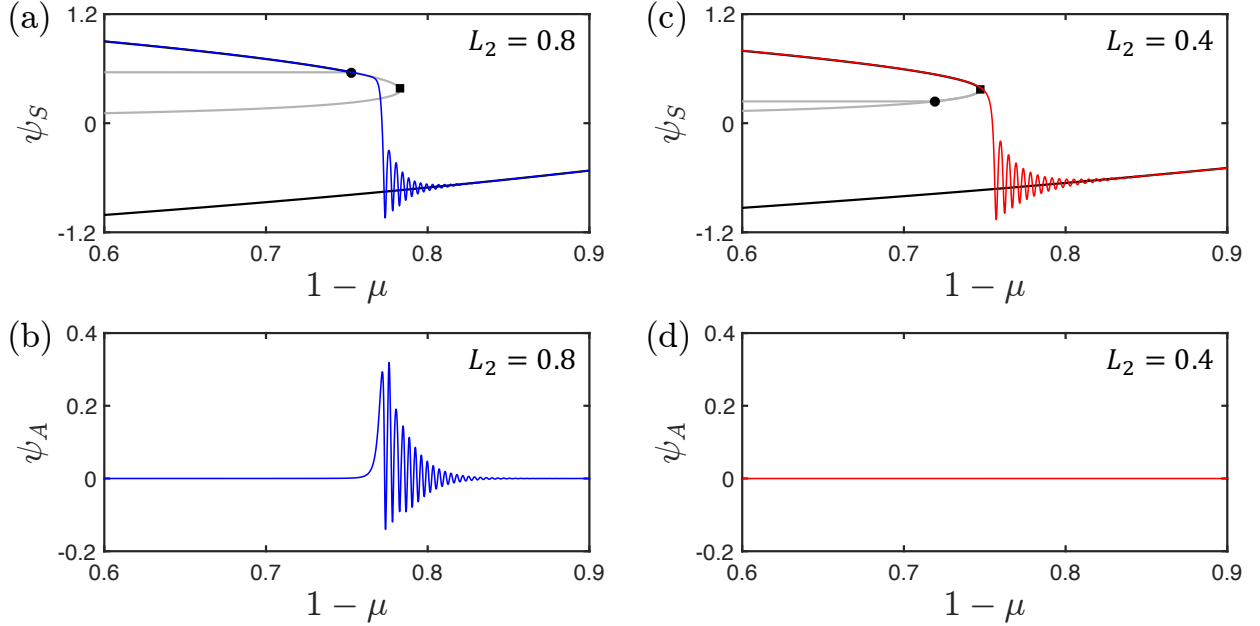


Figure 7: Dynamic trajectories of the double-mass von Mises truss model. (a,b) Evolution of the symmetric coordinate ψ_S and asymmetric coordinate ψ_A as functions of the control parameter $1 - \mu$ for $L_2 = 0.8$. In this case, the pitchfork bifurcation is encountered before the saddle-node bifurcation, and the growth of ψ_A during snap-through indicates an asymmetric transition. (c,d) Evolution of ψ_S and ψ_A for $L_2 = 0.4$. Here, the saddle-node bifurcation occurs before the pitchfork bifurcation, and ψ_A remains zero throughout the dynamic response, indicating a symmetry-preserving snap-through transition.

Figure 7(a,b) show the dynamic response for $L_2 = 0.8$. In this case, the pitchfork bifurcation is encountered before the saddle-node bifurcation. As ψ_S passes the pitchfork point, the asymmetric coordinate ψ_A grows from zero, indicating that the system loses symmetry before snap-through and transitions through an asymmetric pathway. Figure 7(c,d) show the corresponding response for $L_2 = 0.4$. Here, the saddle-node bifurcation is encountered first, and ψ_A remains zero throughout the transition. The system therefore snaps

while preserving symmetry, consistent with the saddle-node-first ordering predicted by the static analysis. These dynamic simulations therefore provide a time-dependent validation of the static bifurcation analysis, confirming that the relative ordering of the pitchfork and saddle-node bifurcations controls whether snap-through proceeds through an asymmetric or symmetric pathway.

6. Conclusions

In this work, we investigated the shear-induced snap-through behavior of folded elastic ribbons and demonstrated that the symmetry of the transition is governed by the competition between distinct instability mechanisms. By combining discrete simulations, experiments and a reduced-order model, we showed that asymmetric and symmetric snap-through pathways are selected by the relative ordering of a symmetry-breaking pitchfork bifurcation and a saddle-node bifurcation on the symmetry-preserving branch. When the pitchfork bifurcation is encountered first, the ribbon loses symmetry before snap-through and follows an asymmetric transition pathway. Conversely, when the saddle-node bifurcation occurs first, the ribbon loses stability while remaining on the symmetry-preserving branch, leading to a symmetric snap-through transition. These results establish bifurcation reordering as the key mechanism by which localized folds control snap-through symmetry.

We further showed that fold placement acts as a sensitive geometric control parameter that reorganizes the stability landscape, even when the overall ribbon geometry remains symmetric. Numerical continuation and stability analysis enabled systematic exploration of the design space and revealed a clear phase boundary separating asymmetric and symmetric snap-through regimes. The reduced-order double-mass von Mises truss model captured the same exchange in bifurcation ordering, indicating that the observed behavior is not specific to the detailed ribbon geometry but reflects a more general consequence of instability competition and energy-landscape restructuring. Collectively, these findings highlight the global mechanical role of localized folds and provide a purely geometric strategy for programming nonlinear transition pathways in slender elastic structures. This principle may inform the design of programmable multistable systems, morphing structures and instability-driven mechanical devices.

CRedit authorship contribution statement

Weicheng Huang: Writing – Original draft, Methodology, Software, Formal analysis, Investigation, Data curation, Validation, Project administration, Funding acquisition, Conceptualization. **Qun Zhang:** Formal analysis, Investigation, Validation. **Bohan Zhang:** Writing – Review & Editing, Formal analysis, Investigation, Validation. **Mingchao Liu:** Writing – Review & Editing, Methodology, Formal analysis, Investigation, Validation, Supervision, Project administration, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

W.H. acknowledges the start-up funding from Newcastle University, UK. M.L. acknowledges start-up funding from The University of Birmingham, UK.

Appendix A. Video

We provide two videos as supplementary material to illustrate our dynamic results.

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