

Model-Informed Joint Material-Structural Optimization of Hard-Magnetic Soft Materials

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Abstract

This work develops a model-informed framework for predictive analysis and optimal design of hard-magnetic soft materials (hMSMs). These materials undergo contact-free, field-driven deformation, making them attractive for soft robotics, adaptive structures, and bio-inspired systems. Accurate prediction requires effective structure–property relations, while optimal design requires simultaneous control of structural density, magnetic particle distribution, and remanent magnetization direction. To address these issues, this work makes two main contributions. First, classical rigid-inclusion relations, a Hill self-consistent relation, and constrained-kinematics models are placed into a unified effective shear-modulus framework for particle-filled elastomers. With one default control relation, seven shear-modulus relations are combined with three strain-energy density functions to obtain 21 constitutive models. The results show that the strain-energy density form has a relatively small effect for the actuation problems considered, whereas the effective shear-modulus relation can significantly affect deformation when magnetic material overlaps with highly deforming regions. Experimental stress–strain data are then used to select a representative shear-modulus relation, with the Mooney relation giving the best overall agreement. Second, using the selected constitutive model, a joint material–structural optimization framework is developed for simultaneous design of structural density, magnetic particle volume fraction, and remanent magnetization direction. Rotational, translational, and restorative examples show that the framework handles different active design fields, objectives, and single- or multi-load-case formulations, producing non-intuitive hMSM designs with prescribed deformation responses. The framework is implemented in the open-source CEADpx/top_optim repository.

Keywords: Hard-magnetic soft materials, topology optimization, constitutive model selection, structure–property relations, magneto–mechanical coupling, joint material–structural design
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Contents

1	Introduction	3
1.1	Constitutive modeling of hMSMs: Effective shear-modulus and strain-energy density functions	3

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1.2	Topology optimization and joint material–structural design of hMSMs	5
1.3	Scope and contributions	6
1.4	Organization of the article	7
1.5	Software availability	8
2	Continuum formulation of hard-magnetic soft materials (hMSMs)	8
2.1	Notation, assumptions, and kinematics	9
2.2	Magneto-mechanical energy and equilibrium	11
2.3	Elastic strain-energy density functions	12
2.4	Effective shear-modulus relations	13
2.5	Constitutive relations summary	17
3	Numerical investigations of model dependence in hMSM deformation	17
3.1	Cantilever beam benchmarks	17
3.1.1	Problem setup	17
3.1.2	Results	18
3.2	Wheel and gripper benchmarks	20
3.2.1	Problem setup	20
3.2.2	Results	20
3.3	Three-dimensional beam actuation tests	22
3.3.1	Problem setup	22
3.3.2	Results	23
3.4	Summary of key observations	23
4	Constitutive model selection	24
4.1	Selection metric 1: shear-modulus data	25
4.2	Selection metric 2: Stress–strain data	25
4.3	Summary of results and selected model	27
5	Joint material–structural optimization formulation	27
5.1	Design variable parameterization	29
5.1.1	Density field	29
5.1.2	Magnetic particle volume fraction field	30
5.1.3	Remanent magnetization direction field	31
5.2	Energy interpolation and design variable coupling	31
5.2.1	Density-penalized moduli and effective material response	32
5.2.2	Magnetic energy coupling	32
5.3	Optimization formulation	32
5.4	Adjoint-based gradient computation	34
6	Design of hMSM structures using joint material–structural optimization	36
6.1	Key computational details	36
6.2	Rotational actuator	38
6.3	Translational actuator	39
6.4	Restorative beam	40
7	Conclusion	42

References	43
Appendix A Rigid-inclusion limit of Hill’s self-consistent shear-modulus relation	47
Appendix B Mesh convergence studies	49
Appendix B.1 Beam benchmark	49
Appendix B.2 Wheel and gripper benchmarks	49
Appendix C Details of constitutive model selection	50
Appendix C.1 Experimental data extraction and processing	50
Appendix C.2 Metric 1 details	51
Appendix C.3 Uniaxial simulation setup for Metric 2	51
Appendix C.4 Metric 2 averaging definitions	52
Appendix C.5 Additional Metric 2 summary at fixed G_0	52

1. Introduction

Hard-magnetic soft materials (hMSMs) are magnetically active composites consisting of hard-magnetic particles embedded within a compliant elastomeric matrix. Unlike soft-magnetic materials, hMSMs retain a remanent magnetization after magnetization and can therefore undergo large, programmable deformations when subjected to external magnetic fields [Kim and Zhao \(2022\)](#), [Lu et al. \(2024\)](#), [Narayanan et al. \(2024\)](#), [Zhao et al. \(2019\)](#). Their contact-free actuation, high compliance, and potential compatibility with soft biological environments make them attractive for applications in soft robotics, shape-morphing structures, adaptive systems, biomedical devices, bio-inspired mechanisms, and soft adhesives [Abbasi et al. \(2023\)](#), [Alapan et al. \(2020\)](#), [Bastola and Hossain \(2021\)](#), [Chen et al. \(2022\)](#), [Li et al. \(2025\)](#), [Lucarini et al. \(2022\)](#), [Wu et al. \(2020b\)](#), [Zhao et al. \(2022\)](#).

The design of hMSM structures raises two closely connected issues. The first is predictive modeling: how constitutive choices, including magnetic energy, strain-energy density, and effective structure–property relations, influence the predicted deformation response. This issue is particularly important here because the work brings together several effective-modulus relations for particle-filled elastomers, which can predict different stiffness values at the same particle volume fraction. The second is optimal design: how structural density, magnetic particle distribution, and remanent magnetization direction should be varied together to achieve prescribed actuation responses. The following subsections organize the introduction around these two issues and then summarize the scope and contributions of this work.

1.1. Constitutive modeling of hMSMs: Effective shear-modulus and strain-energy density functions

Predictive modeling of hMSMs requires a finite-strain magneto-mechanical formulation together with structure–property relations that describe how magnetic particle volume fraction modifies the effective elastic moduli. Energy-based continuum formulations have provided a useful foundation for modeling large magnetically induced deformation [Kadapa and Hossain \(2022\)](#), [Zhao et al. \(2019\)](#), while magneto-viscoelastic and microstructure-informed formulations have been developed to capture additional features of magneto-active and hard-magnetic soft materials [Garcia-Gonzalez \(2019\)](#), [Garcia-Gonzalez and Hossain \(2021\)](#), [Rambausek et al. \(2022\)](#), [Saxena et al. \(2013, 2014\)](#),

Stewart and Anand (2023). However, the effective material response also depends strongly on particle volume fraction, particle arrangement, and particle–matrix interactions. Experimental and modeling studies show that filler content and microstructure can substantially modify stiffness, hysteresis, and magnetic actuation response Garcia-Gonzalez et al. (2021), Kramarenko et al. (2015), Moreno-Mateos et al. (2022), Sorokin et al. (2015). Thus, predictive hMSM modeling requires not only a magneto-mechanical energy formulation, but also appropriate structure–property relations for the effective elastic moduli.

The choice of magnetic energy in a free-energy-based magneto-mechanical formulation is another important source of constitutive modeling. Many hMSM formulations use a Zeeman (magnetostatics) energy to represent the interaction between the remanent magnetization embedded in the material and the externally applied magnetic field Mukherjee et al. (2021), Stewart and Anand (2023), Zhao et al. (2019). Even within this class, different kinematic choices are possible. For example, the referential remanent magnetic field may be mapped to the current configuration using the full deformation gradient $\mathbf{F}$, or using only the rotational part $\mathbf{R}$ from the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$, where $\mathbf{U}$ is the right stretch tensor. The $\mathbf{F}$ -based form couples the magnetic energy directly to the full deformation gradient and therefore includes the effect of stretch on the mapped magnetic vector, whereas the $\mathbf{R}$ -based form emphasizes rotation of the embedded magnetic dipoles without changing their magnitude through stretch. Beyond Zeeman-type energies, dipole–dipole interactions between magnetized particles may also be included, introducing deformation-dependent magnetic interactions through changes in particle spacing and relative orientation Garcia-Gonzalez and Hossain (2021), Ivaneyko et al. (2012). Such terms can be important when particle arrangement, particle spacing, and local magnetic interactions strongly influence the response. In the present work, the $\mathbf{F}$ -based Zeeman form is employed so that the sensitivity associated with elastic strain-energy densities and effective shear-modulus relations can be isolated. Extensions to dipole–dipole interaction energies, $\mathbf{R}$ -based Zeeman-type formulations, and viscoelastic models can be accommodated within the same energy-based analysis framework.

Several structure–property relations have been proposed for particle-filled elastomers and related composites. Classical rigid-inclusion theories describe the effect of particle volume fraction on effective stiffness through modified elastic moduli Guth (1945), Kerner (1956), Mooney (1951); a review of several classical results is provided in Ahmed and Jones (1990). Hill’s self-consistent theory Hill (1965) provides general relations between the effective elastic moduli, the elastic moduli of the constituent phases, and their respective volume fractions. This type of effective modulus–volume fraction relation has also been used to model magnetic-elastomer-based adhesives Zhao et al. (2022). A complementary viewpoint is provided by homogenization-based and constrained-kinematics models, which start from the observation that the deforming matrix surrounding rigid inclusions need not experience the same deformation invariant as the macroscopic composite Alshammari et al. (2019), Lopez-Pamies et al. (2013a,b). In these models, particle reinforcement is represented by modifying the invariant supplied to the matrix strain-energy function, typically the first invariant of the right Cauchy–Green deformation tensor, together with a matrix-volume-fraction factor. This contrast is useful for hMSMs because the magnetic particle volume fraction affects both elastic stiffness and magnetic actuation strength. In this work, these two viewpoints are brought into a common effective shear-modulus representation, enabling direct comparison and model selection.

In addition to the effective shear-modulus relation, the selected strain-energy density function can influence both predicted deformation and numerical robustness. Understanding whether different strain-energy forms and effective modulus models produce similar or distinct responses is there-

fore important for predictive accuracy and computational efficiency. For example, the isochoric–volumetric neo-Hookean form (**NH1**) in Equation (14) may require smaller load increments or may fail to converge beyond moderate applied magnetic fields, whereas the reformulated neo-Hookean form (**NH2**) in Equation (15) can permit larger load steps and larger maximum applied fields while producing similar deformation responses in the cases considered.

Motivated by these considerations, this work fixes the magnetic energy to the $\mathbf{F}$ -based Zeeman form and focuses on the elastic strain-energy density and effective shear-modulus relation. A central contribution is a unified effective-modulus framework in which classical rigid-inclusion models, Hill’s self-consistent relation, and constrained-kinematics models are recast as seven candidate shear-modulus relations $G(\phi)$. Combined with three strain-energy density functions, this gives 21 candidate constitutive models and provides a basis for assessing when model choice changes the predicted hMSM response. The sensitivity study is therefore not limited to a single benchmark: uniformly magnetized and tip-localized beams separate the effects of magnetic loading and particle placement, while wheel, gripper, and three-dimensional actuation examples test whether the same trends persist across different deformation modes. These studies show that the two neo-Hookean forms, **NH1** and **NH2**, and the St. Venant–Kirchhoff model, **SVK**, give nearly identical deformation measures in the cases considered, whereas the selected $G(\phi)$ relation becomes important when magnetic material overlaps with mechanically active regions. The subsequent model-selection step uses experimental stress–strain data from Garcia-Gonzalez et al. (2021) to compare the candidate $G(\phi)$ relations against both extracted shear-modulus values and full true stress–strain responses. The Mooney relation gives the best combined performance and is therefore used with **NH2** in the joint material–structural optimization examples.

1.2. Topology optimization and joint material–structural design of hMSMs

Topology optimization provides a systematic framework for designing hMSM structures by distributing structural and magnetic properties over a design domain. Classical density-based topology optimization has been widely used to obtain non-intuitive structures for prescribed objectives and constraints Bendsoe and Sigmund (2013), Sigmund (2001). For magnetic soft actuators and robots, prior works have optimized magnetization distributions, external fields, voxel encodings, and prescribed material structures to realize programmable deformation and locomotion Lloyd et al. (2020), Lum et al. (2016), Wang et al. (2021), Wu et al. (2020a). Related topology optimization formulations have also been developed for magnetically active actuators and ferromagnetic soft robots, including designs based on prescribed magnetization distributions or simplified mechanical models Sundaram et al. (2019), Tian et al. (2020). For hMSMs, Zhao and Zhang (2022) developed a large-deformation topology optimization framework in which matrix topology, remanent magnetization distribution, and applied magnetic fields are optimized using a discrete set of candidate magnetization directions Zhao and Zhang (2022). This direction was later extended to encoded reprogrammable magneto-mechanical properties Zhao and Zhang (2023). These efforts provide important computational design tools for magnetic soft materials.

The present work is complementary to these advances but focuses on a different design question: how to optimize structural density (topology), magnetic particle concentration, and remanent magnetization direction when the particle volume fraction affects both stiffness and magnetic actuation. In prior large-deformation topology optimization of hMSMs, the remanent magnetization direction is selected from a finite set of prescribed candidate directions, with the optimization algorithm promoting the selection of one direction at each design element Zhao and Zhang (2022). If no magnetized state is selected, the corresponding region effectively has no remanent magnetic material.

Thus, in terms of magnetic material content, the description is closer to a discrete choice between $\phi = 0$ and a prescribed magnetized material state, rather than a continuous particle-volume-fraction field. This provides a powerful discrete encoding of magnetic architecture, but it does not treat the magnetic particle volume fraction as a continuous material-composition field that modifies the effective elastic response. In the present formulation, the magnetic particle volume fraction ϕ is introduced as an independent continuous design variable. This field scales the local magnetic interaction energy and also modifies the effective shear modulus through the selected structure–property relation $G(\phi)$. Thus, optimizing ϕ introduces a direct competition between increasing magnetic actuation and changing local stiffness. When combined with the structural density ρ and the continuous remanent magnetization direction θ_{Br} , this leads to a joint material–structural optimization problem in which topology, particle concentration, and magnetic orientation interact through the same nonlinear finite-strain equilibrium equations.

A central difficulty in joint material–structural optimization is that the three design fields do not have the same physical status. The structural density ρ determines whether material exists locally, whereas the magnetic particle volume fraction ϕ and remanent magnetization direction θ_{Br} are meaningful only within material-filled regions. If these fields are introduced independently without additional coupling, then regions with $\rho \approx 0$ can still carry arbitrary values of ϕ and θ_{Br} . In such regions, the magnetic variables may have little or no physical effect on the equilibrium response, creating nonunique design descriptions and poorly conditioned optimization directions. Thus, the formulation must allow ρ , ϕ , and θ_{Br} to remain independent design variables while ensuring that magnetic material content and remanent direction are physically tied to the presence of structural material.

This issue is addressed through the energy interpolation in the proposed formulation. Instead of imposing separate constraints on ϕ and θ_{Br} in low-density regions, the physical coupling among structural density, magnetic particle concentration, and remanent direction is built directly into the interpolated free energy; see [Section 5.2](#). This construction makes the magnetic contribution depend on the presence of structural material while still allowing ρ , ϕ , and θ_{Br} to remain independent design fields. Building on the density-based optimization infrastructure of FEniTop [Jia et al. \(2024\)](#), the resulting joint material–structural optimization framework can optimize any active subset of these fields, or all three simultaneously. The same formulation also accommodates different objective functions and single- or multi-load-case settings, enabling prescribed-structure magnetic design and full joint material–structural design within one framework.

1.3. Scope and contributions

This work has two connected parts. In the first part, effective-modulus relations and constrained-kinematics models for particle-reinforced elastomers are placed into a common shear-modulus form $G(\phi)$, producing six nontrivial structure–property relations together with one default control relation. These seven relations are combined with three strain-energy density functions to obtain 21 constitutive models for sensitivity analysis of hMSM deformation. Experimental stress–strain data from [Garcia-Gonzalez et al. \(2021\)](#) are then used to select a representative shear-modulus relation for the design studies. The second part develops a joint material–structural optimization framework in which magnetic particle volume fraction, remanent magnetization direction, and structural density can be optimized selectively or simultaneously.

Main contributions:

1. A unified effective shear-modulus representation that brings classical rigid-inclusion relations, a Hill self-consistent relation specialized to rigid inclusions and a near-incompressible matrix, and constrained-kinematics models into a common $G(\phi)$ framework.
2. A systematic sensitivity analysis of 21 constitutive models obtained by combining seven effective shear-modulus relations with three strain-energy density functions.
3. Selection of the $G(\phi)$ relation using two data-based metrics: error relative to extracted shear-modulus values and error relative to simulated true stress–strain responses from [Garcia-Gonzalez et al. \(2021\)](#).
4. A joint material–structural optimization framework for hMSMs in which structural density, magnetic particle volume fraction, and remanent magnetization direction are coupled through a nonlinear finite-strain magneto-mechanical energy formulation.
5. Demonstrations of rotational, translational, and restorative hMSM designs using different active design fields, objective functions, and single- or multi-load-case formulations.
6. An open-source implementation in `CEADpx/top_optim` to support reproducibility, extension, and broader use in hMSM and field-responsive soft-material design studies.

The effective shear-modulus framework considered here is not limited to hMSMs. Similar particle-volume-fraction-dependent relations are used for conventional filled elastomers and magnetorheological elastomers, where filler content and particle arrangement can strongly affect the effective stiffness [Guth \(1945\)](#), [Ivaneyko et al. \(2011\)](#), [Jolly et al. \(1996\)](#). Different research communities often use different reinforcement or homogenization models for this same problem. Therefore, comparing these relations in a common form and selecting a suitable model using experimental data can also be useful for other filled-elastomer systems.

1.4. Organization of the article

The remainder of this article is organized as follows:

- [Section 2](#) presents the continuum formulation for hMSMs, including the magneto-mechanical free energy, equilibrium equations, elastic strain-energy density functions, and effective shear-modulus relations used to represent the influence of magnetic particle volume fraction.
- [Section 3](#) investigates model dependence in forward actuation problems and compares the effects of strain-energy density functions and effective shear-modulus relations on predicted deformation.
- [Section 4](#) uses experimental stress–strain data from the literature to select a representative effective shear-modulus relation for the optimization studies.
- [Section 5](#) introduces the joint material–structural optimization formulation, including the parameterization of structural density, magnetic particle volume fraction, and remanent magnetization direction, their coupling through the total free energy, and the adjoint-based gradient computation.

- [Section 6](#) demonstrates the proposed joint material–structural topology optimization framework, using the selected constitutive model, through rotational, translational, and restorative actuation examples.
- [Section 7](#) summarizes the main findings and discusses the implications of the proposed framework.
- The appendices provide additional details on the Hill self-consistent shear-modulus relation ([Appendix A](#)), mesh convergence studies ([Appendix B](#)), and the constitutive model-selection procedure ([Appendix C](#)).

1.5. Software availability

The computational framework is implemented in the open-source `top_optim` software repository¹ [Galloway and Jha \(2026\)](#). The repository contains the finite-element implementation, constitutive-model evaluation scripts, and optimization examples used in this work, including the beam, wheel, gripper, translational-actuator, and restorative-beam studies.

2. Continuum formulation of hard-magnetic soft materials (hMSMs)

Deformation of hard-magnetic soft materials depends on finite-strain elasticity, magnetic actuation, and the effective material properties induced by magnetic particle inclusions. As a starting point, an energy-based continuum formulation for hMSMs following [Zhao et al. \(2019\)](#) is considered. To isolate the elastic constitutive effects, the form of the magnetic energy is kept fixed, while two elastic modeling choices are varied: the strain-energy density function and the effective shear-modulus relation $G(\phi)$, where ϕ is the magnetic particle volume fraction.

The strain-energy density controls the nonlinear elastic response for a given set of moduli. To understand the extent to which the strain-energy density form affects the predicted response, three energy models are analyzed. The first two are variants of neo-Hookean energy, and the third is the St. Venant–Kirchhoff model. In contrast, the effective shear modulus $G(\phi)$ accounts for particle-induced stiffening of the elastomeric matrix. Several effective-modulus models have been employed in the literature, and it is important to compare them objectively so that an appropriate model can be identified and selected. With this in mind, a unified effective-modulus form is developed through $G(\phi)$. This includes classical rigid-inclusion relations due to Guth, Mooney, and Kerner [Guth \(1945\)](#), [Kerner \(1956\)](#), [Mooney \(1951\)](#), Hill’s self-consistent theory [Hill \(1965\)](#) specialized to the rigid-inclusion and near-incompressible-matrix limit, and constrained-kinematics models for rigid inclusions in elastomers [Alshammari et al. \(2019\)](#), [Garcia-Gonzalez and Hossain \(2021\)](#), [Lopez-Pamies et al. \(2013a\)](#). A review of several classical particulate-reinforcement models is given in [Ahmed and Jones \(1990\)](#).

A key point of this section is that these different approaches can be written in a common effective-modulus form through $G(\phi)$. In particular, the constrained-kinematics models, which are originally written by modifying the deformation invariant, such as the trace of the right Cauchy–Green deformation tensor, are recast here as effective shear-modulus relations. This gives seven $G(\phi)$ models that can be combined with the three strain-energy density functions, leading to 21 constitutive models. [Section 3](#) studies the sensitivity of forward magnetic actuation to these choices,

¹CEADpx/top_optim: https://github.com/CEADpx/top_optim/tree/v0.1.0

and Section 4 uses experimental stress–strain data to select a representative $G(\phi)$ relation for the joint material–structural optimization studies.

2.1. Notation, assumptions, and kinematics

Table 1 summarizes the main notation used throughout the article. Unless stated otherwise, all fields are defined with respect to the reference configuration Ω_0 . Bold lowercase symbols denote vectors, and bold uppercase symbols denote second-order tensors, with few exceptions, such as using $\mathbf{X}$ to denote the reference position vector. The constitutive formulation uses the following assumptions:

- A1. Quasistatic conservative response.** Deformations are treated within finite-strain continuum mechanics. Inertial, viscous, and other dissipative or time-dependent effects are neglected.
- A2. Fixed magnetic energy model.** The magnetic contribution is modeled using the $\mathbf{F}$ -based Zeeman-type interaction between the remanent magnetic flux density and the externally applied magnetic field. Alternative magnetic energy models, such as $\mathbf{R}$ -based formulations and dipole–dipole interactions, are not considered.
- A3. Prescribed magnetic fields.** The applied magnetic flux density $\mathbf{B}^a$ is prescribed as spatially constant and is not obtained by solving Maxwell’s equations. The remanent magnetic flux density has prescribed magnitude B^r , while its direction $\theta_{Br}(\mathbf{X})$ may be prescribed or treated as a design field.
- A4. Homogenized isotropic elastic response.** Magnetic particles are not resolved individually. Their effect is represented through the local volume fraction $\phi(\mathbf{X})$, which modifies the effective shear modulus $G(\phi)$. Near-incompressibility is imposed by taking the bulk modulus much larger than the shear modulus.

Let $\Omega_0 \subset \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ denote the reference and current configurations of the body, respectively, where $d = 2, 3$ is the spatial dimension. The deformation is described by the motion $\chi : \Omega_0 \rightarrow \Omega$, which maps a material point $\mathbf{X} \in \Omega_0$ to its spatial position $\mathbf{x} = \chi(\mathbf{X})$ in the current configuration. The displacement field and deformation gradient are

$$\mathbf{u}(\mathbf{X}) = \mathbf{x} - \mathbf{X}, \quad \mathbf{F} = \nabla_{\mathbf{X}} \chi = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{u}.$$

The right Cauchy–Green deformation tensor and Green–Lagrange strain tensor are

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}).$$

The Green–Lagrange strain tensor is decomposed into volumetric and deviatoric parts as

$$\mathbf{E}_v = \frac{\text{tr } \mathbf{E}}{3} \mathbf{I}, \quad \mathbf{E}_d = \mathbf{E} - \mathbf{E}_v, \quad \mathbf{E} = \mathbf{E}_v + \mathbf{E}_d. \quad (1)$$

It follows that

$$\text{tr}(\mathbf{E}^2) = \text{tr}(\mathbf{E}_d^2) + \frac{1}{3}(\text{tr } \mathbf{E})^2. \quad (2)$$

The key invariants used throughout the continuum formulation are $J = \det \mathbf{F}$ and $I_1 = \text{tr}(\mathbf{C})$.

Table 1: Notations used in the continuum mechanics and joint material–structural optimization framework.

Symbol	Description
Ω_0, Ω	Reference and current configurations of the body
$\mathbf{X}, \mathbf{x}$	Material and spatial points in the reference and current configurations
$\chi, \mathbf{u}$	Motion map, $\chi : \Omega_0 \rightarrow \Omega$, and displacement field, $\mathbf{u}(\mathbf{X}) = \mathbf{x} - \mathbf{X}$
$\mathbf{F}$	Deformation gradient, $\mathbf{F} = \nabla_{\mathbf{X}} \chi = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{u}$
$\mathbf{C} = \mathbf{F}^T \mathbf{F}$	Right Cauchy–Green deformation tensor
$\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$	Green–Lagrange strain tensor
$\mathbf{E}_v, \mathbf{E}_d$	Volumetric and deviatoric parts of $\mathbf{E}$
J, I_1	Jacobian determinant, $J = \det \mathbf{F}$, and first invariant of $\mathbf{C}$, $I_1 = \text{tr}(\mathbf{C})$
$W, W_{\text{elas}}, W_{\text{magn}}$	Total, elastic, and magnetic free-energy densities
$\widehat{W}, \widehat{W}_{\text{elas}}, \widehat{W}_{\text{magn}}$	Design-dependent total, elastic, and magnetic free-energy densities
$\mathcal{R}, \widehat{\mathcal{R}}^{(\ell)}$	Weak residuals for the forward and design-dependent equilibrium problems
$\mathbf{P}, \widehat{\mathbf{P}}$	First Piola–Kirchhoff stress tensors for the forward and design-dependent formulations
$\mathbf{b}, \bar{\mathbf{t}}, \bar{\mathbf{u}}$	Body force, prescribed traction, and prescribed displacement
Γ_u, Γ_t	Displacement and traction parts of the boundary
$\mathbf{B}^a, \mathbf{B}^r$	Externally applied and remanent magnetic flux densities
B^a, B^r	Magnitudes of the externally applied and remanent magnetic flux densities
$\phi(\mathbf{X}), \theta_{B^r}(\mathbf{X})$	Magnetic particle volume fraction and remanent magnetization direction
$\rho(\mathbf{X})$	Structural density design variable
$\bar{\rho}$	Filtered and projected physical density field
$\phi_{\text{phys}}, \theta_{B^r, \text{phys}}$	Filtered physical magnetic particle volume fraction and remanent magnetization direction
G_0, K_0	Shear and bulk moduli of the host matrix
$G(\phi), K(\phi)$	Effective shear and bulk moduli of the particle-filled composite
$\widehat{G}, \widehat{K}$	Design-dependent effective shear and bulk moduli
k_E, ϕ_{max}	Einstein coefficient used in the effective-modulus relations and maximum particle volume fraction
μ_0	Magnetic permeability of vacuum
Q_m, D_m	Displacement measure for model m and its relative percentage difference from the default model
rE, rRMSE	Relative error in extracted $G(\phi)$ data and relative root-mean-square error in stress–strain response
$Q_h, \mathcal{H}, \mathcal{P}_\rho$	Raw design space, Helmholtz-filter operator, and density projection operator
r_ρ, r_ϕ, r_θ	Filter radii for ρ, ϕ , and θ_{B^r}
$v_{\text{max}}, \Phi_{\text{max}}$	Structural material-volume bound and domain-averaged magnetic material bound
$f, f_{\text{rot}}, f_{\text{tr}}, f_{\text{comp}}$	Generic, rotational, translational, and compliance objective functions

2.2. Magneto-mechanical energy and equilibrium

The material response is derived from a total free-energy density consisting of elastic and magnetic contributions,

$$W(\mathbf{X}, \mathbf{F}, \phi, \mathbf{B}^r, \mathbf{B}^a) = W_{\text{elas}}(\mathbf{X}, \mathbf{F}, \phi) + W_{\text{magn}}(\mathbf{X}, \mathbf{F}, \phi, \mathbf{B}^r, \mathbf{B}^a). \quad (3)$$

The elastic energy is introduced in [Section 2.3](#). Following [Zhao et al. \(2019\)](#), the magnetic interaction energy density is taken as

$$W_{\text{magn}} = -\frac{1}{\mu_0} \phi(\mathbf{X}) \mathbf{F} \mathbf{B}^r(\mathbf{X}) \cdot \mathbf{B}^a, \quad (4)$$

where $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$ is the magnetic permeability of vacuum, $\phi : \Omega_0 \rightarrow [0, \phi_{\text{max}}]$ is the magnetic particle volume fraction, $\mathbf{B}^r$ is the remanent magnetic flux density, and $\mathbf{B}^a$ is the externally applied magnetic flux density. In the optimization examples, $\phi_{\text{max}} = 0.3$ is set.

The applied magnetic flux density $\mathbf{B}^a$ is prescribed. The remanent magnetic flux density has prescribed magnitude B^r and orientation $\theta_{B^r}(\mathbf{X})$, which may be prescribed or treated as a design field. Although the model-dependence study includes a three-dimensional benchmark, the joint material–structural optimization examples are two-dimensional, where the remanent magnetic flux density is expressed as

$$\mathbf{B}^r(\mathbf{X}) = B^r \begin{bmatrix} \cos \theta_{B^r}(\mathbf{X}) \\ \sin \theta_{B^r}(\mathbf{X}) \end{bmatrix}. \quad (5)$$

A schematic of the hMSM structure and the design variables used later in the joint optimization formulation is shown in [Figure 1](#).

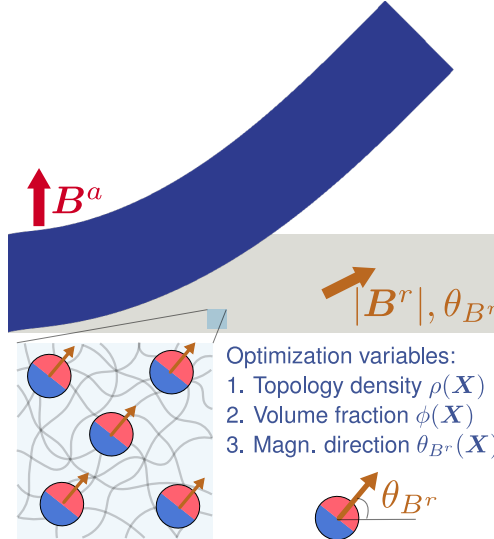


Figure 1: Undeformed and deformed hard-magnetic soft material beam with a representative volume element containing polymer fibers and magnetic particles. In the joint material–structural optimization formulation, the design fields are the structural density ρ , magnetic particle volume fraction ϕ , and remanent magnetization direction θ_{B^r} .

Given $(\phi, \mathbf{B}^r, \mathbf{B}^a, \mathbf{b}, \bar{\mathbf{t}}, \bar{\mathbf{u}})$, the strong form is to find $\mathbf{u}$ such that

$$\begin{aligned} \nabla_{\mathbf{X}} \cdot \mathbf{P} + \mathbf{b} &= \mathbf{0}, & \mathbf{X} &\in \Omega_0, \\ \mathbf{u} &= \bar{\mathbf{u}}, & \mathbf{X} &\in \Gamma_u \subset \partial\Omega_0, \\ \mathbf{P}\mathbf{n} &= \bar{\mathbf{t}}, & \mathbf{X} &\in \Gamma_t := \partial\Omega_0 \setminus \Gamma_u. \end{aligned} \quad (6)$$

Here, $\mathbf{b}$ is the body force per unit reference volume, $\bar{\mathbf{u}}$ is the prescribed displacement on Γ_u , $\bar{\mathbf{t}}$ is the prescribed traction on Γ_t , and $\mathbf{n}$ is the outward unit normal on the reference boundary. The first Piola–Kirchhoff stress tensor is obtained from the total free energy as

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}. \quad (7)$$

To state the weak form, the trial and test spaces are defined as

$$V_u = \left\{ \mathbf{u} \in [H^1(\Omega_0)]^d : \mathbf{u} = \bar{\mathbf{u}} \text{ on } \Gamma_u \right\}, \quad V_v = \left\{ \delta \mathbf{u} \in [H^1(\Omega_0)]^d : \delta \mathbf{u} = \mathbf{0} \text{ on } \Gamma_u \right\}. \quad (8)$$

Here, the prescribed displacement $\bar{\mathbf{u}}$ enters through the trial space V_u , while the displacement boundary Γ_u enters both the trial and test spaces. The Sobolev space $H^1(\Omega_0)$ is defined as

$$H^1(\Omega_0) = \left\{ v \in L^2(\Omega_0) : \nabla v \in [L^2(\Omega_0)]^d \right\},$$

and $[H^1(\Omega_0)]^d$ denotes the corresponding vector-valued space.

For $\mathbf{u} \in V_u$ and $\delta \mathbf{u} \in V_v$, the residual functional associated with the prescribed data $(\phi, \mathbf{B}^r, \mathbf{B}^a, \mathbf{b}, \bar{\mathbf{t}}, \bar{\mathbf{u}})$ is given by

$$\begin{aligned} \mathcal{R}(\mathbf{u}, \delta \mathbf{u}; \phi, \mathbf{B}^r, \mathbf{B}^a, \mathbf{b}, \bar{\mathbf{t}}, \bar{\mathbf{u}}) \\ := \int_{\Omega_0} \mathbf{P} : \nabla_X \delta \mathbf{u} \, dV - \int_{\Omega_0} \mathbf{b} \cdot \delta \mathbf{u} \, dV - \int_{\Gamma_t} \bar{\mathbf{t}} \cdot \delta \mathbf{u} \, dS, \end{aligned} \quad (9)$$

where $\mathbf{P} = \partial W / \partial \mathbf{F}$ and therefore depends on the prescribed fields through the free-energy density W .

The weak form is to find $\mathbf{u} \in V_u$ such that

$$\mathcal{R}(\mathbf{u}, \delta \mathbf{u}; \phi, \mathbf{B}^r, \mathbf{B}^a, \mathbf{b}, \bar{\mathbf{t}}, \bar{\mathbf{u}}) = 0, \quad \forall \delta \mathbf{u} \in V_v. \quad (10)$$

This variational form is used in the nonlinear finite-element implementation for the forward simulations and topology optimization studies. To compute the first Piola–Kirchhoff stress, automatic differentiation in the FEniCSx finite-element framework [Baratta et al. \(2023\)](#) is used.

2.3. Elastic strain-energy density functions

This subsection defines the elastic strain-energy density functions used in the model-dependence study. Three forms are analyzed numerically: two compressible neo-Hookean forms and one St. Venant–Kirchhoff form. The effective shear modulus $G(\phi)$ is left unspecified here and is introduced through the structure–property relations in [Section 2.4](#).

To model near-incompressibility, the bulk modulus of the host material is taken to be much larger than the shear modulus,

$$K_0 = 500G_0, \quad (11)$$

where G_0 and K_0 denote the shear and bulk moduli of the host matrix. This gives

$$\nu_0 = \frac{3K_0 - 2G_0}{6K_0 + 2G_0} = 0.499.$$

Throughout this work, the effective bulk modulus is taken as

$$K(\phi) = K_0. \quad (12)$$

The Lamé parameter is related to the shear and bulk moduli as

$$\lambda(\phi) = K(\phi) - \frac{2G(\phi)}{3}. \quad (13)$$

The first elastic model is the isochoric–volumetric compressible neo-Hookean energy

$$\textbf{(Strain Energy: NH1)} \quad W_{\text{elas}} = \frac{G(\phi)}{2} \left(J^{-2/3} I_1 - 3 \right) + \frac{K(\phi)}{2} (J - 1)^2, \quad (14)$$

where $J = \det \mathbf{F}$ and $I_1 = \text{tr } \mathbf{C}$.

The second elastic model is another compressible neo-Hookean energy,

$$\textbf{(Strain Energy: NH2)} \quad W_{\text{elas}} = \frac{G(\phi)}{2} (I_1 - 3 - 2 \ln J) + \frac{K(\phi)}{2} (J - 1)^2. \quad (15)$$

Compared with **NH1**, this form avoids the isochoric factor $J^{-2/3}$ and was found to be more robust in the numerical examples considered.

The third elastic model is the St. Venant–Kirchhoff energy, which is quadratic in the Green–Lagrange strain:

$$W_{\text{elas}} = G(\phi) \text{tr} (\mathbf{E}^2) + \frac{\lambda(\phi)}{2} (\text{tr } \mathbf{E})^2.$$

Using the split in [Equation \(1\)](#) and the identity in [Equation \(2\)](#), this energy can be written as

$$W_{\text{elas}} = G(\phi) \text{tr} (\mathbf{E}_d^2) + \left[\frac{\lambda(\phi)}{2} + \frac{G(\phi)}{3} \right] (\text{tr } \mathbf{E})^2.$$

Since the term in brackets is $K(\phi)/2$, the equivalent form for the St. Venant–Kirchhoff energy is

$$\textbf{(Strain Energy: SVK)} \quad W_{\text{elas}} = G(\phi) \text{tr} (\mathbf{E}_d^2) + \frac{K(\phi)}{2} (\text{tr } \mathbf{E})^2. \quad (16)$$

Together, **NH1**, **NH2**, and **SVK** define the three strain-energy density functions used in the model-dependence study. The effective shear-modulus relations $G(\phi)$ are introduced next.

2.4. Effective shear-modulus relations

The magnetic particle volume fraction ϕ strongly influences the elastic stiffness of the composite. In this work, this effect is represented through an effective shear modulus $G(\phi)$, while the bulk modulus is kept fixed as in [Equation \(12\)](#). Seven models for $G(\phi)$ are analyzed based on different theoretical approaches: one default or control model, four effective-modulus relations for rigid inclusions, and two relations obtained by recasting constrained-kinematics models into an effective-modulus form.

Rigid-inclusion approach. The first group of models accounts for the stiffening effect of rigid particles by modifying the shear modulus directly. This group includes classical relations due to Guth, Mooney, and Kerner [Guth \(1945\)](#), [Kerner \(1956\)](#), [Mooney \(1951\)](#), together with Hill's self-consistent formula for the effective elastic moduli of two-phase composites [Hill \(1965\)](#). The Hill relation used here is derived in [Appendix A](#) by taking the rigid-inclusion and near-incompressible-matrix limits.

The resulting effective shear-modulus models are

$$\begin{aligned}
(\text{G Model: Default}) \quad & G(\phi) = G_0, \\
(\text{G Model: Guth}) \quad & G(\phi) = G_0 (1 + k_E \phi + 14.1 \phi^2), \\
(\text{G Model: Mooney}) \quad & G(\phi) = G_0 \exp \left(\frac{k_E \phi}{1 - 1.35 \phi} \right), \\
(\text{G Model: Kerner}) \quad & G(\phi) = G_0 \left(1 + \frac{A \phi}{1 - \phi} \right), \\
(\text{G Model: Hill}) \quad & G(\phi) = \frac{G_0}{1 - k_E \phi}.
\end{aligned} \tag{17}$$

Here, $k_E = 2.5$. The **Default** model is included as a control case and ignores particle-induced stiffening. For the Kerner relation, the coefficient A is given by ([Ahmed and Jones, 1990](#), Equation 8) for $\nu = 0.5$

$$A = \frac{15(1 - \nu)}{8 - 10\nu} = k_E = 2.5. \tag{18}$$

Remark 1. To ensure finite and positive shear-modulus values for all models considered, ϕ is restricted to $\phi < \min\{1/1.35, 1/2.5\} = 0.4$. The two bounds correspond to the singular limits of the **Mooney** and **Hill** relations, respectively. Such a restriction is consistent with the literature, where the magnetic particle volume fraction remains below 0.4. When ϕ varies spatially, this condition is understood pointwise.

Constrained-kinematics approach. The second strategy accounts for rigid inclusions by modifying the invariant I_1 to represent constrained matrix deformation. In contrast to the rigid-inclusion models above, this approach is originally expressed at the level of the strain-energy density rather than directly through an effective shear modulus. The first model considered in this group is the homogenized incompressible I_1 -based strain-energy model of Lopez-Pamies [Lopez-Pamies et al. \(2013a\)](#),

$$\Psi_{\text{LP}}(I_1, \phi) = (1 - \phi) \Psi_0 \left(\frac{I_1 - 3}{(1 - \phi)^{7/2}} + 3 \right), \tag{19}$$

where $\Psi_0(I_1)$ is an I_1 -based strain-energy function for the host material. The factor $(1 - \phi)$ accounts for the volume fraction of deforming matrix material, while the modified invariant passed to $\Psi_0(\cdot)$ accounts for the constrained kinematics induced by rigid inclusions.

To connect this constrained-kinematics form to an effective shear-modulus relation, consider the neo-Hookean-type deviatoric energy used in **NH2**, ignoring the volumetric penalty term,

$$\Psi_0(I_1) = G_0(I_1 - 3). \tag{20}$$

Substituting this choice into [Equation \(19\)](#) gives

$$\Psi_{\text{LP}}(I_1, \phi) = (1 - \phi) G_0 \left(\frac{I_1 - 3}{(1 - \phi)^{7/2}} \right) = \frac{G_0}{(1 - \phi)^{5/2}} (I_1 - 3).$$

This naturally leads to an effective shear modulus, referred to here as **LP**,

$$\text{(G Model: LP)} \quad G(\phi) = \frac{G_0}{(1 - \phi)^{5/2}}, \quad (21)$$

so that the corresponding effective strain energy simplifies to

$$\Psi_{\text{LP}}(I_1, \phi) = G(\phi)(I_1 - 3).$$

Thus, the constrained-kinematics model has the same form as $\Psi_0 = G_0(I_1 - 3)$, but with G_0 replaced by the effective shear modulus $G(\phi)$.

This observation allows the constrained-kinematics model to be interpreted as a structure–property relation for $G(\phi)$, placing it in the same framework as the models in Equation (17). This transition from a modified-invariant formulation to an effective-modulus formulation is important because it allows the **LP** relation to be combined with different strain-energy density functions, including **NH1** in Equation (14) and **SVK** in Equation (16).

More recently, the Lopez-Pamies approach was extended in Garcia-Gonzalez and Hossain (2021) to incorporate the structure–property relation proposed in Alshammari et al. (2019). Considering again the example $\Psi_0(I_1) = G_0(I_1 - 3)$, the effective strain energy, referred to here as **LPA**, takes the form

$$\Psi_{\text{LPA}}(I_1, \phi) = (1 - \phi)\Psi_0(\chi(\phi)(I_1 - 3) + 3), \quad (22)$$

where $\chi(\phi)$ captures the influence of ϕ on the modified invariant. It is given by Alshammari et al. (2019)

$$\chi(\phi) = 1 + 0.67g\phi + 1.62(g\phi)^2,$$

where g is a coefficient fixed below. As in the case of Ψ_{LP} , for the specific choice $\Psi_0(I_1) = G_0(I_1 - 3)$, the dependence on ϕ can be moved to an effective shear modulus. Substituting this Ψ_0 into Equation (22) gives

$$\Psi_{\text{LPA}}(I_1, \phi) = G_0(1 - \phi)\chi(\phi)(I_1 - 3).$$

The above expression motivates the definition of an effective shear modulus, referred to here as **LPA**,

$$\text{(G Model: LPA)} \quad G(\phi) = G_0(1 - \phi)\chi(\phi). \quad (23)$$

With the above definition of $G(\phi)$, the effective strain energy becomes

$$\Psi_{\text{LPA}}(I_1, \phi) = G(\phi)(I_1 - 3).$$

Again, the effective strain energy has the same form as Ψ_0 , but with the shear modulus replaced by $G(\phi)$. The same argument used above to extend the **LP** relation to **NH1** and **SVK** can therefore be applied to the **LPA** relation. Thus, **LPA** is also treated as a structure–property relation for $G(\phi)$ and is combined with all three strain-energy density functions.

Remark 2. *The recasting of the **LP** and **LPA** constrained-kinematics models as effective shear-modulus relations is a key step in this work because it converts modified-invariant models into additional $G(\phi)$ relations that can be compared directly with rigid-inclusion models. This recasting is not exact. It is based on shifting the ϕ -dependence from the modified invariant into the effective shear modulus $G(\phi)$, thereby placing both approaches within the same effective-modulus framework.*

It remains to fix the constant g in $\chi(\phi)$. In [Alshammari et al. \(2019\)](#), g appears as a shape or cluster factor in the reinforcement function

$$\chi(\phi) = 1 + 0.67g\phi + 1.62(g\phi)^2.$$

In the present work, the **LPA** form is fixed by setting the constant g rather than fitting it using the data. The value of g is chosen so that the linear term in the amplification function $\chi(\phi)$ matches the classical Einstein/Guth coefficient $k_E = 2.5$ used in the rigid-inclusion relations. This gives

$$0.67g = k_E, \quad g = \frac{k_E}{0.67} \approx 3.73.$$

Table 2: Summary of structure–property models for the effective shear modulus $G(\phi)$. These G models are used in three strain energies (**NH1**, **NH2**, and **SVK**), with the total energy including the magnetic contribution W_{magn} in (4). The table reports model selection metrics from [Section 4](#). Metric 1 reports the relative error rE between $G(\phi)$ models and G data from [Garcia-Gonzalez et al. \(2021\)](#). Metric 2 reports the relative root-mean-square error (rRMSE) between uniaxial simulations on dog-bone specimens and reference stress–strain curves. The model selection and the error calculations are discussed in [Section 4](#). Mooney (blue shading) gives the lowest mean in both metrics and is used in the optimization studies. Reference abbreviations: Guth: [Guth \(1945\)](#); Kerner: [Kerner \(1956\)](#); Mooney: [Mooney \(1951\)](#); A&J: [Ahmed and Jones \(1990\)](#); Hill: [Hill \(1965\)](#); LP: [Lopez-Pamies et al. \(2013a\)](#); Gonzalez: [Garcia-Gonzalez and Hossain \(2021\)](#); and Alsham: [Alshammari et al. \(2019\)](#); Gonzalez data: [Garcia-Gonzalez et al. \(2021\)](#).

G model	Approach	$G(\phi)$	Refs.	Model error using Gonzalez data			
				Metric 1		Metric 2	
				rE of $G(\phi)$		rRMSE of σ_{xx} (uniaxial)	
				Mean	Std	Mean	Std
Default	—	G_0	—	0.335	0.313	0.455	0.189
Guth	Effective modulus	$G_0 (1 + k_E\phi + 14.1\phi^2)$ $k_E = 2.5$	Guth A&J	0.062	0.061	0.255	0.021
Kerner	Effective modulus	$G_0 (1 + A\phi/(1 - \phi))$ $A = k_E = 2.5$	Kerner A&J	0.170	0.182	0.327	0.090
Mooney	Effective modulus	$G_0 \exp(k_E\phi/(1 - 1.35\phi))$	Mooney A&J	0.043	0.051	0.248	0.019
Hill	Effective modulus	$G_0/(1 - k_E\phi)$	Hill App. A	0.075	0.058	0.252	0.014
LP	Constrained kinematics	$G_0/(1 - \phi)^{5/2}$	LP	0.135	0.135	0.300	0.057
LPA	Constrained kinematics	$G_0(1 - \phi)\chi(\phi)$ $\chi(\phi) = 1 + 0.67g\phi + 1.62(g\phi)^2$ $g = 3.73$	Gonzalez Alsham	0.110	0.105	0.281	0.040

Remark 3. With $g = 3.73$, the amplification function $\chi(\phi)$ has the same first-order reinforcement coefficient as the classical rigid-inclusion model of Guth [Guth \(1945\)](#). The final **LPA** effective modulus $G(\phi) = G_0(1 - \phi)\chi(\phi)$ also includes the matrix-volume factor $(1 - \phi)$ inherited from the constrained-kinematics formulation. Therefore, **LPA** with $g = 3.73$ is not intended to match the dilute-limit slope of the Guth relation; however, it provides a conservative structure-property relation that retains the classical first-order amplification inside $\chi(\phi)$. The parameter g may also be determined by fitting to experimental data as done in [Alshammari et al. \(2019\)](#). In this work, we keep $g = 3.73$ fixed.

2.5. Constitutive relations summary

In summary, this work considers three elastic strain-energy density functions and seven effective shear-modulus relations $G(\phi)$. Combining each strain-energy density with each $G(\phi)$ relation gives a total of 21 candidate constitutive models. [Table 2](#) summarizes the shear-modulus models and reports the model-selection metrics from [Section 4](#).

3. Numerical investigations of model dependence in hMSM deformation

This section investigates how the constitutive choices introduced in [Section 2](#) affect magnetically actuated deformation of hMSMs. The magnetic energy formulation and governing equations are kept fixed, while the elastic strain-energy density function and effective shear-modulus relation $G(\phi)$ are varied. The goal is to determine when these constitutive choices produce meaningful differences in the predicted response. The numerical studies begin with cantilever beams, where loading magnitude and magnetic-material placement can be varied in a controlled way, and then consider wheel and gripper geometries with more complex deformation modes. Model dependence is quantified by comparing deformation fields and averaged displacement metrics across the candidate constitutive models.

3.1. Cantilever beam benchmarks

The cantilever beam benchmarks are used to separate two possible sources of constitutive model sensitivity. The first study varies the applied magnetic field magnitude for a uniformly magnetized beam, while the second localizes the magnetic material near the beam tip. Together, these cases distinguish sensitivity caused by actuation strength from sensitivity caused by spatial overlap between magnetic material and the primary bending region.

3.1.1. Problem setup

The geometry, boundary conditions, magnetic field direction, remanent magnetization direction, and magnetic particle distribution for the two beam benchmarks are provided in [Figure 2](#). For each loading condition, the strain-energy density functions in [Section 2.3](#) and the effective shear-modulus relations in [Section 2.4](#) are evaluated.

For the uniformly magnetized beam in [Figure 2\(a\)](#), three applied magnetic field magnitudes are considered: 25, 100, and 400 mT, representing small, medium, and large loading, respectively. For the tip-magnetized beam in [Figure 2\(b\)](#), the applied magnetic field magnitude is fixed at 400 mT.

A mesh convergence study for the beam benchmark is provided in [Appendix B](#). Based on that study, a characteristic mesh size of $l_c = 0.08$ mm is used for all beam simulations in this subsection.

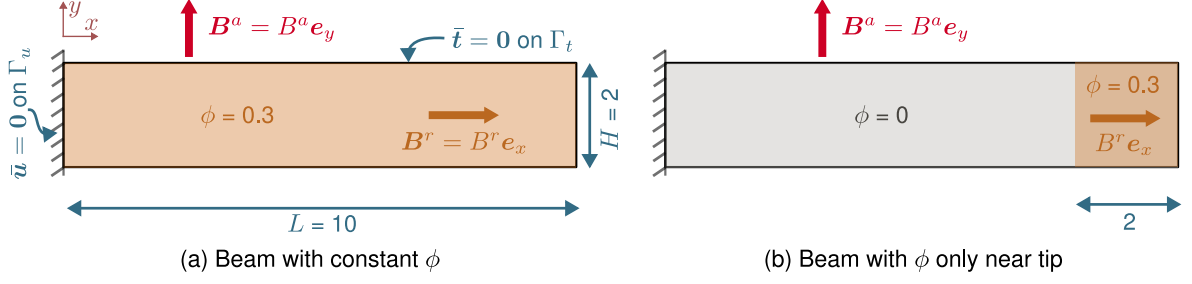


Figure 2: Cantilever beam benchmarks used to study model dependence. Both beams have the same dimensions and are clamped on the left boundary with the remaining boundary traction free. A spatially uniform applied magnetic field is imposed upward, while the remanent magnetic flux density is prescribed along the beam axis with $B^r = 100$ mT. Matrix shear modulus is $G_0 = 100$ kPa, while the bulk modulus is given by Equation (11). In (a), magnetic material is uniformly distributed with $\phi = 0.3$. In (b), magnetic material is localized near the beam tip, with $\phi = 0.3$ for $X > 8$ mm and $\phi = 0$ otherwise. Lengths are in millimeters.

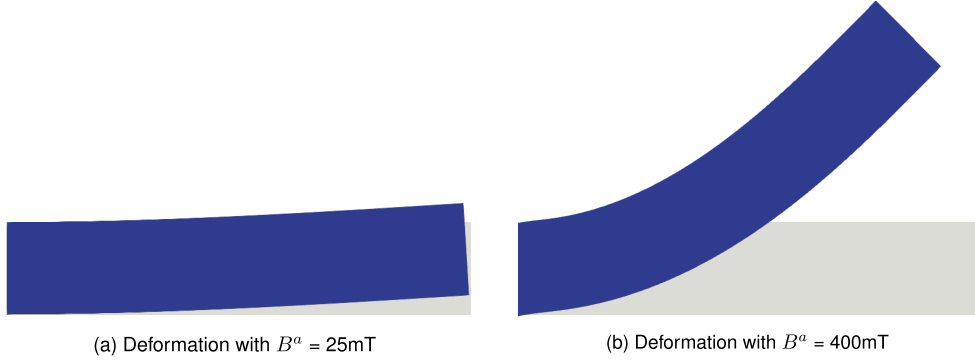


Figure 3: Representative deformation response of the uniformly magnetized cantilever beam under small magnetic actuation (a) and large magnetic actuation (b). The results correspond to the Mooney model with **NH2** strain energy and the uniformly magnetized beam in Figure 2(a).

3.1.2. Results

To compare the deformation responses, a quantity of interest is defined. First, a measurement part of the reference boundary is prescribed, denoted by $\Gamma_{\text{meas}} \subset \partial\Omega_0$. In the finite-element implementation, this boundary is represented by the corresponding set of selected measurement points or nodes, denoted by the index set $\mathcal{I}_{\text{meas}}$. For a given constitutive model m , let $\mathbf{u}_m$ denote the equilibrium displacement field. The magnitude of the average displacement vector over the measurement boundary is taken as the quantity of interest:

$$Q_m = \left\| \frac{1}{|\Gamma_{\text{meas}}|} \int_{\Gamma_{\text{meas}}} \mathbf{u}_m(\mathbf{X}) \, dS \right\| \approx \left\| \frac{1}{N_{\text{meas}}} \sum_{i \in \mathcal{I}_{\text{meas}}} \mathbf{u}_m(\mathbf{X}_i) \right\|, \quad (24)$$

where $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ and $N_{\text{meas}} = |\mathcal{I}_{\text{meas}}|$ is the number of measurement points. The corresponding quantity for the **Default** model is denoted by Q_D .

The absolute relative percentage difference between model m and the default model is then defined as

$$D_m = \left| \frac{Q_m - Q_D}{Q_D} \right| \times 100. \quad (25)$$

The quantity D_m measures the relative effect of the selected structure–property model on the predicted deformation, using the default model $G(\phi) = G_0$ as the control case.

For the cantilever beam benchmark, the measurement boundary is the right edge of the beam,

$$\Gamma_{\text{meas}} = \{(L, Y) \in \partial\Omega_0 : 0 \leq Y \leq H\}, \quad (26)$$

where H is the beam height; see Figure 2. The corresponding discrete set $\mathcal{I}_{\text{meas}}$ consists of the measurement points located on this right edge.

Comparisons among the strain-energy density functions for a fixed $G(\phi)$ relation show little variation: **NH1** and **NH2** give nearly identical responses, while **SVK** differs by less than 1%. Therefore, only the results from the **NH2** energy model are presented in the following discussion to isolate the influence of the effective shear-modulus relation $G(\phi)$. The average free-end displacement magnitude Q_m predicted by the different effective shear-modulus relations is presented in Figure 4.

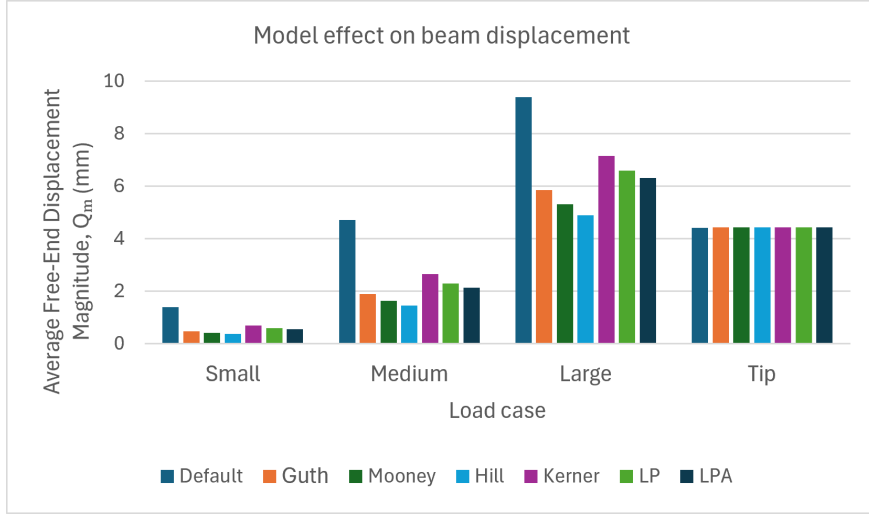


Figure 4: Average free-end displacement magnitude Q_m from Equation (24) predicted by the different structure–property models for the cantilever beam benchmarks. The first three results, from left to right, correspond to the uniformly magnetized beam in Figure 2(a) subjected to increasing applied magnetic field magnitudes B^a . The rightmost result corresponds to the tip-magnetized beam in Figure 2(b) subjected to $B^a = 400$ mT.

For the uniformly magnetized beam in Figure 2(a), all loading cases in Figure 4 exhibit the same qualitative ordering. The default model predicts the largest displacement, followed by the Kerner, LP, LPA, Guth, Mooney, and Hill models in decreasing order of displacement magnitude. This ordering is consistent with the effective stiffness predicted by the different $G(\phi)$ relations: larger values of $G(\phi)$ produce smaller deformation under the same magnetic loading.

The rightmost result in Figure 4 corresponds to the tip-magnetized beam in Figure 2(b). In contrast to the uniformly magnetized cases, the predicted displacements are nearly identical across the structure–property models. This result suggests that the overlap between magnetic material and highly deforming bending regions plays an important role in determining the sensitivity to the selected $G(\phi)$ relation.

The metric D_m is summarized in Table 3. A clear reduction in structure–property model sensitivity is observed as the loading magnitude increases, with the largest deviations occurring for the small loading condition and progressively smaller differences observed for the medium and large

loading cases. The Hill and Mooney models consistently produce the largest deviations from the default model, while the Kerner model remains the closest among the modified structure–property relations. The LP and LPA models exhibit intermediate behavior, producing larger displacements than the Guth, Mooney, and Hill models but smaller displacements than the Kerner model. These results indicate that model sensitivity becomes increasingly important for small actuation responses and fine deformation control.

Table 3: Relative percentage change D_m in the beam displacement measure Q_m with respect to the default model, computed using Equation (25). The results are for **NH2** strain energy.

Beam setup	B^a (mT)	Guth	Mooney	Hill	Kerner	LP	LPA
Uniform, Fig. 2(a)	25 (small)	65.1	70.0	73.4	49.6	57.2	60.4
Uniform, Fig. 2(a)	100 (medium)	59.9	65.3	69.1	43.4	51.3	54.7
Uniform, Fig. 2(a)	400 (large)	37.6	43.4	47.9	23.6	29.8	32.7
Tip-localized, Fig. 2(b)	400	0.4	0.4	0.5	0.3	0.3	0.4

The tip-magnetized beam setup further shows that model sensitivity depends strongly on the spatial overlap between magnetic material and mechanically active bending regions. As highlighted in the final row of Table 3, all structure–property models predict nearly identical displacements for the tip-magnetized case, with relative differences below 0.5%. Thus, when magnetic material is removed from the primary bending region, the influence of the selected $G(\phi)$ relation becomes negligible for this benchmark.

3.2. Wheel and gripper benchmarks

The wheel and gripper benchmarks extend the cantilever study to geometries with different deformation modes and different magnetic-material placement.

3.2.1. Problem setup

The wheel and gripper setups are provided in Figure 5. For both benchmarks, the strain-energy density functions in Section 2.3 and the effective shear-modulus relations in Section 2.4 are evaluated. The wheel is subjected to a spatially uniform upward applied magnetic field of $B^a = 125$ mT, and the remanent magnetic flux density magnitude is fixed at $B^r = 100$ mT. For the gripper, the applied field magnitude is $B^a = 80$ mT, and the remanent magnetic flux density magnitude is $B^r = 40$ mT. Mesh convergence studies for the wheel and gripper benchmarks are provided in Appendix B. Based on those studies, characteristic mesh sizes of $l_c = 0.06$ mm and $l_c = 0.08$ mm are used for the wheel and gripper benchmarks, respectively.

3.2.2. Results

The same displacement measure Q_m and relative percentage change D_m defined in Equations (24) and (25) are used for the wheel and gripper benchmarks. Only the measurement set changes between the two cases. For the wheel, the measurement points are selected near the right outer rim,

$$\Gamma_{\text{meas}}^{\text{wheel}} = \{ \mathbf{X} = (X, Y) \in \partial\Omega_0 : X^2 + Y^2 = R^2, X > 0, |Y| \leq 4t \} , \quad (27)$$

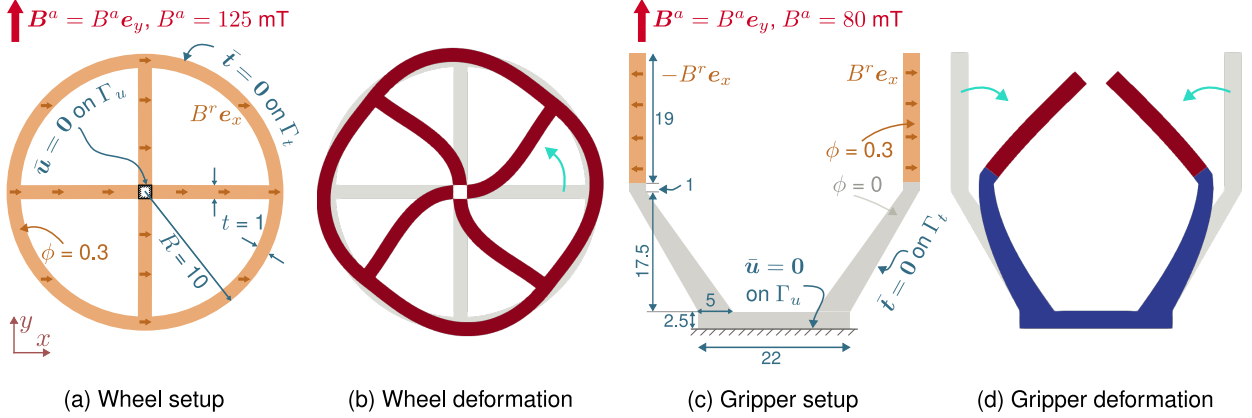


Figure 5: Setup and deformed configurations for the wheel and gripper benchmarks. Panel (a) presents the wheel setup, with the hub boundary fixed and all remaining boundaries traction free. The remanent magnetic flux density magnitude is $B^r = 100$ mT. Panel (b) gives the resulting rotational deformation. Panel (c) presents the gripper setup, where magnetic material is localized in the gripping arms. The bottom boundary is fixed, while all remaining boundaries are traction free. Opposite remanent magnetic flux density directions are prescribed in the two gripping arms with $B^r = 40$ mT. Panel (d) gives the resulting closing deformation. For both benchmarks, G_0 is fixed to $G_0 = 100$ kPa and K_0 is given by Equation (11). All lengths are in millimeters.

where R is the outer radius and t is the rim thickness. For the gripper, the measurement points are selected on the top edge of the right gripping arm,

$$\Gamma_{\text{meas}}^{\text{gripper}} = \{\mathbf{X} = (X, Y) \in \partial\Omega_0 : Y = Y_{\text{tip}}, X_{\min} \leq X \leq X_{\max}\}, \quad (28)$$

where $X_{\min} = 18.5$ mm, $X_{\max} = 21$ mm, and $Y_{\text{tip}} = 39.82$ mm. The corresponding discrete sets $\mathcal{I}_{\text{meas}}$ consist of the selected measurement points in these regions.

The displacement measures Q_m predicted by the different strain-energy density functions and effective shear-modulus relations are presented in Figure 6. Similar to the cantilever beam benchmarks, the strain-energy density function has a relatively small effect: **NH1** and **NH2** give nearly identical responses, while **SVK** differs only slightly. Therefore, the quantitative comparison in Table 4 uses **NH2** to isolate the influence of the effective shear-modulus relation $G(\phi)$.

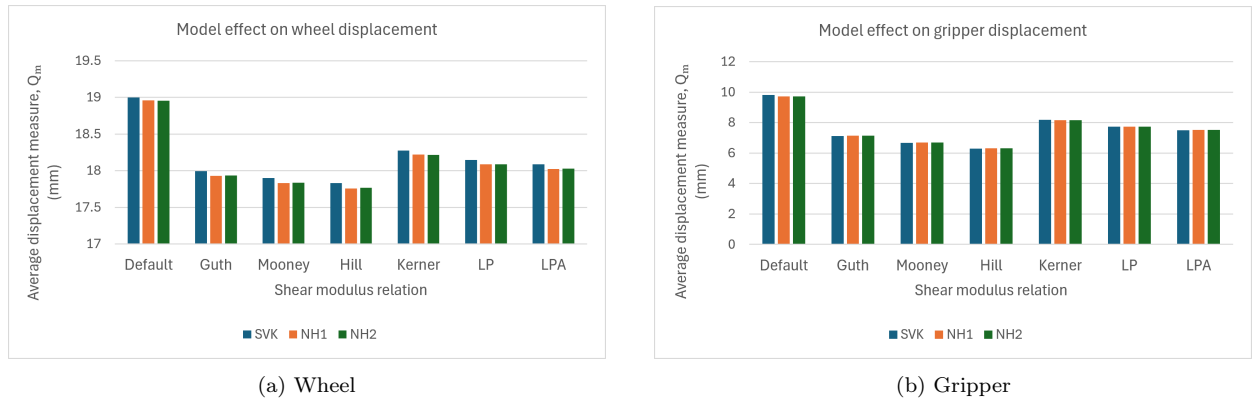


Figure 6: Average displacement magnitude predicted by different strain-energy density functions and effective shear-modulus relations for the wheel and gripper benchmarks.

The results in Figure 6 confirm the qualitative ordering observed in the cantilever beam benchmarks. The default model predicts the largest deformation measure, while the modified structure–property models predict smaller values because they increase the effective stiffness through $G(\phi)$. Visibly larger spread among the structure–property models is observed for the wheel benchmark compared to the gripper benchmark.

The metric D_m is summarized in Table 4. The wheel benchmark is substantially more sensitive to the selected structure–property relation than the gripper benchmark. For the wheel, the relative percentage change ranges from 15.9% for the Kerner model to 35.1% for the Hill model. For the gripper, the corresponding values remain much smaller, ranging from 3.9% to 6.3%.

Table 4: Relative percentage change D_m from Equation (25) in the displacement measure Q_m with respect to the **Default** model for the wheel and gripper benchmarks.

Geometry	Guth	Mooney	Hill	Kerner	LP	LPA
Wheel	26.6	31.2	35.1	15.9	20.5	22.7
Gripper	5.4	5.9	6.3	3.9	4.6	4.9

The larger sensitivity of the wheel is consistent with the magnetic-material distribution in Figure 5(a). Magnetic material is distributed throughout the wheel, including regions that undergo substantial deformation during actuation. Therefore, changes in $G(\phi)$ directly affect mechanically active regions and produce noticeable differences in Q_m .

In contrast, the gripper setup in Figure 5(c) localizes magnetic material in the gripping arms, while much of the deformation occurs in the compliant hinge regions near the base. As shown in Table 4, this separation reduces the relative effect of the selected $G(\phi)$ relation. This behavior is consistent with the tip-magnetized cantilever beam in Figure 2(b), where the absence of magnetic material in the primary bending region makes the model dependence negligible.

3.3. Three-dimensional beam actuation tests

The preceding benchmarks examine model dependence in two-dimensional geometries. As a final numerical test, a three-dimensional beam actuation problem is considered to check whether the same trends persist in a fully three-dimensional deformation setting.

3.3.1. Problem setup

The setup, based on the benchmark example in Zhao et al. (2019), is provided in Figure 7(a). The magnetic material is uniformly distributed with volume fraction $\phi \in \{0.01, 0.05, 0.15, 0.30\}$. The response is measured using the normalized vertical tip deflection

$$Q_m = \frac{\delta_m}{L} = \frac{u_{y,m}(\mathbf{X}_{\text{tip}})}{L}, \quad (29)$$

where $u_{y,m}(\mathbf{X}_{\text{tip}})$ is the vertical displacement of the monitored free-end point $\mathbf{X}_{\text{tip}} = (L, W, C)$ (see Figure 7(a)) predicted by model m .

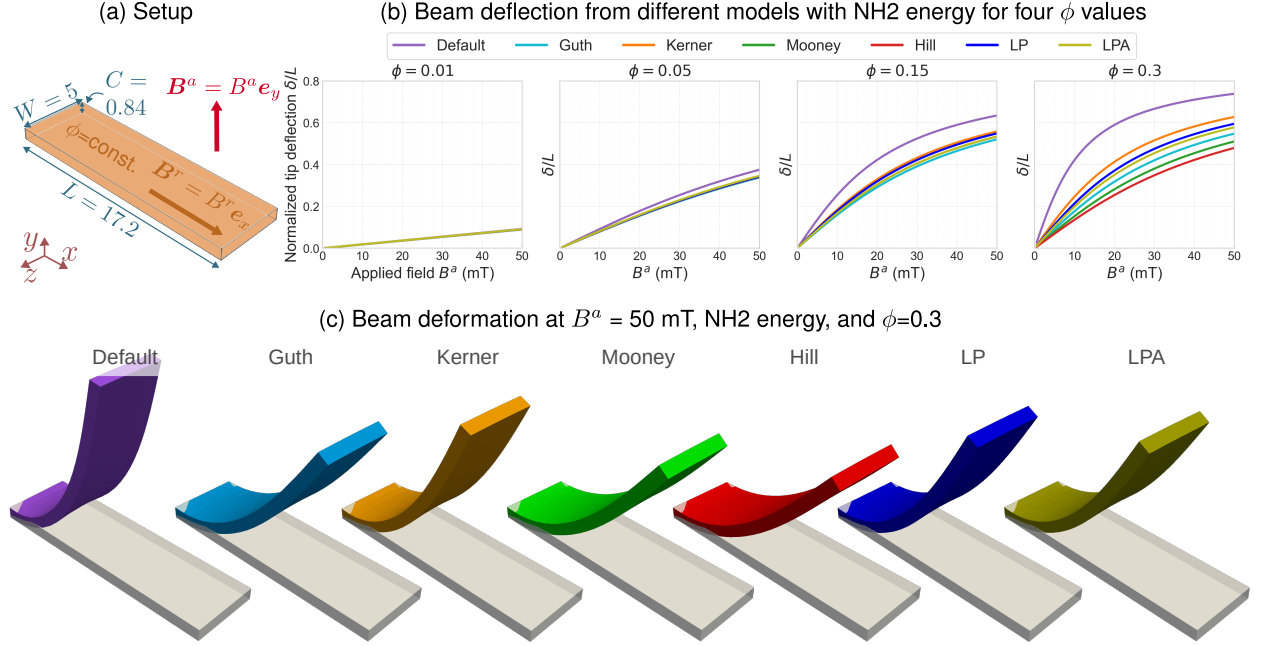


Figure 7: Three-dimensional actuation cuboid benchmark. Panel (a) presents the problem setup. Here, $G_0 = 303$ kPa, K_0 is given by Equation (11), $B^r = 143$ mT, and $B^a = 50$ mT is applied over 100 load increments. The domain is discretized with a structured mesh of $20 \times 4 \times 4$ elements. The left face (at $x = 0$) is fixed and the remaining boundary is traction free. Lengths are in millimeters. Panel (b) includes the normalized tip deflection δ_m/L versus B^a plots for **NH2** with the seven $G(\phi)$ models. Panel (c) visualizes the deformed configurations at $B^a = 50$ mT and $\phi = 0.30$ for **NH2**.

3.3.2. Results

The effect of the strain-energy density function is presented in Figure 8. For each ϕ and each $G(\phi)$ model, the bars for **NH1**, **NH2**, and **SVK** are nearly identical. Thus, for this three-dimensional benchmark, the predicted actuation response is much more sensitive to the selected $G(\phi)$ relation than to the selected strain-energy density function, which is consistent with the two-dimensional beam, wheel, and gripper benchmarks. The ordering of the $G(\phi)$ models is dependent on ϕ , as expected.

For **NH2**, the normalized deflection as a function of the applied magnetic field is plotted in Figure 7(b). At small particle volume fractions, the model predictions are close. As ϕ increases, the spread among the $G(\phi)$ models becomes more pronounced. Figure 7(c) compares the corresponding deformed configurations at $B^a = 50$ mT and $\phi = 0.3$.

3.4. Summary of key observations

The numerical studies in this section lead to the following observations:

- **Strain-energy model effect is small.** Across the beam, wheel, gripper, and three-dimensional cuboid benchmarks, **NH1**, **NH2**, and **SVK** give nearly identical deformation measures for the same structure–property model.
- **The $G(\phi)$ model effect is significant.** The effective shear-modulus relation changes the predicted deformation, especially when magnetic material overlaps with mechanically active regions.

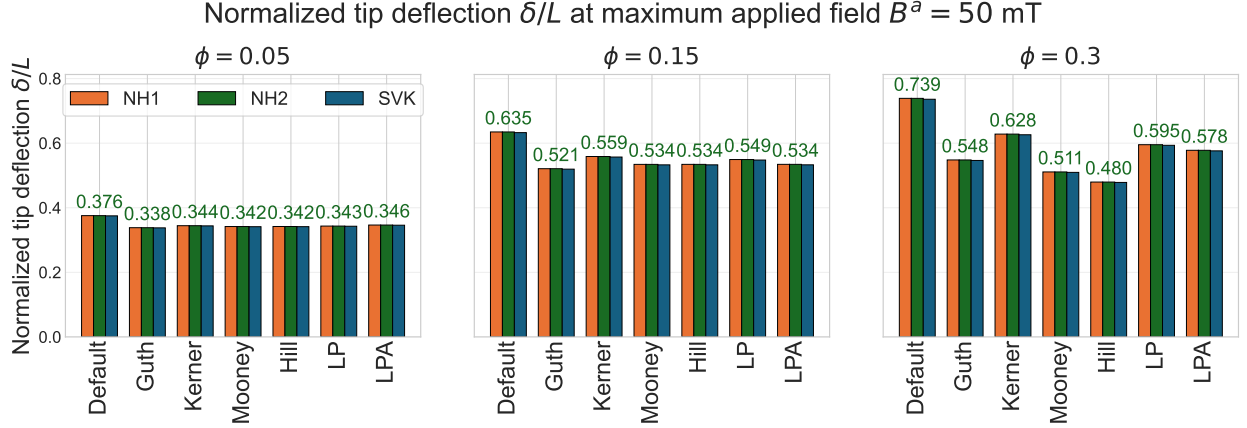


Figure 8: Normalized tip deflection δ_m/L at $B^a = 50$ mT for $\phi = 0.05, 0.15$, and 0.30 . For each $G(\phi)$ model, the grouped bars compare **NH1**, **NH2**, and **SVK**. Numerical values above the green bars indicate the corresponding δ_m/L for **NH2**.

- **Spatial placement matters.** The uniformly magnetized beam and the wheel demonstrate clear sensitivity to the selected $G(\phi)$ relation because magnetic material overlaps with regions undergoing substantial deformation.
- **Sensitivity is reduced when magnetic material is separated from the dominant deformation regions.** This behavior is observed in the tip-localized beam and in the gripper, where the relative changes remain much smaller than in the uniformly magnetized beam and wheel.
- **The default model gives the largest deformation.** Since the default model uses $G(\phi) = G_0$, it ignores particle-induced stiffening and consistently predicts the largest displacement. Therefore, a structure–property relation informed by particle reinforcement is needed for the model-selection and optimization studies that follow.
- **NH2 is selected for later sections.** Since **NH2** gives responses similar to **NH1** and **SVK** while providing better numerical robustness in the examples considered, it is used as the fixed strain-energy density function in [Section 4](#) and in the topology optimization studies.

4. Constitutive model selection

This section selects the effective shear-modulus relation $G(\phi)$ used in the joint material–structural optimization studies. The selection is based on experimental uniaxial true stress–strain data from [Garcia-Gonzalez et al. \(2021\)](#) for particle volume fractions $\phi \in \{0, 0.05, 0.15, 0.30\}$. Although the experiments in [Garcia-Gonzalez et al. \(2021\)](#) were performed on soft-magnetic composites filled with carbonyl iron powder (CIP), the loading is purely mechanical and no applied magnetic field is involved. Therefore, the comparison does not test the magnetic actuation model or the remanent magnetization assumptions. Instead, it isolates the particle-induced stiffening response, which is the specific constitutive effect represented by the candidate $G(\phi)$ relations. On this basis, the dataset is used to identify a representative effective shear-modulus relation for the subsequent design studies. The same selection procedure can be applied to other mechanical datasets for filled elastomers.

The stress–strain curves reported in (Garcia-Gonzalez et al., 2021, Fig. 1) for the mixing ratio 5:1 were digitized and processed as described in Appendix C. Because the small-strain region is sensitive to digitization and curve regularization, two digitization passes were used and averaged to obtain a reference dataset. Figure 9 shows the extracted stress–strain data and the corresponding small-strain shear-modulus estimates. Using the calibration window $\varepsilon_{xx} \in [0, 0.15]$, the reference estimates for $G(\phi)$ in kPa are

$$G(0) = G_0 = 373.7, \quad G(0.05) = 454.8, \quad G(0.15) = 670.1, \quad G(0.30) = 1321.8, \quad (30)$$

where G_0 is the shear modulus of the host matrix.

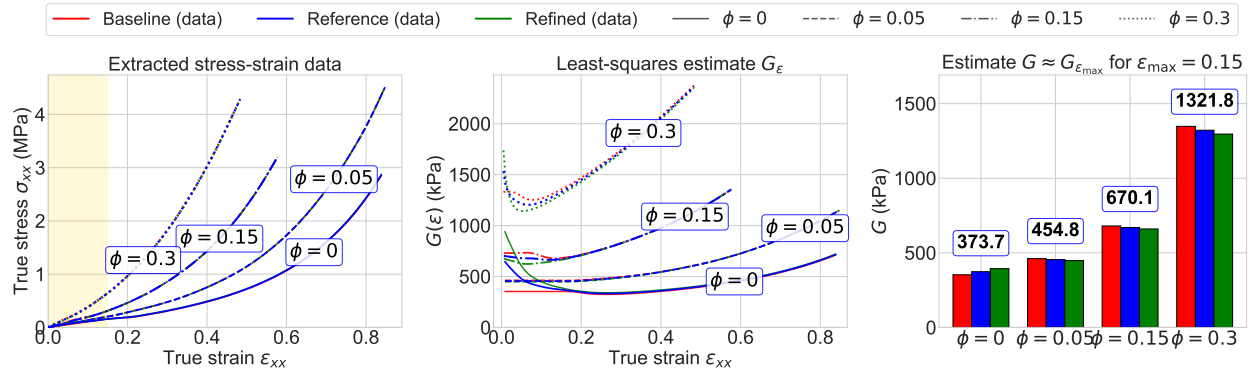


Figure 9: Experimental data used for constitutive model selection. Left: extracted true stress–strain curves at four particle volume fractions, with the shaded region indicating the calibration window $\varepsilon_{xx} \in [0, 0.15]$. Center: least-squares shear-modulus estimate as a function of the upper strain limit; see Section Appendix C.1. Right: shear-modulus estimates at $\varepsilon_{\max} = 0.15$; the labels correspond to the reference dataset.

Two complementary comparisons are used. They are summarized below, with additional details provided in Appendix C.2 and Appendix C.4.

4.1. Selection metric 1: shear-modulus data

Each $G(\phi)$ model is compared directly with the extracted shear-modulus data. For each volume fraction, the relative error is defined as

$$\text{rE}(\phi, \text{model}) = \frac{|G_{\text{model}}(\phi) - G_{\varepsilon_{\max}}^{\text{data}}(\phi)|}{|G_{\varepsilon_{\max}}^{\text{data}}(\phi)|}. \quad (31)$$

The mean and standard deviation of rE over $\phi \in \{0, 0.05, 0.15, 0.30\}$ are reported in Table 2. Figure 10 shows the same comparison visually after normalizing by G_0 . The **Default** model matches the data at $\phi = 0$ by construction but cannot capture the stiffening at larger ϕ . Among the non-default models, **Mooney** gives the lowest mean relative error, followed by **Guth** and **Hill**.

4.2. Selection metric 2: Stress–strain data

The second metric compares simulated and extracted stress–strain curves directly. For each prescribed particle volume fraction ϕ , effective shear-modulus relation $G(\phi)$, and matrix shear modulus G_0 , a dog-bone specimen similar to that in Garcia-Gonzalez et al. (2021) is simulated using the **NH2** energy model and a spatially uniform particle volume fraction. The resulting true stress–strain curve is then compared with the extracted experimental curve. This comparison identifies

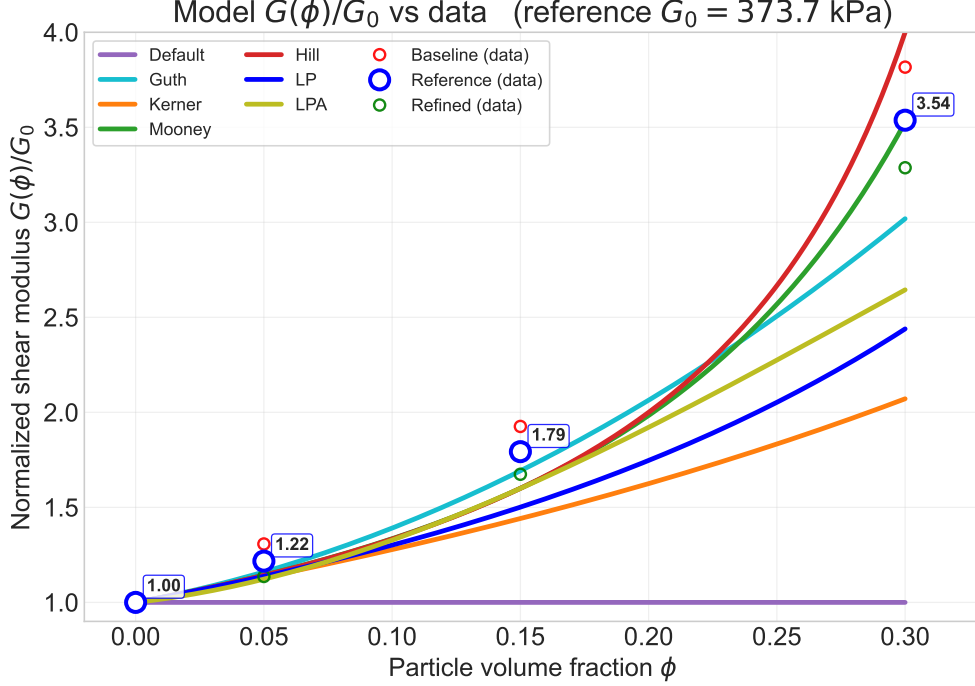


Figure 10: Comparison of the normalized shear-modulus relations $G(\phi)/G_0$ with the extracted modulus data. Solid curves denote the effective shear-modulus relations, while markers denote the baseline, refined, and reference estimates from the digitized experimental data.

the $G(\phi)$ relation that gives the smallest error across the tested values of ϕ and representative values of G_0 .

For each ϕ , G_0 , and $G(\phi)$ model, the simulated curve is compared with the reference data using

$$\text{rRMSE}(\phi, G_0, \text{model}) = \sqrt{\frac{1}{N} \sum_{i=1}^N \left(\frac{\sigma^{\text{model}}(\varepsilon_i^{\text{data}}) - \sigma_i^{\text{data}}}{\sigma_i^{\text{data}}} \right)^2}, \quad (32)$$

where $\sigma^{\text{model}}(\varepsilon_i^{\text{data}})$ denotes the simulated stress interpolated at the experimental strain value $\varepsilon_i^{\text{data}}$. Points with very small reference stress are excluded to avoid division by values near zero. Simulations are performed for $\phi \in \{0, 0.05, 0.15, 0.30\}$, $G_0 \in \{373.7, 400, 420\}$ kPa, and seven $G(\phi)$ models. Details of the dog-bone finite-element setup, averaging over G_0 and ϕ , and additional fixed- G_0 comparisons are provided in [Appendix C.3](#) and [Appendix C.4](#).

[Figure 11](#) summarizes the full stress-strain comparison after averaging over the tested volume fractions and matrix-modulus calibrations. The default model gives the largest mean rRMSE, confirming that a ϕ -dependent modulus relation is required. Mooney, Hill, and Guth form the lowest-error group. Mooney gives the smallest mean rRMSE, while Hill and Guth remain close.

[Figure 12](#) shows representative stress-strain curves for the three lowest-error candidates, namely Mooney, Hill, and Guth, using $G_0 = 373.7$ kPa. The three models are close at small and moderate volume fractions, but their differences become more visible at $\phi = 0.30$. Hill gives a slightly stiffer response at the largest volume fraction, while Mooney provides the best overall balance across the full set of volume fractions and G_0 values.

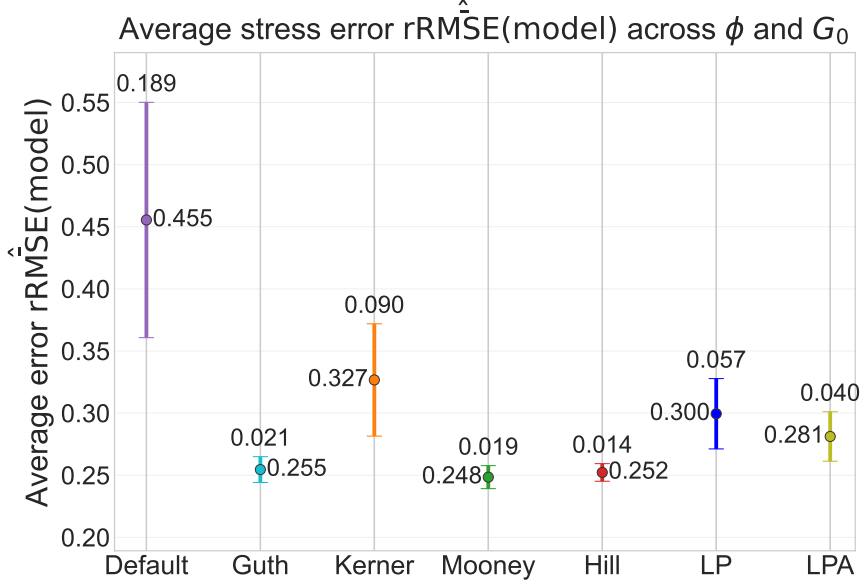


Figure 11: Summary of the rRMSE for seven $G(\phi)$ models. The plot reports the mean rRMSE over $\phi \in \{0, 0.05, 0.15, 0.30\}$ and $G_0 \in \{373.7, 400, 420\}$ kPa. Labels denote the mean values, and vertical bars denote the standard deviation over ϕ .

4.3. Summary of results and selected model

Taken together, the two comparisons show that **Mooney** is the most consistent choice among the available effective shear-modulus relations; see the Metric 1 and Metric 2 entries in Table 2. It gives the lowest mean error in the direct $G(\phi)$ comparison and the lowest mean rRMSE in the full stress–strain comparison. **Hill** and **Guth** are close alternatives in the stress–strain metric, but they do not improve the combined outcome. **Kerner**, **LP**, and **LPA** show larger discrepancies, while the **Default** model performs poorly because it ignores particle-induced stiffening. Therefore, the **Mooney** effective shear-modulus relation is selected for the joint material–structural optimization studies in the remainder of this work.

5. Joint material–structural optimization formulation

This section introduces the joint material–structural optimization formulation used in the design examples. The formulation builds on the density-based topology optimization framework of FEniTop Jia et al. (2024), which provides filtering, projection, adjoint sensitivity analysis, and MMA-based design updates. The present work extends this framework from standard structural topology optimization to nonlinear finite-strain magneto-mechanical optimization of hard-magnetic soft materials. In particular, the equilibrium equations are governed by the energy-based hMSM formulation in Section 2, and the design variables include not only structural density but also magnetic particle volume fraction and remanent magnetization direction.

The formulation is also related to the topology optimization framework of Zhao and Zhang (2022), where topology, remanent magnetization distribution, and applied magnetic fields are optimized for hMSMs. A key difference is the parameterization of the magnetic variables. In Zhao and Zhang (2022), the local remanent magnetization is selected from a finite set of prescribed candidate

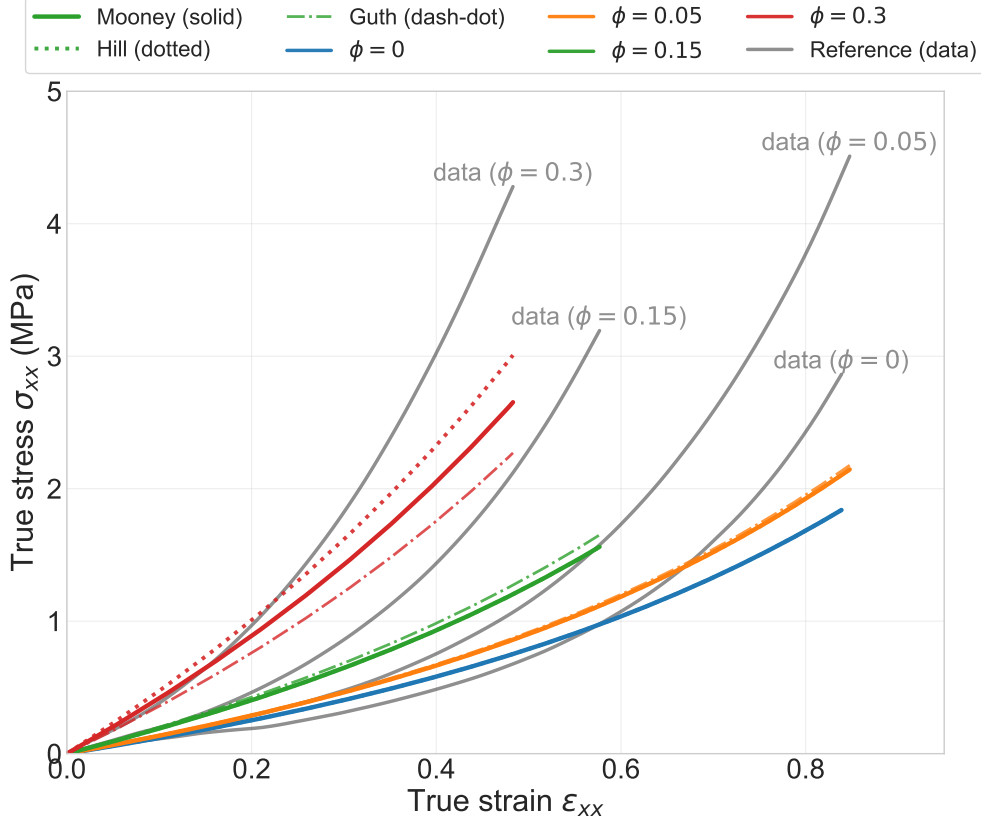


Figure 12: Reference stress–strain data and **NH2** simulations at $G_0 = 373.7$ kPa for Mooney, Hill, and Guth. Color encodes the particle volume fraction ϕ , and line style encodes the $G(\phi)$ model.

directions through interpolation weights, with penalization used to promote the selection of a single direction. In the present work, the remanent magnetization direction is represented by a continuous angular field, while the magnetic particle volume fraction is introduced as an independent continuous material-composition design field. Thus, magnetic material placement and magnetization direction are separated: ϕ controls the amount of magnetic material, and θ_{Br} controls its local remanent direction.

This separation makes the formulation broader than conventional topology optimization, but it also creates the central coupling issue addressed by the proposed methodology. The three design fields do not have the same physical status: ρ controls the presence of structural material, whereas ϕ and θ_{Br} are physically meaningful only in material-filled regions. Therefore, the design variables must remain independent for optimization, while their mechanical and magnetic effects must remain tied to the presence of material. This physical consistency is imposed through the design-dependent free-energy density. The density field scales the elastic and magnetic energy contributions, while ϕ modifies the effective shear modulus and also scales the magnetic interaction energy. In this way, the magnetic contribution depends on both the structural density and the local magnetic particle content, without imposing separate constraints on ϕ or θ_{Br} in low-density regions.

The resulting problem is therefore a joint material–structural optimization problem in which structural density, material composition, and remanent magnetization direction are coupled through

the same nonlinear finite-strain magneto-mechanical energy. The framework can optimize any active subset of the design fields, or all three simultaneously, under single- or multi-load-case combinations of mechanical and magnetic loading. Continuous optimized fields can be used directly for computational design studies or postprocessed into discrete particle concentrations and magnetization directions when required for manufacturing.

5.1. Design variable parameterization

Three design fields are considered on the reference domain Ω_0 : the structural density ρ , the magnetic particle volume fraction ϕ , and the remanent magnetization direction θ_{Br} . Any subset of these variables may be optimized, while the remaining fields are prescribed.

In the discrete implementation, the raw design variables are stored element-wise. Let $\{\Omega_e\}_{e=1}^{N_e}$ denote the finite-element partition of Ω_0 . One density value ρ_e , one magnetic particle volume fraction value ϕ_e , and one remanent magnetization direction value $\theta_{Br,e}$ are associated with each element Ω_e . These element-wise values define piecewise-constant fields on Ω_0 . For notational simplicity, the same symbols ρ , ϕ , and θ_{Br} are used for these piecewise-constant design fields.

The common discrete space for the raw scalar design fields is

$$Q_h = \{q \in L^2(\Omega_0) : q|_{\Omega_e} \in \mathbb{P}_0(\Omega_e), \quad e = 1, \dots, N_e\}. \quad (33)$$

Here, $\mathbb{P}_k(\Omega_e)$ denotes the space of polynomial functions of degree at most k on the element Ω_e . In particular, $\mathbb{P}_0(\Omega_e)$ denotes the space of constant functions on Ω_e .

For any raw scalar design field $q \in Q_h$ and filter radius $r_q > 0$, the Helmholtz-filter operator is written as

$$\mathcal{H}(\cdot; r_q) : Q_h \rightarrow V_h^{\text{HH}}. \quad (34)$$

The filtered field

$$\tilde{q} = \mathcal{H}(q; r_q) \quad (35)$$

is the unique function in V_h^{HH} satisfying

$$\int_{\Omega_0} (r_q^2 \nabla \tilde{q} \cdot \nabla v + \tilde{q} v) dV = \int_{\Omega_0} q v dV, \quad \forall v \in V_h^{\text{HH}}. \quad (36)$$

Here,

$$V_h^{\text{HH}} = \{v \in H^1(\Omega_0) : v|_{\Omega_e} \in \mathcal{L}_1(\Omega_e), \quad e = 1, \dots, N_e\}, \quad (37)$$

where $\mathcal{L}_1(\Omega_e)$ denotes the local first-order Lagrange finite-element space on Ω_e . For simplex elements, this corresponds to $\mathbb{P}_1(\Omega_e)$; for quadrilateral or hexahedral elements, it corresponds to the usual first-order tensor-product Lagrange space. No essential boundary condition is imposed on $\tilde{q}$, so Equation (36) corresponds to the natural homogeneous Neumann condition: $\nabla \tilde{q} \cdot \mathbf{n} = 0$ on $\partial\Omega_0$.

5.1.1. Density field

The density field defines the local occupancy of structural material within the design domain. With the convention introduced above, the element-wise density values $\{\rho_e\}_{e=1}^{N_e}$ define the piecewise-constant field

$$\rho(\mathbf{X}) = \sum_{e=1}^{N_e} \rho_e \chi_e(\mathbf{X}), \quad (38)$$

where χ_e is the characteristic function of element Ω_e . The design space for the raw density field is Q_h defined in Equation (33). We additionally impose the element-wise bounds

$$\rho_{\min} \leq \rho_e \leq 1, \quad e = 1, \dots, N_e.$$

In the implementation, a small positive lower bound $\rho_{\min} \ll 1$ is used for the raw density variable to avoid singular stiffness matrices.

The density field is regularized using the Helmholtz-filter operator defined in Equation (36). Specifically,

$$\tilde{\rho} = \mathcal{H}(\rho; r_\rho), \quad (39)$$

where r_ρ is the density filter radius.

The filtered density field is then passed through a Heaviside projection to obtain the physical density field $\bar{\rho}$:

$$\bar{\rho} = \mathcal{P}_\rho(\tilde{\rho}) := \frac{\tanh(\beta_\rho \eta_\rho) + \tanh(\beta_\rho (\tilde{\rho} - \eta_\rho))}{\tanh(\beta_\rho \eta_\rho) + \tanh(\beta_\rho (1 - \eta_\rho))}. \quad (40)$$

Here, η_ρ is the projection threshold, and β_ρ controls the sharpness of the transition. During optimization, β_ρ is progressively increased to drive $\bar{\rho}$ toward a nearly binary distribution. The physical density field $\bar{\rho}$ is the density variable that enters the free-energy interpolation introduced in Section 5.2.

5.1.2. Magnetic particle volume fraction field

The magnetic particle volume fraction field controls the local concentration of hard-magnetic particles within the design domain. As in the case of the density field,

$$\phi(\mathbf{X}) = \sum_{e=1}^{N_e} \phi_e \chi_e(\mathbf{X}) \quad \in Q_h,$$

where $\{\phi_e\}_{e=1}^{N_e}$ are the element-wise values of the raw magnetic particle volume fraction field. The local element-wise bounds are imposed as

$$0 \leq \phi_e \leq \phi_{\max}, \quad e = 1, \dots, N_e,$$

where $\phi_{\max} < 1$ is the maximum allowable magnetic particle volume fraction.

The magnetic particle volume fraction field is regularized using the same Helmholtz-filter operator defined in Equation (36). Specifically,

$$\phi_{\text{phys}} = \mathcal{H}(\phi; r_\phi), \quad (41)$$

where r_ϕ is the filter radius for the magnetic particle volume fraction field.

Unlike the density field, the magnetic particle volume fraction field is not passed through a Heaviside projection and is not SIMP-penalized. This choice reflects the interpretation of ϕ_{phys} as a continuous material-composition field rather than a void–solid structural density. The filtered field ϕ_{phys} enters the free-energy interpolation introduced in Section 5.2: it modifies the effective shear modulus through the selected structure–property relation and scales the magnetic interaction according to the local concentration of magnetic particles.

5.1.3. Remanent magnetization direction field

The remanent magnetization direction field controls the local orientation of the embedded remanent magnetic flux density. As before,

$$\theta_{Br}(\mathbf{X}) = \sum_{e=1}^{N_e} \theta_{Br,e} \chi_e(\mathbf{X}) \quad \in Q_h,$$

where $\{\theta_{Br,e}\}_{e=1}^{N_e}$ are the element-wise values of the raw remanent magnetization direction field. The local element-wise bounds are enforced as

$$-\pi \leq \theta_{Br,e} \leq \pi, \quad e = 1, \dots, N_e.$$

Similar to the previous design fields, the remanent magnetization direction field is regularized using the Helmholtz-filter operator to obtain a filtered field $\theta_{Br,\text{phys}}$:

$$\theta_{Br,\text{phys}} = \mathcal{H}(\theta_{Br}; r_\theta), \quad (42)$$

where r_θ is the filter radius.

Remark 4. The Helmholtz filter in Equation (42) is applied directly to the scalar angular field θ_{Br} , with angles represented in the interval $[-\pi, \pi]$. This convention is used in the present examples. If optimized directions contain neighboring values near both ends of this interval, a periodic representation based on filtering $\cos \theta_{Br}$ and $\sin \theta_{Br}$ would be more appropriate.

The filtered direction field is then used to define the local remanent magnetic flux density:

$$\mathbf{B}^r(\theta_{Br,\text{phys}}) = B^r \begin{bmatrix} \cos \theta_{Br,\text{phys}} \\ \sin \theta_{Br,\text{phys}} \end{bmatrix}. \quad (43)$$

Similar to ϕ , θ_{Br} is not subjected to Heaviside projection or SIMP penalization, allowing continuous variation of the remanent direction. The filtered field $\theta_{Br,\text{phys}}$ enters the free energy through $\mathbf{B}^r$.

5.2. Energy interpolation and design variable coupling

The filtered and projected design fields from Section 5.1 enter the nonlinear finite-element formulation through the design-dependent total free-energy density. This is the step where the physical coupling among structural density, magnetic particle concentration, and remanent magnetization direction is imposed. For a load case ℓ with applied magnetic flux density $\mathbf{B}^{a,(\ell)}$, the design-dependent free-energy density is written as

$$\widehat{W}^{(\ell)}(\mathbf{F}; \bar{\rho}, \phi_{\text{phys}}, \theta_{Br,\text{phys}}, \mathbf{B}^{a,(\ell)}) = \widehat{W}_{\text{elas}}(\mathbf{F}; \bar{\rho}, \phi_{\text{phys}}) + \widehat{W}_{\text{magn}}^{(\ell)}(\mathbf{F}; \bar{\rho}, \phi_{\text{phys}}, \theta_{Br,\text{phys}}, \mathbf{B}^{a,(\ell)}). \quad (44)$$

The two terms are defined below to make explicit how each design field enters the elastic and magnetic parts of the energy.

5.2.1. Density-penalized moduli and effective material response

The density field is incorporated through a SIMP-type interpolation of the reference elastic moduli. Define

$$s_\rho(\bar{\rho}) = \epsilon_\rho + (1 - \epsilon_\rho)\bar{\rho}^{p_\rho}, \quad (45)$$

where $\epsilon_\rho > 0$ is a small stiffness floor and p_ρ is the density penalization exponent. The density-penalized matrix shear and bulk moduli are

$$\hat{G}_0(\bar{\rho}) = s_\rho(\bar{\rho})G_0, \quad \hat{K}_0(\bar{\rho}) = s_\rho(\bar{\rho})K_0. \quad (46)$$

The selected effective shear-modulus relation is then evaluated using the density-penalized matrix shear modulus. The design-dependent effective shear modulus is written in the general form

$$\hat{G}(\bar{\rho}, \phi_{\text{phys}}) = G_{\text{sel}}(\phi_{\text{phys}}; \hat{G}_0(\bar{\rho})), \quad (47)$$

where G_{sel} denotes the selected effective shear-modulus relation. Based on the model-selection study in [Section 4](#), the optimization examples use the Mooney relation. Therefore,

$$\hat{G}(\bar{\rho}, \phi_{\text{phys}}) = \hat{G}_0(\bar{\rho}) \exp\left(\frac{k_E \phi_{\text{phys}}}{1 - 1.35\phi_{\text{phys}}}\right). \quad (48)$$

Consistent with the constitutive formulation in [Section 2.3](#), the effective bulk modulus is not modified by ϕ_{phys} and is taken as

$$\hat{K}(\bar{\rho}) = \hat{K}_0(\bar{\rho}). \quad (49)$$

Using the selected **NH2** strain-energy density, the design-dependent elastic free-energy density is

$$\widehat{W}_{\text{elas}}(\mathbf{F}; \bar{\rho}, \phi_{\text{phys}}) = \frac{\hat{G}(\bar{\rho}, \phi_{\text{phys}})}{2} (I_1 - 3 - 2 \ln J) + \frac{\hat{K}(\bar{\rho})}{2} (J - 1)^2. \quad (50)$$

5.2.2. Magnetic energy coupling

For load case ℓ , the design-dependent magnetic free-energy density is

$$\widehat{W}_{\text{magn}}^{(\ell)}(\mathbf{F}; \bar{\rho}, \phi_{\text{phys}}, \theta_{B^r, \text{phys}}, \mathbf{B}^{a, (\ell)}) = -\frac{1}{\mu_0} \bar{\rho} \phi_{\text{phys}} \mathbf{F} \mathbf{B}^r(\theta_{B^r, \text{phys}}) \cdot \mathbf{B}^{a, (\ell)}. \quad (51)$$

Here, $\mathbf{B}^r(\theta_{B^r, \text{phys}})$ is defined in [Equation \(43\)](#). The factor $\bar{\rho}$ attenuates the magnetic energy in low-density regions, while ϕ_{phys} scales the magnetic interaction according to the local magnetic particle concentration. Thus, magnetic actuation is active only where structural material and magnetic particles are present, while $\theta_{B^r, \text{phys}}$ controls the local remanent direction.

The relations [Equations \(50\)](#) and [\(51\)](#), together with [Equation \(44\)](#), complete the definition of the design-dependent total free-energy density.

5.3. Optimization formulation

This subsection defines the finite-dimensional optimization problem in terms of the raw design fields introduced in [Section 5.1](#). The physical fields that enter the equilibrium equations are

$$\bar{\rho} = \mathcal{P}_\rho(\mathcal{H}(\rho; r_\rho)), \quad \phi_{\text{phys}} = \mathcal{H}(\phi; r_\phi), \quad \theta_{B^r, \text{phys}} = \mathcal{H}(\theta_{B^r}; r_\theta), \quad (52)$$

where $\mathcal{H}(\cdot; r)$ denotes the Helmholtz filter operator with radius r in Equation (34) and $\mathcal{P}_\rho$ denotes the Heaviside projection in Equation (40). These physical fields determine the design-dependent free-energy density through Equations (44), (50) and (51).

For each load case $\ell = 1, \dots, N_\ell$, the applied magnetic flux density $\mathbf{B}^{a,(\ell)}$, body force $\mathbf{b}^{(\ell)}$, prescribed traction $\bar{\mathbf{t}}^{(\ell)}$, and prescribed displacement $\bar{\mathbf{u}}^{(\ell)}$ are specified. The displacement field $\mathbf{u}^{(\ell)}$ satisfies the design-dependent residual equation

$$\widehat{\mathcal{R}}^{(\ell)} \left(\mathbf{u}^{(\ell)}, \delta \mathbf{u}; \bar{\rho}, \phi_{\text{phys}}, \theta_{Br, \text{phys}}, \mathbf{B}^{a,(\ell)}, \mathbf{b}^{(\ell)}, \bar{\mathbf{t}}^{(\ell)}, \bar{\mathbf{u}}^{(\ell)} \right) = 0, \quad \forall \delta \mathbf{u} \in V_v. \quad (53)$$

Here, $\widehat{\mathcal{R}}^{(\ell)}$ is obtained from the residual functional in Equation (9) by replacing W with the design-dependent free-energy density $\widehat{W}^{(\ell)}$ in Equation (44). Equivalently, the first Piola–Kirchhoff stress in load case ℓ is

$$\widehat{\mathbf{P}}^{(\ell)} = \frac{\partial \widehat{W}^{(\ell)}}{\partial \mathbf{F}}. \quad (54)$$

After finite-element discretization, Equation (53) gives the nonlinear algebraic residual equation

$$\widehat{\mathbf{r}}^{(\ell)} \left(\mathbf{u}^{(\ell)}; \bar{\rho}, \phi_{\text{phys}}, \theta_{Br, \text{phys}}, \mathbf{B}^{a,(\ell)}, \mathbf{b}^{(\ell)}, \bar{\mathbf{t}}^{(\ell)}, \bar{\mathbf{u}}^{(\ell)} \right) = \mathbf{0}, \quad \ell = 1, \dots, N_\ell. \quad (55)$$

The optimization problem is written in terms of the raw design variables ρ , ϕ , and θ_{Br} :

$$\begin{aligned} & \min_{\rho, \phi, \theta_{Br}} \quad f \left(\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(N_\ell)} \right) \\ & \text{subject to} \quad \widehat{\mathbf{r}}^{(\ell)} \left(\mathbf{u}^{(\ell)}; \bar{\rho}, \phi_{\text{phys}}, \theta_{Br, \text{phys}}, \mathbf{B}^{a,(\ell)}, \mathbf{b}^{(\ell)}, \bar{\mathbf{t}}^{(\ell)}, \bar{\mathbf{u}}^{(\ell)} \right) = \mathbf{0}, \quad \ell = 1, \dots, N_\ell, \\ & \quad \frac{1}{|\Omega_0|} \sum_{e=1}^{N_e} \bar{\rho}_e v_e \leq v_{\max}, \\ & \quad \frac{1}{|\Omega_0|} \sum_{e=1}^{N_e} \phi_{\text{phys}, e} v_e \leq \Phi_{\max}, \\ & \quad \left. \begin{aligned} \rho_{\min} &\leq \rho_e \leq 1, \\ 0 &\leq \phi_e \leq \phi_{\max}, \\ -\pi &\leq \theta_{Br, e} \leq \pi \end{aligned} \right\} \quad e = 1, \dots, N_e. \end{aligned} \quad (56)$$

Here, $v_e = |\Omega_e|$ denotes the measure (area or volume) of element Ω_e , and $|\Omega_0| = \sum_{e=1}^{N_e} v_e$. The quantities $\bar{\rho}_e$ and $\phi_{\text{phys}, e}$ denote element-wise averages of the corresponding physical fields. The parameter $v_{\max}$ specifies the allowable structural material volume fraction, $\Phi_{\max}$ bounds the domain-averaged magnetic material fraction, $\rho_{\min}$ is the lower bound on the raw density variable used to avoid singular stiffness matrices, and $\phi_{\max}$ specifies the local upper bound on magnetic particle concentration.

The objective function f is problem dependent. In the examples considered in Section 6, it is assembled from one or more load-case responses. A typical form is

$$f \left(\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(N_\ell)} \right) = \sum_{\ell=1}^{N_\ell} w_\ell f^{(\ell)} \left(\mathbf{u}^{(\ell)} \right), \quad (57)$$

where w_ℓ is a prescribed load-case weight. Other choices, such as max-type objectives or single-load-case objectives, can be used without changing the design-variable parameterization or the equilibrium formulation.

Although Equation (56) is written with all three design variables active, the same formulation applies when only a subset of the fields is optimized. Inactive fields are prescribed and held fixed, and MMA updates only the remaining active design variables. This modular structure allows the formulation to represent prescribed-structure magnetic design, material-only design, or full joint material–structural design within the same framework.

5.4. Adjoint-based gradient computation

The optimization problem in Equation (56) is solved using the Method of Moving Asymptotes (MMA), a gradient-based optimization algorithm that requires derivatives of the objective and constraint functions with respect to the raw design variables. These derivatives are computed using an adjoint method, following the general framework described in Jia et al. (2024), Zhao and Zhang (2022).

Let Φ denote a response function, such as the objective or one of the constraints. For each load case ℓ , let $\mathbf{u}^{(\ell)}$ denote the discrete displacement vector and let

$$\hat{\mathbf{r}}^{(\ell)} \left(\mathbf{u}^{(\ell)}; \bar{\rho}, \phi_{\text{phys}}, \theta_{Br, \text{phys}}, \mathbf{B}^{a, (\ell)}, \mathbf{b}^{(\ell)}, \bar{\mathbf{t}}^{(\ell)}, \bar{\mathbf{u}}^{(\ell)} \right) = \mathbf{0}$$

denote the discrete residual equation from Equation (55). The tangent stiffness matrix for load case ℓ is

$$\hat{\mathbf{K}}_{\text{T}}^{(\ell)} = \frac{\partial \hat{\mathbf{r}}^{(\ell)}}{\partial \mathbf{u}^{(\ell)}}. \quad (58)$$

The corresponding adjoint vector $\boldsymbol{\lambda}^{(\ell)}$ is obtained by solving

$$\left(\hat{\mathbf{K}}_{\text{T}}^{(\ell)} \right)^T \boldsymbol{\lambda}^{(\ell)} = - \frac{\partial \Phi}{\partial \mathbf{u}^{(\ell)}}. \quad (59)$$

For each response function requiring design sensitivities, this adjoint system is solved once per load case. The same adjoint vector can then be used to evaluate derivatives with respect to all active design variables.

Let η_e denote a representative physical design variable, where

$$\eta_e \in \{ \bar{\rho}_e, \phi_{\text{phys}, e}, \theta_{Br, \text{phys}, e} \}.$$

The total derivative of Φ with respect to η_e is

$$\frac{d\Phi}{d\eta_e} = \left. \frac{\partial \Phi}{\partial \eta_e} \right|_{\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(N_\ell)}} + \sum_{\ell=1}^{N_\ell} \left(\boldsymbol{\lambda}^{(\ell)} \right)^T \frac{\partial \hat{\mathbf{r}}^{(\ell)}}{\partial \eta_e}. \quad (60)$$

The first term accounts for any explicit dependence of the response function on the design field, while the second term accounts for the dependence through the equilibrium equations. If a response depends on only one load case, the summation in Equation (60) is restricted to that load case. For state-independent constraints, such as the volume constraints in Equation (56), the adjoint contribution is zero and the derivative is computed directly.

The adjoint calculation gives derivatives with respect to the physical fields that enter the finite-element formulation. Since MMA updates the raw design variables, these derivatives are propagated backward through the filtering and projection operations using the chain rule.

For the density field, the raw, filtered, and projected variables are related by

$$\tilde{\rho} = \mathcal{H}(\rho; r_\rho), \quad \bar{\rho} = \mathcal{P}_\rho(\tilde{\rho}).$$

In vector form, let $\boldsymbol{\rho}$, $\tilde{\boldsymbol{\rho}}$, and $\bar{\boldsymbol{\rho}}$ denote the corresponding discrete fields. The chain rule gives

$$\frac{d\Phi}{d\boldsymbol{\rho}} = \left(\frac{\partial \tilde{\boldsymbol{\rho}}}{\partial \boldsymbol{\rho}} \right)^T \left(\frac{\partial \bar{\boldsymbol{\rho}}}{\partial \tilde{\boldsymbol{\rho}}} \right)^T \frac{d\Phi}{d\bar{\boldsymbol{\rho}}}. \quad (61)$$

The projection map $\mathcal{P}_\rho$ is applied pointwise to the filtered density field (see Equation (40)). Therefore, its discrete derivative is diagonal:

$$\frac{\partial \bar{\rho}_i}{\partial \tilde{\rho}_j} = \mathcal{P}'_\rho(\tilde{\rho}_i) \delta_{ij}, \quad (62)$$

where δ_{ij} is the Kronecker delta. Thus, the projection part of the chain rule reduces to an entry-wise multiplication.

The Helmholtz filter is a linear PDE-based map. In discrete form, the filtered density is obtained from

$$\mathbf{A}_\rho \tilde{\boldsymbol{\rho}} = \mathbf{B}_\rho \boldsymbol{\rho}, \quad (63)$$

where $\mathbf{A}_\rho$ is the assembled finite-element matrix associated with the Helmholtz-filter bilinear form, and $\mathbf{B}_\rho$ is the assembled matrix whose product with the raw element-wise density vector $\boldsymbol{\rho}$ gives the discretized right-hand side. Therefore, the discrete filter map is

$$\tilde{\boldsymbol{\rho}} = \mathbf{H}_\rho \boldsymbol{\rho}, \quad \mathbf{H}_\rho = \mathbf{A}_\rho^{-1} \mathbf{B}_\rho. \quad (64)$$

Substituting this relation into Equation (61) gives

$$\frac{d\Phi}{d\boldsymbol{\rho}} = \mathbf{H}_\rho^T \mathbf{D}_\rho \frac{d\Phi}{d\bar{\boldsymbol{\rho}}}, \quad \mathbf{D}_\rho = \text{diag}(\mathcal{P}'_\rho(\tilde{\rho}_1), \dots, \mathcal{P}'_\rho(\tilde{\rho}_{n_\rho})). \quad (65)$$

Equivalently, this can be evaluated without explicitly forming $\mathbf{H}_\rho$. First, compute

$$\mathbf{w}_\rho = \mathbf{D}_\rho \frac{d\Phi}{d\bar{\boldsymbol{\rho}}},$$

then solve the transpose Helmholtz-filter system

$$\mathbf{A}_\rho^T \mathbf{z}_\rho = \mathbf{w}_\rho,$$

and finally compute

$$\frac{d\Phi}{d\boldsymbol{\rho}} = \mathbf{B}_\rho^T \mathbf{z}_\rho. \quad (66)$$

This is the backpropagation step through the density filter and projection.

For the magnetic particle volume fraction and remanent magnetization direction, the procedure is the same except that no Heaviside projection is applied. Thus, for

$$\phi_{\text{phys}} = \mathcal{H}(\phi; r_\phi), \quad \theta_{Br, \text{phys}} = \mathcal{H}(\theta_{Br}; r_\theta),$$

the gradients are propagated only through the transpose Helmholtz-filter maps. In other words, the same steps in Equations (63) and (66) are used with ρ replaced by ϕ or θ_{Br} and with the projection derivative omitted.

The resulting gradients with respect to the raw design variables ρ_e , ϕ_e , and $\theta_{Br,e}$ are supplied to MMA for the next design update. When a design field is inactive, its gradient is not used and the corresponding field is held fixed during optimization.

The complete optimization procedure is summarized in the flowchart in Figure 13. The next section presents numerical examples demonstrating how the formulation handles different active design fields, objectives, and loading scenarios.

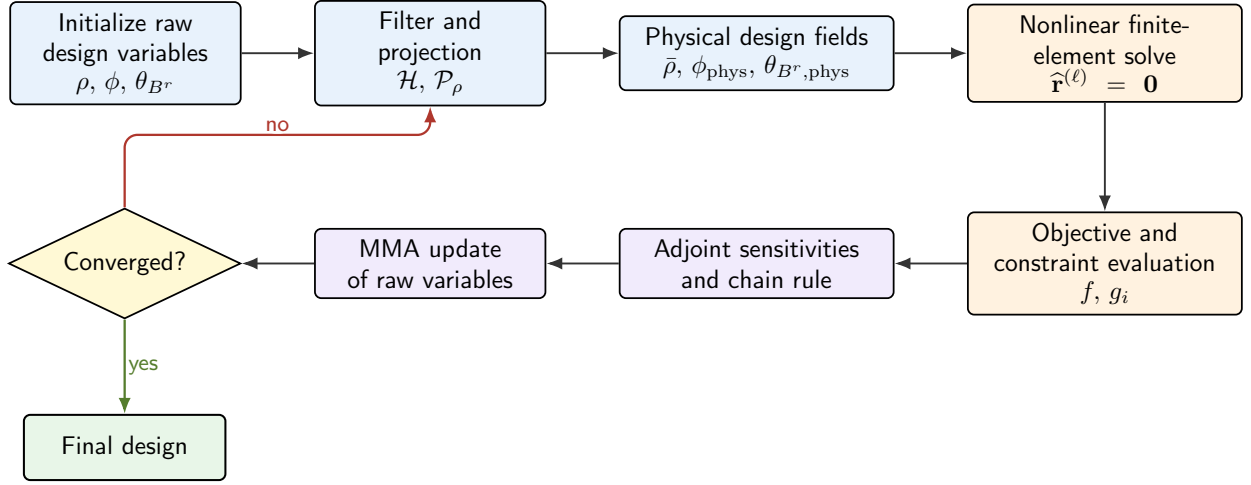


Figure 13: Flowchart of the joint material–structural optimization procedure. Starting from the raw design variables, filtering and projection are applied to obtain the physical design fields. The nonlinear finite-element problem is then solved, objective and constraint values are evaluated, adjoint sensitivities are computed, and MMA updates the raw design variables. The loop continues until the convergence criterion is satisfied.

6. Design of hMSM structures using joint material–structural optimization

This section demonstrates the joint material–structural optimization framework developed in Section 5. All examples use the constitutive model selected in Section 4, namely the **NH2** strain-energy density together with the **Mooney** effective shear-modulus relation. The examples are organized to progress from prescribed-structure magnetic design to full joint material–structural design, as summarized in Table 5. Together, these examples highlight the central feature of the proposed framework: structural density, magnetic material concentration, and remanent magnetization direction can be optimized independently or simultaneously within the same nonlinear magneto-mechanical formulation. This allows both prescribed-structure magnetic design and fully coupled material–structural design to be treated in a unified way.

6.1. Key computational details

Table 6 summarizes the key parameters used in the three optimization examples. The implementation is based on the FEniTop library² Jia et al. (2024) and the FEniCSx finite-element library (version 0.10.0) Baratta et al. (2023).

² FEniTop: <https://github.com/missionlab/fenitop>

Table 5: Summary of the optimization examples considered in this section.

Example	Optimized fields	Main demonstration
Rotational actuator	ϕ, θ_{Br}	Magnetic design in a fixed wheel geometry: redistribute magnetic material and orient remanent magnetization to increase rotational output.
Translational actuator	ϕ, θ_{Br}	Magnetic design in a fixed scissor-like geometry: produce targeted horizontal motion while suppressing undesired vertical displacement.
Restorative beam	ρ, ϕ, θ_{Br}	Full joint material–structural design: optimize structural density, magnetic particle distribution, and remanent magnetization direction for bidirectional restoration.

For each optimization iteration and load case, the nonlinear equilibrium equation is solved for the displacement field using the DOLFINx Newton solver. The Jacobian is obtained by automatic differentiation of the nonlinear residual. An incremental convergence criterion is used, with the absolute and relative tolerances both set to 10^{-4} . To improve robustness for the finite-deformation response, the applied magnetic field and mechanical traction are increased linearly from zero to their target values over N_{step} equal load increments. The converged displacement from each increment is used as the initial guess for the following increment. At the beginning of each load case, the displacement field and applied loads are reset to zero so that the load cases are solved independently.

Further implementation details are available in the `top_optim` software repository³ Galloway and Jha (2026), particularly in the Python input scripts contained in the `opt` directory.

Table 6: Optimization parameters that differ across the three design examples. Common settings, not listed in the table body, are the **NH2** strain-energy density, Mooney effective shear-modulus relation, $G_0 = 100$ kPa, $\phi_{\text{max}} = 0.30$, $\rho_{\text{min}} = 0.05$, $p_\rho = 3$, $\epsilon_\rho = 10^{-6}$, a maximum of 100 optimization iterations, and an optimization tolerance of 10^{-5} . For the restorative beam, the density-projection parameters are $\beta_{\text{max}} = 4$ and $\beta_{\text{int}} = 25$. In the rotational and translational actuator examples, $\rho \equiv 1$ is fixed and $v_{\text{max}} = 1.00$ therefore denotes a fully solid structure rather than an active volume constraint. Here, $\mathcal{D}_{\text{act}}$ is the set of active design fields, B^r is the fixed magnitude of the remanent magnetic flux density, $\mathbf{B}^a$ is the applied magnetic flux-density vector, $\mathbf{t}$ is the applied traction vector, N_{LC} is the number of load cases, N_{step} is the number of load steps per load case, v_{max} is the structural material-volume bound, Φ_{max} is the domain-averaged magnetic material bound, $r = r_\rho = r_\phi = r_\theta$ is the common filter radius, and m_{MMA} is the MMA move limit. The entries for B^r and $\mathbf{B}^a$ are reported in mT, $\mathbf{t}$ in mN/mm², and r in mm. For the restorative beam, entries on corresponding lines in the $\mathbf{B}^a$ and $\mathbf{t}$ columns define the two paired load cases.

Example	$\mathcal{D}_{\text{act}}$	B^r	$\mathbf{B}^a$	$\mathbf{t}$	N_{LC}	N_{step}	v_{max}	Φ_{max}	r	m_{MMA}	f
Rotational actuator	$\{\phi, \theta_{Br}\}$	100	(0, 100)	(0, 0)	1	100	1.00	0.30	1.00	0.01	rotational rim-band displacement
Translational actuator	$\{\phi, \theta_{Br}\}$	100	(125, 0)	(0, 0)	1	40	1.00	0.15	0.20	0.01	displacement tracking
Restorative beam	$\{\rho, \phi, \theta_{Br}\}$	200	(0, 25) (0, -25)	(0, -0.50) (0, 0.50)	2	50	0.50	0.10	1.00	0.005	compliance

³top_optim: https://github.com/CEADpx/top_optim/tree/v0.1.0

6.2. Rotational actuator

The first example optimizes the magnetic material distribution and remanent magnetization direction within the fixed wheel geometry shown in Figure 14(a). The structural density is prescribed as $\rho \equiv 1$, while the magnetic particle volume fraction ϕ and remanent direction θ_{Br} are treated as design fields. The objective is to maximize the counterclockwise tangential displacement near the outer rim under a uniform upward applied magnetic field, $\mathbf{B}^a = (0, 100)$ mT.

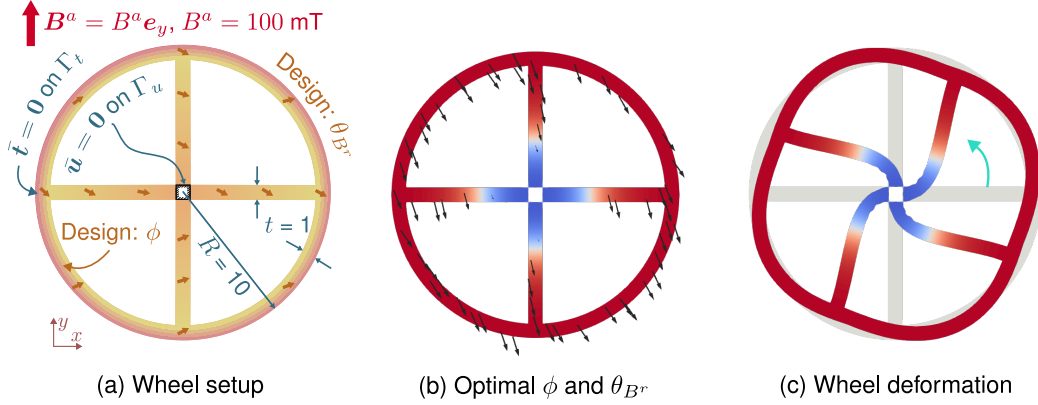


Figure 14: Optimization of the rotational actuator. (a) Fixed wheel geometry with prescribed structural density $\rho \equiv 1$, remanent magnetic flux-density magnitude $B^r = 100$ mT, and uniform upward applied magnetic field $\mathbf{B}^a = (0, 100)$ mT. The magnetic particle volume fraction ϕ and remanent direction θ_{Br} are optimized. (b) Optimized magnetic particle volume fraction and remanent magnetization direction. The arrows indicate the optimized local direction θ_{Br} . (c) Deformed configuration under the applied magnetic field, with the undeformed configuration shown in the background.

To quantify the rotational response, the tangential displacement of material points located in a narrow annular band near the outer rim of the wheel is measured. This avoids relying on a small set of boundary nodes and gives a smoother objective for optimization. For a material point $\mathbf{X} = (X, Y)$, the counterclockwise unit tangent vector about the wheel center is

$$\boldsymbol{\tau}(\mathbf{X}) = \frac{1}{\sqrt{X^2 + Y^2}} \begin{bmatrix} -Y \\ X \end{bmatrix}. \quad (67)$$

The rotational objective is then defined as

$$f_{\text{rot}}(\mathbf{u}) = - \int_{\Omega_0} w_{\text{rim}}(\mathbf{X}) \mathbf{u}(\mathbf{X}) \cdot \boldsymbol{\tau}(\mathbf{X}) \, dV. \quad (68)$$

Here, w_{rim} is a smooth radial weighting function given by $w_{\text{rim}} = \exp[-((r - r_0)/\sigma)^2]$, where $r = \sqrt{X^2 + Y^2}$, $r_0 = 0.95R$, and $\sigma = 0.75$ mm. This weighting localizes the objective to a narrow annular band near the outer rim. The quantity $\mathbf{u} \cdot \boldsymbol{\tau}$ is the counterclockwise tangential displacement. Therefore, the negative sign in Equation (68) makes the minimization problem favor larger counterclockwise rotation. Other computational parameters are listed in Table 6.

The optimized fields are shown in Figure 14(b). Despite the fixed structural geometry, the optimization produces a nonuniform magnetic particle distribution and a spatially varying remanent magnetization direction. Because ϕ influences both the magnetic coupling and the effective

shear modulus through the Mooney relation, its distribution controls the local balance between magnetic actuation and mechanical stiffness. Simultaneously, θ_{Br} controls the local orientation of the remanent magnetic flux density. The optimizer therefore coordinates the two fields to promote the desired counterclockwise motion rather than producing a uniformly magnetized wheel.

The resulting deformation is shown in Figure 14(c). For comparison, the intuitive wheel design introduced in Section 3.2 is used as the reference design. This design has a uniform magnetic particle volume fraction $\phi = \phi_{\max} = 0.30$ and a uniform horizontal remanent magnetization direction, $\theta_{Br} = 0$ rad. Although the optimization is driven by the weighted tangential-displacement objective in Equation (68), the final designs are compared using the total displacement magnitude, $\|\mathbf{u}(\mathbf{X}_m)\|$, at the material point $\mathbf{X}_m = (0, 10)$ mm located at the top of the undeformed wheel. When evaluated under the same applied magnetic loading, the intuitive design produces a displacement magnitude of 6.35 mm, whereas the optimized design achieves 10.14 mm. This represents a 59.7% increase in the measured displacement magnitude and demonstrates that optimizing ϕ and θ_{Br} can substantially enhance rotational actuation even when the structural geometry is fixed.

6.3. Translational actuator

The second example optimizes the magnetic material distribution and remanent magnetization direction within the fixed scissor-like geometry shown in Figure 15(a). As in the rotational actuator, the structural density is prescribed as $\rho \equiv 1$, while ϕ and θ_{Br} are treated as design fields. Under the uniform rightward applied magnetic field $\mathbf{B}^a = (125, 0)$ mT, the objective is to produce a prescribed rightward displacement of the output plate while suppressing undesired vertical motion. The corresponding optimization parameters are listed in the translational-actuator row of Table 6.

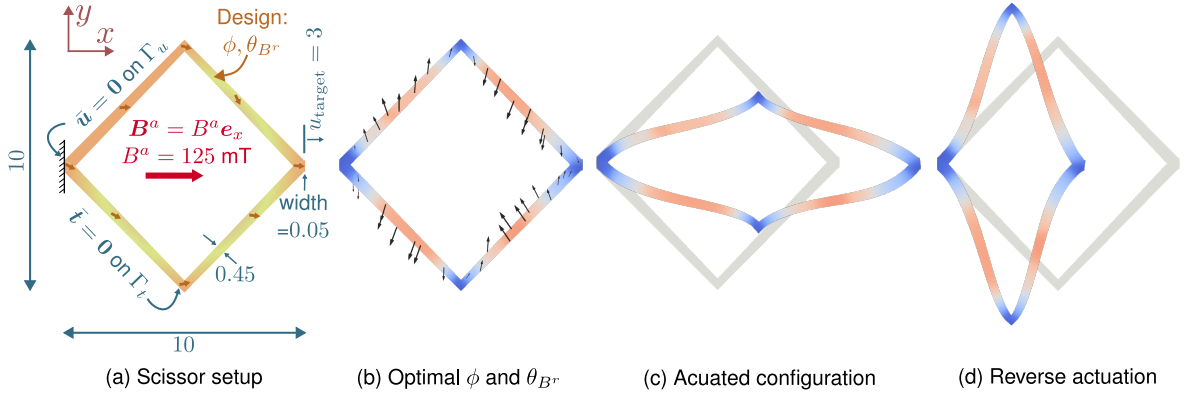


Figure 15: Optimization of the translational actuator. (a) Fixed scissor-like geometry with prescribed structural density $\rho \equiv 1$, remanent magnetic flux-density magnitude $B^r = 100$ mT, and uniform rightward applied magnetic field $\mathbf{B}^a = (125, 0)$ mT. The magnetic particle volume fraction ϕ and remanent direction θ_{Br} are optimized to produce the target output displacement $\mathbf{u}_{\text{target}} = (3, 0)$ mm. (b) Optimized magnetic particle volume fraction and remanent magnetization direction. The arrows indicate the optimized local direction θ_{Br} . (c) Deformed configuration under the rightward applied magnetic field. (d) Deformed configuration under the reversed applied magnetic field $\mathbf{B}^a = (-125, 0)$ mT.

A displacement-tracking objective localized near the output point is used to optimize ϕ and θ_{Br} so that the output plate undergoes the prescribed horizontal displacement while suppressing vertical motion:

$$f_{\text{tr}}(\mathbf{u}) = \int_{\Omega_0} w_{\text{out}}(\mathbf{X}) \|\mathbf{u}(\mathbf{X}) - \mathbf{u}_{\text{target}}\|^2 dV, \quad (69)$$

where w_{out} is a smooth Gaussian weighting function centered near the output location. In the implementation, the target displacement is

$$\mathbf{u}_{\text{target}} = \begin{bmatrix} 3 \text{ mm} \\ 0 \end{bmatrix}. \quad (70)$$

Thus, minimizing Equation (69) drives the displacement field toward the desired motion while penalizing deviations from the prescribed displacement vector.

Several non-intuitive features emerge from the optimized design. As shown in Figure 15(b), magnetic material is concentrated near the central portions of the scissor members, with comparatively little material placed in the compliant hinge regions. The output plate is prescribed as a nonmagnetic region, $\phi = 0$, and is therefore excluded from the magnetic material optimization. The optimizer also produces two dominant remanent magnetization orientations, approximately aligned with the upper-left and lower-left diagonal directions. Together, the optimized particle distribution and remanent directions generate a coordinated deformation response that promotes the prescribed horizontal output motion while reducing parasitic vertical displacement.

The forward-field deformation shown in Figure 15(c) is quantified at the material point $\mathbf{X}_m$ located at the vertical midpoint of the outer edge of the output plate. Under the rightward applied magnetic field $\mathbf{B}^a = (125, 0)$ mT, the optimized design produces $\mathbf{u}(\mathbf{X}_m) = (2.62, 0.039)$ mm, confirming that the output motion is predominantly horizontal. For the reversed-field evaluation shown in Figure 15(d), the optimized distributions of ϕ and θ_{Br} are held fixed while the applied field is reversed to $\mathbf{B}^a = (-125, 0)$ mT. The resulting displacement is $\mathbf{u}(\mathbf{X}_m) = (-4.00, -0.075)$ mm, indicating motion in the opposite horizontal direction. The forward and reversed displacement magnitudes are not expected to be equal because the response is geometrically nonlinear and only the forward-field load case is included in the optimization. Nevertheless, the reversed response demonstrates directional reversibility within the model and shows that the proposed framework can design magnetic architectures for prescribed and controllable modes of motion.

6.4. Restorative beam

The final example demonstrates the full joint material–structural formulation by optimizing the structural density ρ , magnetic particle volume fraction ϕ , and remanent magnetization direction θ_{Br} simultaneously. In contrast to the rotational and translational actuators, the structural geometry is no longer prescribed. The goal is to obtain a beam that resists mechanical loading while using magnetic actuation to restore the structure toward its undeformed configuration.

The problem setup is presented in Figure 16. The key feature of this example is the use of two opposing load cases. In the first load case, a downward traction is paired with an upward applied magnetic field; in the second, both directions are reversed. This multi-load-case formulation prevents the design from being specialized to a single bending direction and instead requires the optimized structural and magnetic fields to provide restorative behavior under loading in either direction. The optimization parameters for this case are listed in the restorative-beam row of Table 6.

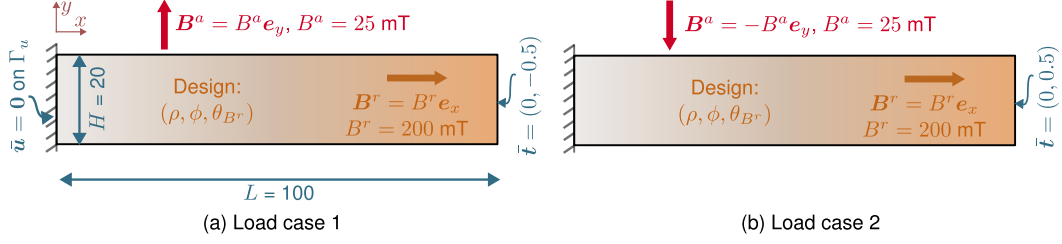


Figure 16: Multi-load-case restorative beam setup with $B^r = 200$ mT. (a) Downward traction $\bar{\mathbf{t}} = (0, -0.50)$ kPa paired with an upward applied magnetic field $\mathbf{B}^a = (0, 25)$ mT. (b) Reversed loading with upward traction $\bar{\mathbf{t}} = (0, 0.50)$ kPa and downward applied magnetic field $\mathbf{B}^a = (0, -25)$ mT. The design variables are the structural density ρ , magnetic particle volume fraction ϕ , and remanent magnetization direction θ_{B^r} . Lengths are in millimeters.

The objective is the weighted compliance over the load cases,

$$f_{\text{comp}} = \sum_{\ell=1}^{N_\ell} w_\ell \int_{\Gamma_t^{(\ell)}} \mathbf{u}^{(\ell)} \cdot \bar{\mathbf{t}}^{(\ell)} \, dS, \quad (71)$$

where $\mathbf{u}^{(\ell)}$ and $\bar{\mathbf{t}}^{(\ell)}$ are the displacement field and applied traction for load case ℓ , respectively, and w_ℓ is the corresponding load-case weight. Equal weights, $w_1 = w_2 = 1$, are assigned to the two opposing load cases. Minimizing Equation (71) promotes a structure that remains stiff under the applied tractions while using the optimized magnetic architecture to counteract the mechanical loading.

Figure 17 shows the optimized design. Because the two load cases are equal and opposite and are assigned the same weight, the objective does not favor either bending direction. The resulting structural and magnetic distributions are therefore nearly symmetric about the beam centerline. The optimized structural density forms continuous load-carrying paths along the upper and lower portions of the beam, while material is removed from regions that contribute less effectively to the restorative response. Through the coupling between $\bar{\rho}$ and ϕ_{phys} in the magnetic energy, the effective magnetic particle fraction vanishes as the projected structural density approaches zero. Consequently, magnetic material is confined to structurally occupied regions rather than being placed within the void.

Within the solid regions, magnetic material is concentrated along portions of the upper and lower load paths. This placement serves two coupled purposes: increasing ϕ raises the local effective shear modulus through the Mooney relation, thereby stiffening mechanically important regions, while also strengthening the local magnetic coupling where actuation can generate a restoring response. The spatially varying remanent directions further coordinate this response by allowing different regions to counteract the two opposing mechanical loads under their corresponding applied-field directions.

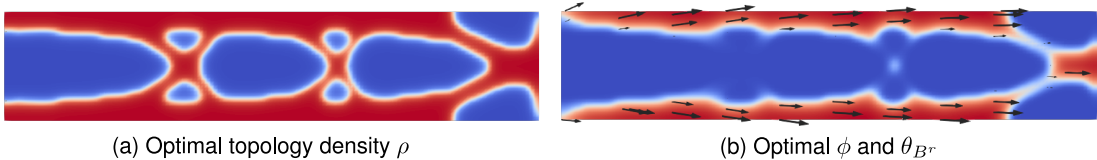


Figure 17: Optimized restorative beam. (a) Optimized structural density ρ . (b) Optimized magnetic particle volume fraction ϕ and remanent magnetization direction θ_{B^r} . Arrows indicate the optimized local direction θ_{B^r} .

To evaluate the response of the optimized design beyond the loading conditions used during optimization, the beam is subjected to a downward traction $\bar{\mathbf{t}} = (0, -1.00)$ kPa, which is twice the magnitude of the corresponding optimization traction. The applied magnetic field remains upward and is increased from 0 to 75 mT in increments of 5 mT. Selected configurations from this field sweep are shown in Figure 18. With no applied magnetic field, the beam bends downward under the mechanical traction. As the applied magnetic field magnitude increases, the magnetically induced response progressively counteracts the mechanical load. At approximately $|\mathbf{B}^a| = 65$ mT, the beam is restored close to its undeformed configuration despite the increased mechanical loading.

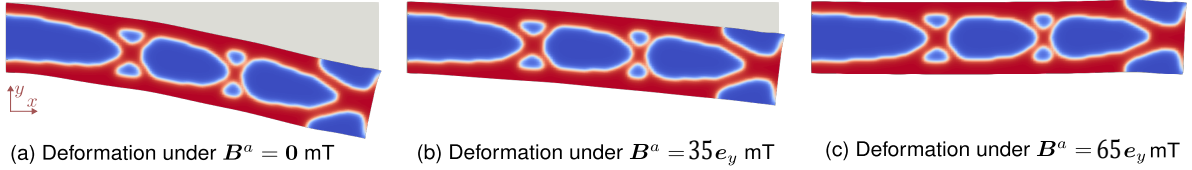


Figure 18: Evaluation of the optimized beam under the fixed downward traction $\bar{\mathbf{t}} = (0, -1.00)$ kPa, which is twice the magnitude used during optimization. Selected configurations are shown as the upward applied magnetic field is increased from 0 to 75 mT. The magnetically induced response progressively counteracts the increased mechanical loading and restores the beam toward its undeformed configuration.

This example demonstrates the benefit of optimizing all three design fields simultaneously. The structural density provides efficient load paths, the magnetic particle distribution places magnetic material where it contributes to both stiffness and actuation, and the remanent magnetization direction controls how different regions respond to the applied magnetic field. Together, these fields produce a coupled structural and magnetic architecture designed to provide restorative behavior under the two opposing optimization load cases. The higher-load evaluation further shows that the optimized architecture retains its restorative capability under a downward traction twice the optimization magnitude when a sufficiently strong applied magnetic field is used.

7. Conclusion

This work developed a model-informed framework for the analysis and joint material–structural optimization of hard-magnetic soft materials. The first part of the work placed classical rigid-inclusion relations, a Hill self-consistent relation, and constrained-kinematics models into a common effective shear-modulus form $G(\phi)$. The numerical studies showed that, for the actuation problems considered, the selected strain-energy density function has a relatively small effect on the predicted deformation, whereas the effective shear-modulus relation can have a significant effect. This sensitivity is strongest when magnetic material overlaps with mechanically active regions, as observed in the uniformly magnetized beam and wheel examples, and is much weaker when magnetic material is separated from the dominant deformation region, as observed in the tip-magnetized beam and gripper examples. These results highlight the importance of selecting an appropriate structure–property relation when particle concentration influences both stiffness and magnetic actuation.

Experimental stress–strain data were then used to select a representative $G(\phi)$ relation. Among the candidate models, the **Mooney** relation gave the best combined agreement with the extracted shear-modulus data and full uniaxial stress–strain responses, with **Guth** and **Hill** very close behind. The **Mooney** relation was used with the **NH2** strain-energy density in the optimization studies.

Building on this selected constitutive model, a joint material–structural optimization formulation was developed in which structural density, magnetic particle volume fraction, and remanent magnetization direction are coupled through the nonlinear finite-strain magneto-mechanical energy. The rotational, translational, and restorative examples showed that the framework can optimize different active subsets of the design fields, handle different objective functions and loading scenarios, and produce non-intuitive hMSM designs with prescribed deformation responses.

Although the examples focused on hMSMs, the formulation is more broadly applicable to filled and field-responsive elastomers in which local material composition affects both stiffness and actuation. The same computational structure can be extended to dielectric elastomers, magnetorheological elastomers, and shape-memory-alloy-elastomer composites by replacing the magnetic energy and effective-modulus relation with the corresponding electro-mechanical, magneto-mechanical, or thermomechanical material model. In this sense, the open-source CEADpx/top_optim implementation provides a starting point for a more general nonlinear joint optimization framework for coupled material composition, structural density, and actuation fields.

Future work will focus on further validation, model extension, and computational acceleration. Additional experimental datasets are needed to test the selected effective-property relations under different filler types, volume fractions, loading modes, and actuation conditions. Extensions to viscoelastic and dissipative material models will be important for capturing rate dependence, hysteresis, and cyclic response in practical soft-material systems. Since nonlinear finite-strain optimization with multiple design fields and load cases can be computationally expensive, another important direction is the development of neural-operator and other surrogate models to accelerate repeated forward and adjoint solves. Recent work on neural operators, residual-based error correction, and neural-operator acceleration of inverse and optimization problems provides useful tools for this direction [Cao et al. \(2023\)](#), [Jha \(2024, 2026\)](#). Together, these directions will strengthen the use of experimentally informed constitutive modeling and joint material–structural optimization as a systematic route for designing field-responsive soft structures with tailored actuation behavior.

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Appendix A. Rigid-inclusion limit of Hill’s self-consistent shear-modulus relation

This section derives the rigid-inclusion limit of Hill’s self-consistent shear-modulus relation used in [Section 2.4](#). Let K_0 and G_0 denote the bulk and shear moduli of the matrix, and let K_1 and G_1 denote the bulk and shear moduli of the particle-inclusion phase. The particle volume fraction is denoted by ϕ . The effective bulk and shear moduli of the composite are denoted by $K(\phi)$ and $G(\phi)$, respectively.

For an isotropic dispersion of spherical inclusions, Hill’s self-consistent equations [Hill \(1965\)](#) can be written in the present notation as

$$\frac{1}{K(\phi) + \frac{4}{3}G(\phi)} = \frac{\phi}{K_1 + \frac{4}{3}G(\phi)} + \frac{1 - \phi}{K_0 + \frac{4}{3}G(\phi)} \quad (\text{A.1})$$

and

$$\frac{\phi}{G(\phi) - G_0} + \frac{1 - \phi}{G(\phi) - G_1} = \frac{\beta(\phi)}{G(\phi)}. \quad (\text{A.2})$$

The parameter $\beta(\phi)$ is defined through

$$\alpha(\phi) = 3 - 5\beta(\phi), \quad \alpha(\phi) = \frac{K(\phi)}{K(\phi) + \frac{4}{3}G(\phi)}. \quad (\text{A.3})$$

Equivalently, combining the above two equations gives

$$\beta(\phi) = \frac{3 - \alpha(\phi)}{5} = \frac{6(K(\phi) + 2G(\phi))}{5(3K(\phi) + 4G(\phi))}. \quad (\text{A.4})$$

The bulk-modulus equation can be rearranged as

$$K(\phi) = \frac{K_0 - 4\phi A_K(\phi)G(\phi)}{1 + 3\phi A_K(\phi)}, \quad A_K(\phi) = \frac{K_0 - K_1}{3K_1 + 4G(\phi)}. \quad (\text{A.5})$$

Similarly, the shear-modulus equation can be written as

$$G(\phi) = \frac{G_0}{1 + \phi A_G(\phi)}, \quad A_G(\phi) = \frac{G_0 - G_1}{(1 - \beta(\phi))G(\phi) + \beta(\phi)G_1}. \quad (\text{A.6})$$

Equations (A.5) and (A.6) remain implicit because $A_K(\phi)$ and $A_G(\phi)$ depend on the effective moduli.

Next, consider the rigid-inclusion limit,

$$K_1 \rightarrow \infty, \quad G_1 \rightarrow \infty. \quad (\text{A.7})$$

In this limit,

$$A_K(\phi) \rightarrow -\frac{1}{3}, \quad A_G(\phi) \rightarrow -\frac{1}{\beta(\phi)}. \quad (\text{A.8})$$

Substitution into Equations (A.5) and (A.6) gives

$$K(\phi) = \frac{K_0 + \frac{4}{3}\phi G(\phi)}{1 - \phi}, \quad (\text{A.9})$$

and

$$G(\phi) = \frac{G_0}{1 - \phi/\beta(\phi)}. \quad (\text{A.10})$$

Finally, take the nearly incompressible matrix limit, $K_0 \rightarrow \infty$. From Equation (A.9), this implies $K(\phi) \rightarrow \infty$ for $0 \leq \phi < 1$. Substituting this limit into Equation (A.4) gives

$$\beta(\phi) \rightarrow \frac{2}{5}. \quad (\text{A.11})$$

Therefore, Equation (A.10) reduces to

$$G(\phi) = \frac{G_0}{1 - \frac{5}{2}\phi}. \quad (\text{A.12})$$

This is the **Hill** effective shear-modulus relation used in Equation (17). The expression is singular at $\phi = 0.4$, so it is used only for $\phi < 0.4$. In the optimization studies, the maximum magnetic particle volume fraction is $\phi_{\max} = 0.30$, which remains below this singular limit.

Table B.7: Beam benchmark convergence study.

l_c (mm)	Elements	Uniform magnetization		Tip-magnetized	
		$ \mathbf{u} _{\text{avg}}$	Diff. (%)	$ \mathbf{u} _{\text{avg}}$	Diff. (%)
0.15	2,334	7.76	15.4	2.91	29.8
0.10	4,934	9.15	0.2	4.32	4.4
0.08	7,562	9.23	0.6	4.36	5.3
0.06	13,494	9.17	Reference	4.14	Reference

Appendix B. Mesh convergence studies

Mesh convergence studies were performed to verify that the numerical predictions presented throughout this work were not significantly influenced by spatial discretization. For each benchmark problem, progressively refined meshes were analyzed, and the resulting displacement responses were compared against a reference solution obtained using the finest mesh considered. The mesh resolutions adopted in the main text were selected to provide a balance between computational cost and solution accuracy.

Appendix B.1. Beam benchmark

The cantilever beam geometry introduced in [Section 3.1.1](#) was analyzed using characteristic mesh sizes $l_c = 0.15, 0.10, 0.08$, and 0.06 mm, where l_c denotes the target element size prescribed during mesh generation. The domain was discretized using unstructured first-order triangular finite elements.

Convergence was assessed using the average displacement magnitude measured along the right edge of the beam. Two representative cases were considered: the uniformly magnetized beam subjected to the largest applied field, $B^a = 400$ mT, and the tip-magnetized beam shown in [Figure 2\(b\)](#). The resulting displacement predictions and relative differences with respect to the reference mesh are summarized in [Table B.7](#).

For the uniformly magnetized beam, the selected mesh differed from the reference solution by only 0.6%. For the tip-magnetized beam, the corresponding difference was 5.3%. In contrast, the coarsest mesh produced errors of 15.4% and 29.8%, respectively. Although the displacement predictions do not converge monotonically with mesh refinement, the results indicate that the deformation response becomes largely mesh independent for mesh sizes of $l_c \leq 0.10$ mm.

Based on these results, a characteristic mesh size of $l_c = 0.08$ mm was adopted for all subsequent beam evaluation studies. The selected mesh contained approximately 3,780 nodes and 7,562 elements, while the reference mesh contained approximately 6,746 nodes and 13,494 elements. This discretization provided a suitable balance between computational cost and solution accuracy while maintaining a consistent mesh across all constitutive and structure–property model comparisons.

Appendix B.2. Wheel and gripper benchmarks

The wheel and gripper geometries shown in [Figure 5](#) were analyzed using progressively refined meshes generated with Gmsh. The domains were discretized using unstructured first-order triangular finite elements, and the finest mesh for each benchmark was used as the reference solution.

Convergence was assessed using the average displacement magnitude measured along the evaluation boundary associated with each benchmark. For the wheel benchmark, characteristic mesh sizes

Table B.8: Wheel and gripper benchmark convergence study.

Wheel				Gripper			
l_c (mm)	Elements	$ \mathbf{u} _{\text{avg}}$	Diff. (%)	l_c (mm)	Elements	$ \mathbf{u} _{\text{avg}}$	Diff. (%)
0.10	24,717	6.73	15.4	0.10	68,514	17.87	6.0
0.08	36,973	7.16	10.0	0.08	107,268	18.48	2.8
0.06	65,583	8.02	0.8	0.06	190,252	19.00	Reference
0.045	113,483	7.96	Reference	—	—	—	—

of $l_c = 0.10, 0.08, 0.06$, and 0.045 mm were considered. For the gripper benchmark, characteristic mesh sizes of $l_c = 0.10, 0.08$, and 0.06 mm were considered. The resulting displacement predictions and relative differences with respect to the reference meshes are summarized in [Table B.8](#).

For the wheel benchmark, the selected mesh differed from the reference solution by 0.8%, whereas the coarsest mesh produced an error of 15.4%. For the gripper benchmark, the selected mesh differed from the reference solution by 2.8%, while the coarsest mesh produced an error of 6.0%. These results indicate that both benchmark problems exhibit a largely mesh-independent response for the selected discretizations.

Based on this study, characteristic mesh sizes of $l_c = 0.06$ mm and $l_c = 0.08$ mm were adopted for the wheel and gripper benchmarks, respectively. The selected wheel mesh contained approximately 32,460 nodes and 65,583 elements, while the selected gripper mesh contained approximately 53,627 nodes and 107,268 elements. These discretizations provided a suitable balance between computational cost and solution accuracy while maintaining consistent meshes across all constitutive and structure–property model comparisons.

Appendix C. Details of constitutive model selection

The data-processing and simulation details used in [Section 4](#) are provided here.

Appendix C.1. Experimental data extraction and processing

In ([Garcia-Gonzalez et al., 2021](#), Fig. 1), true stress–strain curves are reported for dog-bone specimens with a mixing ratio of 5:1 and four particle volume fractions $\phi \in \{0, 0.05, 0.15, 0.30\}$. The figure was digitized using image-based extraction based on axis calibration followed by curve tracing. Because digitization is sensitive to axis scaling and to missing or noisy points at very small strain, two independent digitization passes were constructed and combined rather than relying on a single extraction. These two digitization passes are referred to as the baseline and refined datasets. The extracted data were cleaned and regularized to obtain true stress–strain curves over a common strain range and strain interval. The two datasets were then averaged to obtain the reference dataset. The extracted curves and resulting modulus estimates are presented in [Figure 9](#).

The shear modulus is estimated using $G = E/3$, corresponding to an incompressible material, where Young’s modulus is obtained using the relation below:

$$E = \frac{d\sigma}{d\varepsilon}.$$

For a fixed $\phi \in \{0, 0.05, 0.15, 0.30\}$, the modulus calculation is restricted to small strain by setting $\varepsilon_{\text{max}} = 0.15$. A direct derivative of the stress–strain curve near zero strain can be sensitive to noise

and to the choice of the first nonzero data point. Therefore, the modulus is estimated by fitting a line through the origin over the interval $(0, \bar{\varepsilon}]$.

For a fixed strain $\bar{\varepsilon}$, let $\{(\varepsilon_i, \sigma_i)\}$ denote all data points with $\varepsilon_i \in (0, \bar{\varepsilon}]$. The line $\sigma(\varepsilon) = E_{\bar{\varepsilon}}^{\text{data}} \varepsilon$ is obtained by minimizing

$$\sum_{i, \varepsilon_i \in (0, \bar{\varepsilon}]} \left(\sigma_i - E_{\bar{\varepsilon}}^{\text{data}} \varepsilon_i \right)^2.$$

The normal equation gives

$$E_{\bar{\varepsilon}}^{\text{data}} = \frac{\sum_{i, \varepsilon_i \in (0, \bar{\varepsilon}]} \varepsilon_i \sigma_i}{\sum_{i, \varepsilon_i \in (0, \bar{\varepsilon}]} \varepsilon_i^2}, \quad G_{\bar{\varepsilon}}^{\text{data}} = \frac{E_{\bar{\varepsilon}}^{\text{data}}}{3} = \frac{1}{3} \frac{\sum_{i, \varepsilon_i \in (0, \bar{\varepsilon}]} \varepsilon_i \sigma_i}{\sum_{i, \varepsilon_i \in (0, \bar{\varepsilon}]} \varepsilon_i^2}. \quad (\text{C.1})$$

Repeating this calculation for each volume fraction gives $E_{\bar{\varepsilon}}^{\text{data}}(\phi)$ and $G_{\bar{\varepsilon}}^{\text{data}}(\phi)$. The shear moduli $G_{\varepsilon_{\text{max}}}^{\text{data}}(\phi)$ with $\varepsilon_{\text{max}} = 0.15$ are shown in [Figure 9](#). For $\phi = 0$, the shear modulus of the host matrix is set to

$$G_0 = G_{\varepsilon_{\text{max}}}^{\text{data}}(0) = 373.7 \text{ kPa}.$$

This value is used as the reference matrix shear modulus in Metric 1, while Metric 2 considers this value together with two nearby values of G_0 to assess whether the selected model remains robust to modest variations in the matrix shear modulus.

Appendix C.2. Metric 1 details

Metric 1 quantifies how well each candidate $G(\phi)$ relation reproduces the data shear modulus $G_{\varepsilon_{\text{max}}}^{\text{data}}(\phi)$. For each model, the mean and standard deviation of [Equation \(31\)](#) are computed over $\phi \in \{0, 0.05, 0.15, 0.30\}$. The results are shown in [Table 2](#) under the column heading ‘‘Metric 1’’. The reference dataset is used to report the error values and to set G_0 . The baseline and refined datasets are included in [Figure 10](#) to show the small variation due to digitization.

Appendix C.3. Uniaxial simulation setup for Metric 2

To test whether the same models reproduce the full uniaxial response, the dog-bone experiments of [Barba et al. \(2020\)](#), [Garcia-Gonzalez \(2019\)](#), [Garcia-Gonzalez et al. \(2021\)](#) are simulated with the finite-element model described in [Section 2](#). [Figure C.19](#) summarizes the problem: an ISO I / ASTM Type I tensile specimen (115 mm $\times$ 19 mm $\times$ 3.2 mm overall; gauge section 27.53 mm $\times$ 6 mm $\times$ 3.2 mm) is meshed with refinement in the gauge ($\ell_c = 1.0$ mm) and coarser discretization in the grips ($\ell_c = 2.5$ mm). The discretization contains 11 976 elements and 3 384 nodes. The left end is fixed, and the right end is subjected to a displacement-controlled stretch in the loading (x) direction with zero applied magnetic field ($\mathbf{B}^a = \mathbf{0}$). Gauge true strain and average Cauchy stress σ_{xx} in the gauge section are recorded during the test. The σ_{xx} field in [Figure C.19](#) shows that the stress is approximately uniform in the gauge region.

Based on the sensitivity studies in [Section 3](#), the strain energy is fixed to the neo-Hookean model **NH2** in [Equation \(15\)](#). The bulk modulus is taken as $K(\phi) = K_0$, consistent with the constitutive formulation in [Section 2.3](#). All seven $G(\phi)$ models in [Table 2](#), including **Default**, are evaluated at $\phi \in \{0, 0.05, 0.15, 0.30\}$. Simulations are run for three calibrations of the matrix modulus,

$$G_0 \in \{373.7, 400, 420\} \text{ kPa},$$

which allows the model comparison to assess sensitivity to modest variations in the matrix shear modulus. The target engineering strain on the overall specimen length is 200%, applied in 58 equal displacement increments, with $\Delta u_x = 4$ mm per step. No magnetic loading is applied, so the test isolates the mechanical $G(\phi)$ model embedded in **NH2**.

Appendix C.4. Metric 2 averaging definitions

For each volume fraction ϕ , each $G(\phi)$ model, and each matrix modulus G_0 , the simulated stress-strain curve is interpolated to the experimental strain values used in the reference dataset from [Section Appendix C.1](#). Points with $|\sigma^{\text{data}}| \leq 10^{-3}$ MPa are excluded to avoid division by values near zero. The relative root-mean-square error is defined in [Equation \(32\)](#).

Averaging over the three G_0 calibrations at fixed ϕ gives

$$\text{rRMSE}(\phi, \text{model}) = \frac{1}{3} \sum_{i=1}^3 \text{rRMSE}(\phi, G_{0,i}, \text{model}), \quad (\text{C.2})$$

with standard deviation $\text{rRMSE}_\sigma(\phi, \text{model})$ computed from the same set $\{\text{rRMSE}(\phi, G_{0,i}, \text{model})\}_{i=1}^3$. This pair summarizes model error at a given ϕ while accounting for uncertainty in G_0 .

Conversely, for fixed G_0 and model, averaging over the four volume fractions gives

$$\text{rRMSE}(G_0, \text{model}) = \frac{1}{4} \sum_{j=1}^4 \text{rRMSE}(\phi_j, G_0, \text{model}), \quad (\text{C.3})$$

with standard deviation $\text{rRMSE}_\sigma(G_0, \text{model})$ over $\{\text{rRMSE}(\phi_j, G_0, \text{model})\}_{j=1}^4$. Finally, the double mean over ϕ and G_0 is

$$\hat{\text{rRMSE}}(\text{model}) = \frac{1}{4} \sum_{j=1}^4 \text{rRMSE}(\phi_j, \text{model}), \quad (\text{C.4})$$

with standard deviation $\hat{\text{rRMSE}}_\sigma(\text{model})$ taken over the same four values $\{\text{rRMSE}(\phi_j, \text{model})\}_{j=1}^4$.

The Metric 2 columns in [Table 2](#) report $\hat{\text{rRMSE}}(\text{model})$ and $\hat{\text{rRMSE}}_\sigma(\text{model})$ for each $G(\phi)$ model. These values are visualized in [Figure 11](#). At $\phi = 0$, all models reduce to the same pure-matrix response, so $\text{rRMSE}(0, \text{model})$ is identical across models and reflects only the G_0 calibration mismatch rather than model form.

Appendix C.5. Additional Metric 2 summary at fixed G_0

[Figure C.20](#) shows $\text{rRMSE}(G_0, \text{model})$ and $\text{rRMSE}_\sigma(G_0, \text{model})$ for the three G_0 calibrations. **Mooney**, **Hill**, and **Guth** form the lowest-error group at every G_0 , while **Default** gives the largest error. The order within the lowest-error group shifts slightly with G_0 : **Hill** is best at 373.7 kPa, while **Mooney** is best at 400 and 420 kPa. This sensitivity is small compared with the gap between these three models and the **Default**, **Kerner**, **LP**, and **LPA** models.

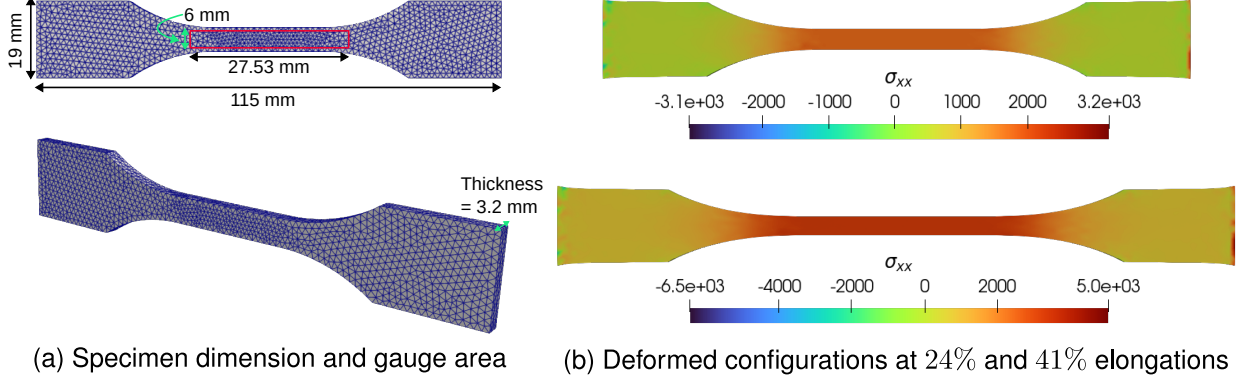


Figure C.19: Uniaxial dog-bone validation setup. **Left:** geometry based on the ISO I / ASTM Type I profile used in Barba et al. (2020), Garcia-Gonzalez (2019), Garcia-Gonzalez et al. (2021); boundary conditions; and finite-element mesh (11 976 elements, 3 384 nodes) with gauge-region refinement. **Right:** Cauchy stress σ_{xx} at two load steps, corresponding to percentage elongations of 24% and 41%. The results correspond to the Mooney model, **NH2** energy, uniform particle volume fraction $\phi = 0.3$, and matrix shear modulus $G_0 = 373.7$ kPa.

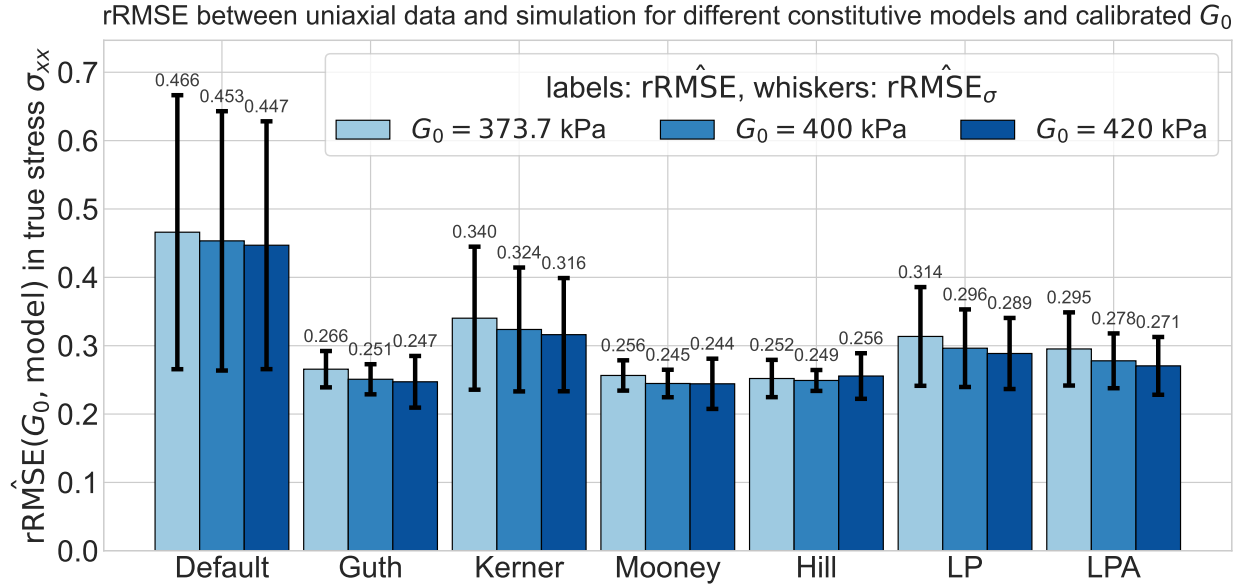


Figure C.20: Metric 2 summary at fixed G_0 : $r\hat{RMSE}(G_0, \text{model})$ from Equation (C.3) and $r\hat{RMSE}_\sigma(G_0, \text{model})$ for $G_0 \in \{373.7, 400, 420\}$ kPa. Bar heights indicate mean values, and error bars indicate standard deviations over the four particle volume fractions.