A compressible isotropic hyperelastic model combining polyconvexity and global true-stress-true-strain monotonicity

Jinchen Xie[a,*], Ali Javili[b], Patrizio Neff[c], Christian Linder[a]

[a] Department of Civil and Environmental Engineering, Stanford University, Stanford, 94305, CA, USA
[b] Department of Mechanical Engineering, Bilkent University, 06800 Ankara, Turkey
[c] Chair for Nonlinear Analysis and Modelling, Faculty of Mathematics, Universität Duisburg–Essen, Thea-Leymann Str. 9 45127 Essen, Germany

Abstract

Formulating of a compressible hyperelastic model that satisfies both polyconvexity and strict monotonicity of the true stress true strain relationship across the entire deformation domain has been a long-standing problem in isotropic nonlinear elasticity. This paper presents, an explicit two-parameter solution to this problem in both two- and three-dimensional cases, with $n \in \{2, 3\}$ being the spatial dimension. The proposed strain energy is objective and isotropic, smooth and finite over the entire $\mathrm{GL}^+(n)$, and it has the rotation group $\mathrm{SO}(n)$ as its unique set of natural states. By constructing a jointly convex extension of the deformation gradient and its determinant, we rigorously prove that this energy is polyconvex and rank-one convex. The two independent parameters in the model correspond precisely to given (positive) shear modulus and (positive) bulk modulus at the natural state. By directly evaluating the complete fourth-order logarithmic stress tangent, we show that its self-adjoint symmetric part is strictly positive definite everywhere. We further demonstrate that the left stretch and the Cauchy stress form a globally smooth diffeomorphism. This paper provides a constructive solution to the unification of polyconvexity, true stress true strain monotonicity (TSTS-M^{++}), and global stress reversibility relevant to Truesdell's Hauptproblem.

Keywords: Compressible isotropic hyperelasticity, Polyconvexity, True stress-true strain monotonicity, Diffeomorphism, Truesdell's Hauptproblem

1. Introduction

Truesdell (Truesdell, 1956; Truesdell and Noll, 2004) formulated his Hauptproblem of finite elasticity as follows "What is the class of functions, which can serve as strain energy densities, for perfectly elastic materials?" This question remains a foundational criterion in material modeling of ideal isotropic nonlinear elasticity (Ball, 2002). In the past decades, two families of *variational*, or rather polyconvexity-driven, and *constitutive*, or rather monotonicity-driven, admissibility requirements have been developed.

The first family emerges from *variational restrictions* in that the strain energy must remain sufficiently convex for the equilibrium to be well posed. The notion of Polyconvexity in this context dates back to the seminal work of Ball (1977), and it requires the energy to be a convex function of the minors of the deformation gradient. Polyconvexity together with coercivity can guarantee the existence of energy minimizers, and it implies quasiconvexity in the sense of Morrey (1952), which in turn implies rank-one convexity, or rather Legendre–Hadamard ellipticity (Ciarlet, 1988;

[*] Corresponding author.
Email address: jinchenx@stanford.edu (Jinchen Xie)

Šilhavý, 1997; Dacorogna, 2008). Thus, polyconvexity has become the underpinning guideline of constitutive modeling well beyond its variational isotropic origin (Schröder and Neff, 2003; Hartmann and Neff, 2003), and has been introduced to neural network constitutive models that enforce it by construction (Klein et al., 2022).

The second family driven by *constitutive inequalities* on the other hand requires that the stress response must not soften in an unphysical manner at any finite strain. Constitutive inequalities demanding monotone stress-strain response in appropriate conjugate pairs date back to Hill (1968, 1970) and Ogden (1970). For finite strain elasticity, various strain measures have been introduced. The logarithmic strain measure (Hencky, 1928; Richter, 1948) stands out as a natural strain measure of isotropic elasticity (Anand, 1979; Neff et al., 2016) in that its trace yields the true volumetric strain, while its deviatoric part quantifies the true distortion. For the pair of Cauchy (true) stress and logarithmic (true) strain, the monotonicity condition, referred to as true-stress-true-strain monotonicity TSTS-M^{++}, was introduced by Leblond (1992), employed also in Jog and Patil (2013), and has been investigated extensively and systematically by Neff et al. (Neff et al., 2015; d'Agostino et al., 2025). Furthermore, TSTS-M^{++} is significant also because it is essentially equivalent to the positive incremental stress power for positive corotational rate, and it guarantees positive incremental Cauchy stress moduli in all homogeneous deformation tests with fixed principal axes (Neff et al., 2025c), see also (Xiao et al., 1997; Neff et al., 2025a,b).

Although both families are inspired by physically meaningful requirements of *variational restrictions* and *constitutive inequalities*, they do not necessarily imply one another. *Polyconvex energies* may exhibit a non-monotone Cauchy stress already in uniaxial tension with traction-free lateral faces (Wollner et al., 2026a). *Monotonicity satisfying energies* constructed for the logarithmic strain, too, fail to be rank-one convex at large distortions (Neff et al., 2015; Ghiba et al., 2015; Martin et al., 2018). The incompressible setting seems to be more forgiving, in that rank-one convexity even implies polyconvexity (Ghiba et al., 2018), in the planar case. See also the very recent contribution (Wollner et al., 2026b) wherein the interplay of both requirements under the incompressibility constraint has been analyzed in detail and enforced with application to neural network constitutive models. In the compressible case, however, fulfilling both requirements have so far proven more challenging (Wollner et al., 2026a). This state of affairs has come to a prized question posed publicly by Neff (2024) as follows. *Find an explicit compressible isotropic energy, defined and smooth on all of* $\mathrm{GL}^+(n)$*, such that it (i) is polyconvex, (ii) linearizes to isotropic elasticity with arbitrarily prescribed* $\mu, \kappa > 0$*, and (iii) satisfies TSTS-M*$^{++}$ *at every finite strain, and renders the Cauchy stress a bijective function of the left stretch tensor; or show that no such energy exists.* In this paper, we propose an explicit two-parameter solution to this problem as

$$W_n(\mathbf{F}) = 4\kappa\,[\exp(\Phi_n(\mathbf{F})) - 1] \quad \text{with} \quad \Phi_n(\mathbf{F}) = \frac{\mu}{8\kappa}\,[J^{-2/n}\,\|\mathbf{F}\|^2 - n] + \frac{1}{32}\,[J^2 + J^{-2} - 2],$$

with $n \in \{2, 3\}$ being the spatial dimension, and $J = \det \mathbf{F}$, and arbitrary $\mu, \kappa > 0$. The construction combines the isochoric invariant $J^{-2/n}\,\|\mathbf{F}\|^2$, with its polyconvexity properties well established (Hartmann and Neff, 2003), under an exponential envelope. The two constitutive coefficients μ and κ are exactly the shear and bulk moduli at the natural state, which precisely complete the monotonicity proof. We prove that W_n is smooth and finite on all of $\mathrm{GL}^+(n)$, objective and isotropic. Our proposed energy density is positive except on its natural state $\mathrm{SO}(n)$, and it is polyconvex, hence rank-one convex. By an exact quadratic form identity for the logarithmic stress tangent, we establish TSTS-M^{++} at every state[1]. Variational restrictions and constitutive monotonicity under arbitrary finite

[1]Based on the monotonicity, supercoercivity and properness of the stress mapping, we further show that both $\log \mathbf{V} \mapsto \widehat{\boldsymbol{\sigma}}$ and $\mathbf{V} \mapsto \boldsymbol{\sigma}$ are global C^∞ diffeomorphisms.

deformation are achieved simultaneously within our model. Beyond the constitutive question itself, we expect such formulations paves the way to better understand rotation (Schek et al., 2026) and higher-gradient fields (Javili et al., 2013) under large deformations, and generalized continua (Xie et al., 2025, 2026).

The remainder of this paper is organized as follows. Section 2 introduces the notation and states the constitutive requirements. Section 3 constructs the model. Section 4 establishes smoothness, invariance and the natural states, and Section 5 proves polyconvexity by a jointly convex extension in the minor variables. Sections 6 and 7 derive the Piola and Cauchy stresses together with the exact logarithmic representation and the linearization at the natural state. Section 8 proves the full true-stress-true-strain monotonicity, Section 9 the properness and global invertibility of the stress mapping. Section 10 concludes this contribution and provides further outlook.

2. Notation and constitutive requirements

This contribution deals with both the two-dimensional and three-dimensional cases. Therefore, we introduce $n \in \{2, 3\}$ as the spatial dimension throughout the following discussion. We first establish the standard notations for the relevant kinematic quantities and tensor spaces, in particular the deformation gradient, the rotation group, and the spaces of symmetric and symmetric positive-definite tensors as follows:

$$\begin{aligned}
&\mathrm{GL}^+(n) = \{\mathbf{F} \in \mathbb{R}^{n\times n} : \det \mathbf{F} > 0\},\\
&\mathrm{SO}(n) = \{\mathbf{R} \in \mathbb{R}^{n\times n} : \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I},\ \det \mathbf{R} = 1\},\\
&\mathrm{Sym}(n) = \{\mathbf{X} \in \mathbb{R}^{n\times n} : \mathbf{X}^{\mathrm{T}} = \mathbf{X}\},\\
&\mathrm{Sym}^{++}(n) = \{\mathbf{X} \in \mathrm{Sym}(n) : \mathbf{X} \text{ is positive definite}\}.
\end{aligned} \tag{1}$$

All tensor spaces are equipped with the Frobenius inner product and its induced norm. Any second-order tensor can be uniquely decomposed into a deviatoric part and a spherical part. The deviatoric part of a symmetric tensor describes shape changes that do not involve changes in volume, while its trace quantifies spherical or volumetric changes. The following notations are used throughout the paper:

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}), \quad \|\mathbf{A}\|^2 = \mathbf{A} : \mathbf{A}, \quad \mathrm{dev}\,\mathbf{A} = \mathbf{A} - \frac{\mathrm{tr}\,\mathbf{A}}{n}\mathbf{I}. \tag{2}$$

For any $\mathbf{F} \in \mathrm{GL}^+(n)$, let J denote the local volume ratio, and $\mathbf{B}$ denote the left Cauchy–Green tensor. To separate changes in shape from changes in volume, we further define the isochoric tensor $\overline{\mathbf{B}}$ as

$$J = \det \mathbf{F}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \overline{\mathbf{B}} = J^{-2/n}\mathbf{B}. \tag{3}$$

Since $\det \mathbf{B} = J^2$, the normalized tensor satisfies $\det \overline{\mathbf{B}} = 1$ since

$$\det \mathbf{B} = J^2, \quad \det \overline{\mathbf{B}} = \det(J^{-2/n}\mathbf{B}) = J^{-2}\det \mathbf{B} = 1. \tag{4}$$

Therefore, $\overline{\mathbf{B}}$ no longer contains local volume information, but only records the distortion level of the material element. To eliminate rigid-body rotation and directly characterize local elongation, the left polar decomposition of the deformation gradient is given by

$$\mathbf{F} = \mathbf{V}\mathbf{R}, \quad \mathbf{V} \in \mathrm{Sym}^{++}(n), \quad \mathbf{R} \in \mathrm{SO}(n). \tag{5}$$

In particular, $\mathbf{R}$ describes only local rigid-body rotation, while $\mathbf{V}$ contains all strain information. The left Cauchy–Green tensor can be expressed as $\mathbf{B} = \mathbf{V}^2$, and the local volume ratio is given by $J = \det \mathbf{V}$. The objective isotropic strain energy can therefore be expressed entirely in terms of $\mathbf{V}$. Here, we denote $\mathbf{X} = \log \mathbf{V}$ as the "true strain", which is a symmetric tensor that describes the logarithmic elongation of the material element, with its trace satisfying $\operatorname{tr} \mathbf{X} = \log J$, and thus precisely furnishing the true volumetric strain. The deviatoric component of $\mathbf{X}$ describes the true strain with constant volume under finite deformation. Then, the corresponding Cauchy stress in terms of the true strain reads

$$\mathbf{X} = \log \mathbf{V} \in \operatorname{Sym}(n), \quad \widehat{\boldsymbol{\sigma}}(\mathbf{X}) = \boldsymbol{\sigma}(\mathrm{e}^{\mathbf{X}}). \tag{6}$$

The positive shear and bulk moduli at the natural state ensure stability under small strains. They cannot, however, rule out the possibility of incremental softening in the material under large deformations. This manuscript targets a stronger global condition. That is, *at any finite strain state* $\mathbf{X}$*, every nonzero true strain increment* $\mathbf{H}$ *must produce a positive increment in stress energy*, such that

$$D\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}] : \mathbf{H} > 0, \quad \forall\, \mathbf{X} \in \operatorname{Sym}(n), \quad \forall\, \mathbf{0} \neq \mathbf{H} \in \operatorname{Sym}(n). \tag{7}$$

Bear in mind that the symbol $D\widehat{\boldsymbol{\sigma}}(\mathbf{X})$ is a fourth-order linear operator, but the Cauchy stress with respect to logarithmic strain is generally not self-adjoint. We define its adjoint operator $(D\widehat{\boldsymbol{\sigma}}(\mathbf{X}))^*$ such that

$$(D\widehat{\boldsymbol{\sigma}}(\mathbf{X}))^*[\mathbf{H}] : \mathbf{K} = \mathbf{H} : D\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{K}], \quad \forall\, \mathbf{H}, \mathbf{K} \in \operatorname{Sym}(n), \tag{8}$$

which readily leads to

$$\begin{aligned}
[\operatorname{sym} D\widehat{\boldsymbol{\sigma}}(\mathbf{X})][\mathbf{H}] : \mathbf{H} &= \tfrac{1}{2}\,[D\widehat{\boldsymbol{\sigma}}(\mathbf{X}) + (D\widehat{\boldsymbol{\sigma}}(\mathbf{X}))^*][\mathbf{H}] : \mathbf{H} \\
&= \tfrac{1}{2}\,[D\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}] : \mathbf{H} + (D\widehat{\boldsymbol{\sigma}}(\mathbf{X}))^*[\mathbf{H}] : \mathbf{H}] \\
&= \tfrac{1}{2}\,[D\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}] : \mathbf{H} + \mathbf{H} : D\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}]] \\
&= D\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}] : \mathbf{H}.
\end{aligned} \tag{9}$$

The incremental work is determined solely by the self-adjoint symmetric part of the tangent operator. Thus, Eq. (7) is equivalent to the self-adjoint symmetric part of the fourth-order tangent operator being strictly positive definite, or, equivalently,

$$\operatorname{sym} D_{\mathbf{X}}\widehat{\boldsymbol{\sigma}}(\mathbf{X}) \in \operatorname{Sym}_4^{++}(\frac{n\,[n+1]}{2}), \quad \forall\, \mathbf{X} \in \operatorname{Sym}(n). \tag{10}$$

The subscript '4' in the symbol $\operatorname{Sym}_4^{++}$ denotes a fourth-order operator, while $n\,[n+1]/2$ in parentheses denotes the finite-dimensional symmetric tensor space on which it actually acts. This space is three-dimensional in 2D and six-dimensional in 3D. Using the relation $\mathbf{X} = \log \mathbf{V}$, this condition is can be written as

$$\operatorname{sym} D_{\log \mathbf{V}}\widehat{\boldsymbol{\sigma}}(\log \mathbf{V}) \in \operatorname{Sym}_4^{++}(\frac{n\,[n+1]}{2}), \quad \forall\, \mathbf{V} \in \operatorname{Sym}^{++}(n). \tag{11}$$

3. Construction of the model

The underlying assumption here is that a compressible solid stores two distinct types of energy: one resulting from changes in the shape of the material, and the other resulting from the change in its local volume. The former should exclude rigid-body rotation and pure volume changes, while the latter should rely solely on the volume ratio J. Based on this physical distinction, we use $[\operatorname{tr}\overline{\mathbf{B}} - n]$ to measure distortion, and $[J^2 + J^{-2} - 2]$ to measure volumetric change. The first quantity reaches its minimum value in the undistorted state. The second quantity is zero when $J = 1$ and increases simultaneously at both ends of the volume collapse $J \to 0$ and infinite expansion $J \to +\infty$. Upon this basis, we propose the following form of energy:

$$\boxed{W_n(\mathbf{F}) = 4\kappa\,[\exp(\Phi_n(\mathbf{F})) - 1], \quad \Phi_n(\mathbf{F}) = \frac{\mu}{8\kappa}\,[J^{-2/n}\,\|\mathbf{F}\|^2 - n] + \frac{1}{32}\,[J^2 + J^{-2} - 2].} \tag{12}$$

The outer exponential function introduces a positive quadratic term in the true stress tangent, thereby preventing any adverse coupling that might arise between the distortional response and the volumetric response. The coefficients in the model are selected precisely to ensure this stable structure. By substituting $n = 2$ and $n = 3$ into the unified expression, respectively, we derive the two-dimensional model and the three-dimensional model as

$$\boxed{\begin{aligned} W_2(\mathbf{F}) &= 4\kappa\,[\exp(\frac{\mu}{8\kappa}\,[J^{-1}\,\|\mathbf{F}\|^2 - 2] + \frac{1}{32}\,[J^2 + J^{-2} - 2]) - 1], \\ W_3(\mathbf{F}) &= 4\kappa\,[\exp(\frac{\mu}{8\kappa}\,[J^{-2/3}\,\|\mathbf{F}\|^2 - 3] + \frac{1}{32}\,[J^2 + J^{-2} - 2]) - 1]. \end{aligned}} \tag{13}$$

The parameters $\{\mu, \kappa\}$ can take any positive values independently of one another. Shortly, we show that the two parameters correspond precisely to the infinitesimal shear modulus and bulk modulus at the natural state, respectively. Therefore, the two independent moduli required for isotropic linear elastic materials are retained. We now state the following main theorem:

Theorem 1 (Main result). *Let $n \in \{2, 3\}$, and let $\mu > 0$ and $\kappa > 0$ be arbitrary. For every $\mathbf{F} \in \mathrm{GL}^+(n)$, define*

$$W_n(\mathbf{F}) = 4\kappa\,[\exp(\frac{\mu}{8\kappa}\,[J^{-2/n}\,\|\mathbf{F}\|^2 - n] + \frac{1}{32}\,[J^2 + J^{-2} - 2]) - 1], \quad J = \det\mathbf{F}.$$

Denoting the corresponding Cauchy stress by $\boldsymbol{\sigma}$ and introducing $\widehat{\boldsymbol{\sigma}}(\mathbf{X}) = \boldsymbol{\sigma}(\mathrm{e}^{\mathbf{X}})$ for $\mathbf{X} \in \mathrm{Sym}(n)$, the following properties hold:

(i) $W_n \in C^\infty(\mathrm{GL}^+(n))$ *and* $W_n(\mathbf{F}) < +\infty$ *for every* $\mathbf{F} \in \mathrm{GL}^+(n)$.

(ii) W_n *is objective and isotropic.*

(iii) $W_n(\mathbf{F}) \geqslant 0$, *and* $W_n(\mathbf{F}) = 0$ *if and only if* $\mathbf{F} \in \mathrm{SO}(n)$. *Every* $\mathbf{R} \in \mathrm{SO}(n)$ *is stress free.*

(iv) The standard extended-valued extension of W_n is polyconvex and hence rank-one convex.

(v) At the natural state, the infinitesimal shear modulus and bulk modulus are exactly μ and κ:

$$D\widehat{\boldsymbol{\sigma}}(\mathbf{0})\,[\mathbf{H}] = 2\mu\,\operatorname{dev}\mathbf{H} + \kappa\,[\operatorname{tr}\mathbf{H}]\,\mathbf{I}.$$

(vi) The full logarithmic Cauchy-stress tangent satisfies

$$D\widehat{\boldsymbol{\sigma}}(\mathbf{X})\,[\mathbf{H}] : \mathbf{H} > 0, \quad \forall\,\mathbf{X} \in \mathrm{Sym}(n), \quad \forall\,\mathbf{0} \neq \mathbf{H} \in \mathrm{Sym}(n).$$

(vii) The mappings

$$\widehat{\boldsymbol{\sigma}} : \mathrm{Sym}\,(n) \longrightarrow \mathrm{Sym}\,(n), \quad \boldsymbol{\sigma} : \mathrm{Sym}^{++}\,(n) \longrightarrow \mathrm{Sym}\,(n)$$

are global C^∞-diffeomorphisms.

The main theorem summarizes the properties of the model at three different levels. (i) Polyconvexity provides the variational structure required for the energy minimization problem. (ii) The complete TSTS-M^{++} condition guarantees a strictly positive incremental Cauchy-stress/logarithmic-strain pairing at every finite strain state. (iii) The global diffeomorphism property of the stress map ensures that every symmetric Cauchy stress corresponds to a unique strain state. Next, we prove the above properties sequentially.

4. Basic properties and natural states

We first verify that this energy function is defined throughout the entire physical deformation domain. For any given $\mathbf{F} \in \mathrm{GL}^+\,(n)$, J is a strictly positive finite real number, and $\|\mathbf{F}\|$ is also a finite quantity, more precisely,

$$0 < J < +\infty, \quad \|\mathbf{F}\|^2 = \sum_{i,j=1}^{n} F_{ij}^2 < +\infty, \quad \forall\, \mathbf{F} \in \mathrm{GL}^+\,(n). \tag{14}$$

Accordingly, the positive and negative powers, as well as the exponents, in the energy function are finite, which implies that

$$W_n\,(\mathbf{F}) = 4\kappa\,[\exp\,(\frac{\mu}{8\kappa}\,[J^{-2/n}\,\|\mathbf{F}\|^2 - n] + \frac{1}{32}\,[J^2 + J^{-2} - 2]) - 1] < +\infty. \tag{15}$$

In addition, it is necessary to verify that the model is sufficiently smooth with respect to the deformation gradient. Specifically, the norm and determinant of the deformation gradient, as well as the power functions on $J > 0$, are all smooth functions, satisfying

$$\mathbf{F} \mapsto \|\mathbf{F}\|^2 \in C^\infty\,(\mathbb{R}^{n\times n}), \quad \mathbf{F} \mapsto J = \det \mathbf{F} \in C^\infty\,(\mathbb{R}^{n\times n}), \quad J \mapsto \{J^{-2/n}, J^2, J^{-2}\} \subset C^\infty\,((0, +\infty)). \tag{16}$$

Thus, the smoothness of the composite function shows that

$$W_n\,(\mathbf{F}) = 4\kappa\,[\exp\,(\frac{\mu}{8\kappa}\,[J^{-2/n}\,\|\mathbf{F}\|^2 - n] + \frac{1}{32}\,[J^2 + J^{-2} - 2]) - 1] \in C^\infty\,(\mathrm{GL}^+\,(n)). \tag{17}$$

Below, we demonstrate that energy is independent of any rotation of the observer and the material's orientation. The Frobenius norm remains invariant under left-right orthogonal transformations, more specifically,

$$\begin{aligned} \|\mathbf{R}_1\mathbf{F}\mathbf{R}_2\|^2 &= \mathrm{tr}\,([\mathbf{R}_1\mathbf{F}\mathbf{R}_2]^{\mathrm{T}}[\mathbf{R}_1\mathbf{F}\mathbf{R}_2]) \\ &= \mathrm{tr}\,(\mathbf{R}_2^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{R}_1^{\mathrm{T}}\mathbf{R}_1\mathbf{F}\mathbf{R}_2) \\ &= \mathrm{tr}\,(\mathbf{R}_2^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{R}_2) \\ &= \mathrm{tr}\,(\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{R}_2\mathbf{R}_2^{\mathrm{T}}) \\ &= \mathrm{tr}\,(\mathbf{F}^{\mathrm{T}}\mathbf{F}) = \|\mathbf{F}\|^2. \end{aligned} \tag{18}$$

The determinant of a proper rotation tensor is equal to one, so rotation does not alter the local volume ratio, namely, $\det(\mathbf{R}_1\mathbf{F}\mathbf{R}_2) = \det\mathbf{F}$. The energy depends only on the two rotation invariants mentioned above. Left-rotation invariance ensures objectivity, while right-rotation invariance ensures isotropy. This yields

$$W_n(\mathbf{R}_1\mathbf{F}\mathbf{R}_2) = W_n(\mathbf{F}) \quad \forall\, \mathbf{R}_1, \mathbf{R}_2 \in \mathrm{SO}(n). \tag{19}$$

Next, we identify the natural state. Let the positive eigenvalues of $\overline{\mathbf{B}}$ be $\{b_1, \ldots, b_n\}$. Since its determinant is equal to one, the product of these eigenvalues is also equal to one, which implies that

$$\det\overline{\mathbf{B}} = 1 \quad \Longleftrightarrow \quad \prod_{i=1}^{n} \overline{b}_i = 1. \tag{20}$$

The arithmetic-geometric mean inequality provides

$$\mathrm{tr}\,\overline{\mathbf{B}} = \sum_{i=1}^{n} \overline{b}_i \geqslant n\left(\prod_{i=1}^{n} \overline{b}_i\right)^{1/n} = n, \quad \mathrm{tr}\,\overline{\mathbf{B}} = n \quad \Longleftrightarrow \quad \overline{\mathbf{B}} = \mathbf{I}. \tag{21}$$

The equality condition requires that all isochoric principal stretches be identical, while the determinant constraint forces them all to be equal to one. The distortional energy therefore vanishes only when $\overline{\mathbf{B}} = \mathbf{I}$. Furthermore, the volumetric term satisfies

$$J^2 + J^{-2} - 2 = [J - J^{-1}]^2 \geqslant 0, \quad J^2 + J^{-2} - 2 = 0 \quad \Longleftrightarrow \quad J = 1. \tag{22}$$

Clearly, it is non-negative on both the compression and expansion sides, and is zero only when $J = 1$. Therefore, zero energy requires that both the distortional part and the volumetric part be zero. This means that the material element has neither changed its shape nor its volume, and thus

$$\Phi_n(\mathbf{F}) = 0 \quad \Longleftrightarrow \quad \overline{\mathbf{B}} = \mathbf{I} \text{ and } J = 1 \quad \Longleftrightarrow \quad \mathbf{B} = \mathbf{I} \quad \Longleftrightarrow \quad \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{I}. \tag{23}$$

For orientation-preserving deformations, $\mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{I}$ is equivalent to $\mathbf{F} \in \mathrm{SO}(n)$. Consequently, pure rotations are the only zero-energy solutions, and any other physical deformation stores positive energy:

$$\Phi_n(\mathbf{F}) \geqslant 0, \quad W_n(\mathbf{F}) \geqslant 0, \quad W_n(\mathbf{F}) = 0 \quad \Longleftrightarrow \quad \mathbf{F} \in \mathrm{SO}(n). \tag{24}$$

A pure rotational state is a global local minimum of the smooth energy on the open set $\mathrm{GL}^+(n)$, and its first-order variation must vanish. As a result, every $\mathbf{R} \in \mathrm{SO}(n)$ is a stress-free natural state, that is

$$D_{\mathbf{F}}\, W_n(\mathbf{R}) = \mathbf{0}, \quad \forall\, \mathbf{R} \in \mathrm{SO}(n). \tag{25}$$

5. Polyconvexity

Polyconvexity is typically defined using extended values in the full tensor space. To exclude non-physical states $J \leqslant 0$, the energy is extended as

$$\overline{W}_n(\mathbf{F}) = \begin{cases} W_n(\mathbf{F}), & \det \mathbf{F} > 0, \\ +\infty, & \det \mathbf{F} \leqslant 0. \end{cases} \tag{26}$$

The value $+\infty$ does not imply the existence of strain constraints within the positive deformation domain, but merely restricts the valid domain of convex analysis to deformations that preserve orientation. To simplify the derivation, we introduce dimensionless parameters $\{p, c, \alpha\}$ as

$$p = \frac{2}{n}, \quad c = \frac{\mu}{8\kappa}, \quad \alpha = \frac{\kappa}{4\mu}. \tag{27}$$

For $n \in \{2, 3\}$, we have $0 < p \leqslant 1$. Furthermore, the choice of parameters satisfies the key identity $\alpha c = 1/32$. This identity will be used later to derive the quadratic formula for the tangent of the true stress. These conditions are

$$p \in \{1, \frac{2}{3}\}, \quad 0 < p \leqslant 1, \quad c > 0, \quad \alpha > 0, \quad \alpha c = \frac{1}{32}. \tag{28}$$

We now define the function $f(\mathbf{A}, \delta)$ as

$$f(\mathbf{A}, \delta) = \delta^{-p} \|\mathbf{A}\|^2, \quad \delta > 0, \tag{29}$$

which is central to the proof of polyconvexity. Consider the second-order directional derivatives of $f(\mathbf{A}, \delta)$ with respect to an arbitrary joint perturbation $(\mathbf{G}, \eta)$, namely

$$\mathbf{A}(\varepsilon) = \mathbf{A} + \varepsilon\mathbf{G}, \quad \delta(\varepsilon) = \delta + \varepsilon\eta, \quad \forall (\mathbf{G}, \eta) \in \mathbb{R}^{n\times n} \times \mathbb{R}. \tag{30}$$

If the quadratic form $D^2 f(\mathbf{A}, \delta)[(\mathbf{G}, \eta), (\mathbf{G}, \eta)]$ is nonnegative everywhere, then $f(\mathbf{A}, \delta)$ is jointly convex. The first-order directional derivative of $f(\mathbf{A}, \delta)$ at $\varepsilon = 0$ is

$$D f(\mathbf{A}, \delta)[\mathbf{G}, \eta] = D(\delta^{-p})[\eta] \|\mathbf{A}\|^2 + \delta^{-p} D(\|\mathbf{A}\|^2)[\mathbf{G}] = -p\delta^{-p-1}\|\mathbf{A}\|^2\eta + 2\delta^{-p}\mathbf{A} : \mathbf{G}. \tag{31}$$

Taking the derivative of the first and second terms again, we arrive at

$$\begin{aligned} &D[-p\delta^{-p-1} \|\mathbf{A}\|^2\eta][\mathbf{G}, \eta] = p[p+1]\delta^{-p-2}\|\mathbf{A}\|^2\eta^2 - 2p\delta^{-p-1}[\mathbf{A} : \mathbf{G}]\eta, \\ &D[2\delta^{-p}\mathbf{A} : \mathbf{G}][\mathbf{G}, \eta] = -2p\delta^{-p-1}[\mathbf{A} : \mathbf{G}]\eta + 2\delta^{-p}\|\mathbf{G}\|^2. \end{aligned} \tag{32}$$

Adding the two expressions above gives

$$D^2 f(\mathbf{A}, \delta)[(\mathbf{G}, \eta), (\mathbf{G}, \eta)] = 2\delta^{-p}\|\mathbf{G}\|^2 - 4p\delta^{-p-1}[\mathbf{A} : \mathbf{G}]\eta + p[p+1]\delta^{-p-2}\|\mathbf{A}\|^2\eta^2. \tag{33}$$

Now, consider the following complete square representation

$$2\delta^{-p}\,\|\mathbf{G}-p\frac{\eta}{\delta}\mathbf{A}\|^2 = 2\delta^{-p}\,\|\mathbf{G}\|^2 - 4p\delta^{-p-1}\,[\mathbf{A}:\mathbf{G}]\,\eta + 2p^2\delta^{-p-2}\,\|\mathbf{A}\|^2\eta^2, \tag{34}$$

from which the exact identity is derived as

$$D^2 f\,(\mathbf{A},\delta)\,[(\mathbf{G},\eta),(\mathbf{G},\eta)] = 2\delta^{-p}\,\|\mathbf{G}-p\frac{\eta}{\delta}\mathbf{A}\|^2 + p\,[1-p]\,\delta^{-p-2}\,\|\mathbf{A}\|^2\eta^2 \geqslant 0. \tag{35}$$

Hence, $f\,(\mathbf{A},\delta)$ is jointly convex with respect to $(\mathbf{A},\delta)$ on the convex set $\mathbb{R}^{n\times n}\times(0,+\infty)$. When $n=2$, $p=1$, and the second term in the above expression vanishes. However, the remaining perfect square term is still sufficient to guarantee convexity. To proceed, we define the function $h\,(\delta)$ and take its second derivative, which is

$$h\,(\delta) = \alpha\,[\delta^2+\delta^{-2}-2], \quad h'\,(\delta) = 2\alpha\,[\delta-\delta^{-3}], \quad h''\,(\delta) = 2\alpha + 6\alpha\delta^{-4} > 0. \tag{36}$$

Accordingly, $h\,(\delta)$ is strictly convex on $(0,+\infty)$. Then, $\mathcal{Q}_n\,(\mathbf{A},\delta)$ is defined as the sum of a jointly convex isochoric term and a strictly convex volumetric term, and is therefore jointly convex with respect to $(\mathbf{A},\delta)$, namely

$$\mathcal{Q}_n\,(\mathbf{A},\delta) = \delta^{-p}\,\|\mathbf{A}\|^2 - n + \alpha\,[\delta^2+\delta^{-2}-2]. \tag{37}$$

We further define $g\,(r)$ and compute its first- and second-order derivatives as

$$g\,(r) = \frac{\mu}{2c}\,[\mathrm{e}^{cr}-1], \quad g'\,(r) = \frac{\mu}{2}\mathrm{e}^{cr} > 0, \quad g''\,(r) = \frac{\mu c}{2}\mathrm{e}^{cr} > 0. \tag{38}$$

Thus, $g\,(r)$ is a convex and strictly monotonically increasing scalar function, and the composite function $g\circ\mathcal{Q}_n$ remains a convex function. More specifically, by the second-order chain rule, it follows that

$$\begin{aligned} D^2(g\circ\mathcal{Q}_n)\,(\mathbf{A},\delta)\,[(\mathbf{G},\eta),(\mathbf{G},\eta)] &= g''\,(\mathcal{Q}_n)\,(D\,\mathcal{Q}_n\,(\mathbf{A},\delta)\,[\mathbf{G},\eta])^2 \\ &\quad + g'\,(\mathcal{Q}_n)\,D^2\mathcal{Q}_n\,(\mathbf{A},\delta)\,[(\mathbf{G},\eta),(\mathbf{G},\eta)] \geqslant 0. \end{aligned} \tag{39}$$

Finally, we define the function $\mathcal{P}_n$ on the complete minor variables by

$$\mathcal{P}_n\,(\mathbf{A},\mathbf{C},\delta) = \begin{cases} \dfrac{\mu}{2c}\,[\exp\,(c\mathcal{Q}_n\,(\mathbf{A},\delta))-1], & \delta>0, \\ +\infty, & \delta\leqslant 0, \end{cases} \qquad \operatorname{dom}\mathcal{P}_n = \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\times(0,+\infty). \tag{40}$$

This expression is independent of the variable $\mathbf{C}$, and a function that is independent of a given independent variable is, by definition, convex with respect to that independent variable. In a three-dimensional polyconvex representation, the second variable corresponds to $\operatorname{cof}\mathbf{F}$, but this model does not need to explicitly depend on it. We also need to examine the limiting behavior near $\delta=0$. As $\delta\to 0^+$, the term $\alpha\delta^{-2}$ drives both $\mathcal{Q}_n$ and $\mathcal{P}_n$ to $+\infty$, that is

$$\mathcal{Q}_n\,(\mathbf{A},\delta) \geqslant \alpha\delta^{-2} - n - 2\alpha, \quad \lim_{\delta\to 0^+}\mathcal{P}_n\,(\mathbf{A},\mathbf{C},\delta) = +\infty. \tag{41}$$

Hence, assigning the value $+\infty$ at $\delta\leqslant 0$ is consistent with the limit behavior of the original function, resulting in a standard lower-semi-continuous convex extension. Now let us restore the abstract variables as $\mathbf{A}\equiv\mathbf{F}$ and $\delta\equiv\det\mathbf{F}$.

From the definition of the parameters, cQ_n exactly reproduces $\Phi_n(\mathbf{F})$ in the model, namely

$$\begin{aligned} cQ_n(\mathbf{F}, \det \mathbf{F}) &= c\,[J^{-p}\,\|\mathbf{F}\|^2 - n + \alpha\,[J^2 + J^{-2} - 2]] \\ &= \frac{\mu}{8\kappa}\,[J^{-2/n}\,\|\mathbf{F}\|^2 - n] + \frac{1}{32}\,[J^2 + J^{-2} - 2] = \Phi_n(\mathbf{F}). \end{aligned} \tag{42}$$

Therefore, the extended energy can be expressed as

$$\overline{W}_n(\mathbf{F}) = \mathcal{P}_n(\mathbf{F}, \operatorname{cof}\mathbf{F}, \det\mathbf{F}). \tag{43}$$

Since $\mathcal{P}_n$ is constructed as a convex function, this naturally establishes the polyconvexity of W_n. As a direct consequence, the present hyperelastic formulation satisfies the condition of rank-one convexity.

6. Piola stress and Cauchy stress

Having established the variational structure of the energy via polyconvexity, we now focus on the material's stress response. To obtain an explicit expression for the Cauchy stress in terms of finite strain, we recall

$$Q(\mathbf{F}) = J^{-p}\,\|\mathbf{F}\|^2 - n + \alpha\,[J^2 + J^{-2} - 2], \quad p = \frac{2}{n}. \tag{44}$$

For any increment $\mathbf{H} \in \mathbb{R}^{n\times n}$, the derivative of the isochoric term is calculated as

$$\begin{aligned} &D(J)[\mathbf{H}] = \operatorname{cof}\mathbf{F} : \mathbf{H} = J\mathbf{F}^{-\mathrm{T}} : \mathbf{H}, \quad D(\|\mathbf{F}\|^2)[\mathbf{H}] = 2\mathbf{F} : \mathbf{H}, \\ &D(J^{-p})[\mathbf{H}] = -\,pJ^{-p-1}D(J)[\mathbf{H}] = -\,pJ^{-p}\mathbf{F}^{-\mathrm{T}} : \mathbf{H}, \\ &D(J^{-p}\,\|\mathbf{F}\|^2)[\mathbf{H}] = D(J^{-p})[\mathbf{H}]\,\|\mathbf{F}\|^2 + J^{-p}D(\|\mathbf{F}\|^2)[\mathbf{H}] = [2J^{-p}\mathbf{F} - pJ^{-p}\|\mathbf{F}\|^2\mathbf{F}^{-\mathrm{T}}] : \mathbf{H}. \end{aligned} \tag{45}$$

At the same time, for the volumetric item, the following relation holds

$$\begin{aligned} &D(J^2)[\mathbf{H}] = 2J\,D(J)[\mathbf{H}] = 2J^2\mathbf{F}^{-\mathrm{T}} : \mathbf{H}, \\ &D(J^{-2})[\mathbf{H}] = -\,2J^{-3}D(J)[\mathbf{H}] = -\,2J^{-2}\mathbf{F}^{-\mathrm{T}} : \mathbf{H}, \\ &D(\alpha\,[J^2 + J^{-2} - 2])[\mathbf{H}] = 2\alpha\,[J^2 - J^{-2}]\,\mathbf{F}^{-\mathrm{T}} : \mathbf{H}. \end{aligned} \tag{46}$$

Combining the isochoric and volumetric derivatives yields the gradient of Q with respect to $\mathbf{F}$ as

$$D_{\mathbf{F}}Q(\mathbf{F}) = 2J^{-p}\mathbf{F} - pJ^{-p}\,\|\mathbf{F}\|^2\mathbf{F}^{-\mathrm{T}} + 2\alpha\,[J^2 - J^{-2}]\,\mathbf{F}^{-\mathrm{T}}. \tag{47}$$

Notably, the original energy can be expressed in an exponential composite form, given by

$$W_n(\mathbf{F}) = \frac{\mu}{2c}\,[\mathrm{e}^{cQ(\mathbf{F})} - 1]. \tag{48}$$

Applying the chain rule to this composite function gives the Piola stress as

$$\mathbf{P}(\mathbf{F}) = D_{\mathbf{F}}W_n(\mathbf{F}) = \frac{\mu}{2}\mathrm{e}^{cQ(\mathbf{F})}D_{\mathbf{F}}Q(\mathbf{F}). \tag{49}$$

To convert the Piola stress to Cauchy stress, we multiply Eq. (47) on the right by $\mathbf{F}^{\mathrm{T}}$. In this way, $\mathbf{F}\mathbf{F}^{\mathrm{T}}$ naturally combines to form $\mathbf{B}$, and $\mathbf{F}^{-\mathrm{T}}\mathbf{F}^{\mathrm{T}} = \mathbf{I}$. It leads to

$$D_{\mathbf{F}} Q\,(\mathbf{F})\,\mathbf{F}^{\mathrm{T}} = 2J^{-p}\mathbf{B} - pJ^{-p}\,||\mathbf{F}||^2\mathbf{I} + 2\alpha\,[J^2 - J^{-2}]\,\mathbf{I}, \tag{50}$$

which yields a preliminary expression for the Cauchy stress. It clearly distinguishes between deviatoric stresses and spherical stresses, that is

$$\boldsymbol{\sigma}\,(\mathbf{F}) = \frac{1}{J}\mathbf{P}\,(\mathbf{F})\,\mathbf{F}^{\mathrm{T}} = \frac{\mu\,\mathrm{e}^{cQ\,(\mathbf{F})}}{J}\,[J^{-p}\mathbf{B} - \frac{p}{2}J^{-p}\,||\mathbf{F}||^2\mathbf{I} + \alpha\,[J^2 - J^{-2}]\,\mathbf{I}]. \tag{51}$$

Since $p = 2/n$, the first two terms happen to form the deviatoric part of $J^{-2/n}\mathbf{B}$, namely

$$\begin{aligned} &\mathrm{tr}\,(J^{-p}\mathbf{B}) = J^{-p}\mathrm{tr}\,\mathbf{B} = J^{-p}\,||\mathbf{F}||^2 = J^{-2/n}\,||\mathbf{F}||^2, \\ &J^{-p}\mathbf{B} - \frac{p}{2}J^{-p}\,||\mathbf{F}||^2\mathbf{I} = J^{-2/n}\mathbf{B} - \frac{1}{n}\mathrm{tr}\,(J^{-2/n}\mathbf{B})\,\mathbf{I} = \mathrm{dev}\,(J^{-2/n}\mathbf{B}). \end{aligned} \tag{52}$$

By restoring the material parameters and making use of

$$\mu\alpha = \frac{\kappa}{4}, \quad cQ\,(\mathbf{F}) = \Phi_n\,(\mathbf{F}), \tag{53}$$

the Cauchy stress can be represented in a more compact form as

$$\boldsymbol{\sigma}\,(\mathbf{F}) = \frac{\mathrm{e}^{\Phi_n\,(\mathbf{F})}}{J}\,[\mu\,\mathrm{dev}\,(J^{-2/n}\mathbf{B}) + \frac{\kappa}{4}\,[J^2 - J^{-2}]\,\mathbf{I}] \quad \text{with} \quad \mathbf{B} = \mathbf{V}^2\,,\ J = \det\mathbf{V}. \tag{54}$$

This expression therefore can be alternatively expressed as

$$\boldsymbol{\sigma}\,(\mathbf{V}) = \frac{\mathrm{e}^{\Phi_n\,(\mathbf{V})}}{\det\mathbf{V}}\,[\mu\,\mathrm{dev}\,((\det\mathbf{V})^{-2/n}\mathbf{V}^2) + \frac{\kappa}{4}\,[(\det\mathbf{V})^2 - (\det\mathbf{V})^{-2}]\,\mathbf{I}]. \tag{55}$$

At the natural state $\mathbf{V} = \mathbf{I}$, both the deviatoric and volumetric terms vanish, so the natural state is strictly stress-free, as demonstrated by

$$\mathbf{V} = \mathbf{I} \quad \Longrightarrow \quad \det\mathbf{V} = 1, \quad \mathrm{dev}\,\mathbf{I} = \mathbf{0}, \quad (\det\mathbf{V})^2 - (\det\mathbf{V})^{-2} = 0 \quad \Longrightarrow \quad \boldsymbol{\sigma}\,(\mathbf{I}) = \mathbf{0}. \tag{56}$$

It is also consistent with the conclusion derived from the energy minimization in Section 4.

7. Exact logarithmic representation and linearization

Here we move to the logarithmic strain space. We decompose $\mathbf{X}$ orthogonally into a deviatoric component $\mathbf{Y}$ and a spherical component $s/n\,\mathbf{I}$. Since $\mathrm{tr}\,\mathbf{Y} = 0$, these two components represent isochoric true strain and volumetric true strain, given by

$$\mathbf{X} = \log\mathbf{V}, \quad s = \mathrm{tr}\,\mathbf{X}, \quad \mathbf{Y} = \mathrm{dev}\,\mathbf{X} = \mathbf{X} - \frac{s}{n}\mathbf{I}, \quad \mathbf{X} = \mathbf{Y} + \frac{s}{n}\mathbf{I}. \tag{57}$$

Since the spherical tensor commutes with any tensor, the tensor exponential can be factored. From this, we can simultaneously obtain exact expressions for

$$\begin{aligned} \mathbf{V} &= \mathrm{e}^{\mathbf{X}} = \exp(\mathbf{Y} + \frac{s}{n}\mathbf{I}) = \mathrm{e}^{\mathbf{Y}}\mathrm{e}^{s\mathbf{I}/n} = \mathrm{e}^{s/n}\mathrm{e}^{\mathbf{Y}}, \\ J &= \det\mathbf{V} = \det(\mathrm{e}^{\mathbf{X}}) = \mathrm{e}^{\mathrm{tr}\,\mathbf{X}} = \mathrm{e}^{s}, \end{aligned} \tag{58}$$

and

$$J^{-2/n}\mathbf{B} = \mathrm{e}^{-2s/n}\mathrm{e}^{2\mathbf{X}} = \mathrm{e}^{-2s/n}\mathrm{e}^{2\mathbf{Y}+2s\mathbf{I}/n} = \mathrm{e}^{-2s/n}\mathrm{e}^{2\mathbf{Y}}\mathrm{e}^{2s\mathbf{I}/n} = \mathrm{e}^{2\mathbf{Y}}. \tag{59}$$

Substituting the above expression into the internal variable of energy Q, we find that the deviatoric term depends only on $\mathbf{Y}$, while the volumetric term depends only on s, that is

$$Q(\mathbf{X}) = \mathrm{tr}(\mathrm{e}^{2\mathbf{Y}}) - n + \alpha\,[\mathrm{e}^{2s} + \mathrm{e}^{-2s} - 2]. \tag{60}$$

This exact separation in logarithmic variables is a key structural element for the subsequent proof of tangential positive definiteness. Incorporating these relationships into the Cauchy stress formula yields a global explicit expression for true stress in terms of true strain as

$$\widehat{\boldsymbol{\sigma}}(\mathbf{X}) = \mu\,\mathrm{e}^{cQ(\mathbf{X})-s}\,[\mathrm{dev}(\mathrm{e}^{2\mathbf{Y}}) + \alpha\,[\mathrm{e}^{2s} - \mathrm{e}^{-2s}]\,\mathbf{I}]. \tag{61}$$

To determine the elastic moduli near the natural state, we perform a Taylor expansion of the tensor exponential associated with the deviatoric part. Since $\mathrm{tr}\,\mathbf{Y} = 0$, the first-order trace term vanishes. Consequently, the distortion energy begins with the quadratic term, yielding

$$\begin{aligned} \mathrm{e}^{2\mathbf{Y}} &= \mathbf{I} + 2\mathbf{Y} + 2\mathbf{Y}^2 + O(\|\mathbf{Y}\|^3), \\ \mathrm{tr}(\mathrm{e}^{2\mathbf{Y}}) - n &= 2\,\mathrm{tr}\,\mathbf{Y} + 2\,\mathrm{tr}(\mathbf{Y}^2) + O(\|\mathbf{Y}\|^3) = 2\,\|\mathbf{Y}\|^2 + O(\|\mathbf{Y}\|^3). \end{aligned} \tag{62}$$

The volume term concerning s is an even function. Its odd-order terms cancel each other out, and the lowest nonzero term is also a second-order term, namely

$$\begin{aligned} \mathrm{e}^{2s} &= 1 + 2s + 2s^2 + \frac{4}{3}s^3 + \frac{2}{3}s^4 + O(|s|^5), \\ \mathrm{e}^{-2s} &= 1 - 2s + 2s^2 - \frac{4}{3}s^3 + \frac{2}{3}s^4 + O(|s|^5), \\ \mathrm{e}^{2s} + \mathrm{e}^{-2s} - 2 &= 4s^2 + \frac{4}{3}s^4 + O(|s|^6) = 4s^2 + O(|s|^4). \end{aligned} \tag{63}$$

Combining these two parts, we see that $Q(\mathbf{X})$ has neither a constant term nor a linear term at the origin, but rather is of order $O(\|\mathbf{X}\|^2)$, which indicates that

$$\begin{aligned} &Q(\mathbf{X}) = 2\,\|\mathbf{Y}\|^2 + 4\alpha s^2 + O(\|\mathbf{X}\|^3), \quad Q(\mathbf{X}) = O(\|\mathbf{X}\|^2), \\ &\mathrm{e}^{cQ(\mathbf{X})} - 1 = cQ(\mathbf{X}) + \frac{c^2}{2}Q(\mathbf{X})^2 + O(Q(\mathbf{X})^3) = cQ(\mathbf{X}) + O(\|\mathbf{X}\|^4). \end{aligned} \tag{64}$$

Incorporating this expansion into the energy function recovers the standard quadratic strain energy for isotropic linear elasticity, expressed as

$$
\begin{aligned}
W_n(\mathrm{e}^{\mathbf{X}}) &= \frac{\mu}{2c}\,[cQ(\mathbf{X}) + O(\|\mathbf{X}\|^4)] \\
&= \frac{\mu}{2}\,[2\,\|\mathbf{Y}\|^2 + 4\alpha s^2 + O(\|\mathbf{X}\|^3)] + O(\|\mathbf{X}\|^4) \\
&= \mu\|\mathbf{Y}\|^2 + 2\mu\alpha s^2 + O(\|\mathbf{X}\|^3) \\
&= \mu\,\|\operatorname{dev}\mathbf{X}\|^2 + 2\mu\alpha\,[\operatorname{tr}\mathbf{X}]^2 + O(\|\mathbf{X}\|^3) \\
&= \mu\,\|\operatorname{dev}\mathbf{X}\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}\mathbf{X}]^2 + O(\|\mathbf{X}\|^3).
\end{aligned}
\tag{65}
$$

Hence, *μ and κ are, precisely, the shear modulus and bulk modulus under infinitesimal deformations*. For the sake of completeness, we will now directly differentiate with respect to stress. To facilitate the calculation of the stress tangent, we express the true stress as the product of a positive scalar factor $K(\mathbf{X})$ and a tensor factor $\mathbf{A}(\mathbf{X})$, namely

$$
\mathbf{A}(\mathbf{X}) = \operatorname{dev}(\mathrm{e}^{2\mathbf{Y}}) + \alpha\,[\mathrm{e}^{2s} - \mathrm{e}^{-2s}]\,\mathbf{I}, \quad K(\mathbf{X}) = \mu\,\mathrm{e}^{cQ(\mathbf{X})-s}, \quad \widehat{\boldsymbol{\sigma}}(\mathbf{X}) = K(\mathbf{X})\,\mathbf{A}(\mathbf{X}).
\tag{66}
$$

At the natural state, the scalar factor is μ, while the tensor factor is zero, more precisely,

$$
\mathbf{X} = \mathbf{0} \quad\Longrightarrow\quad s = 0, \quad \mathbf{Y} = \mathbf{0}, \quad Q(\mathbf{0}) = 0, \quad K(\mathbf{0}) = \mu, \quad \mathbf{A}(\mathbf{0}) = \mathbf{0}.
\tag{67}
$$

For any symmetric increment $\mathbf{H}$, we similarly decompose it into a deviatoric part $\mathbf{Z}$ and a volumetric part v as

$$
v = \operatorname{tr}\mathbf{H}, \quad \mathbf{Z} = \operatorname{dev}\mathbf{H}, \quad \mathbf{H} = \mathbf{Z} + \frac{v}{n}\mathbf{I}, \quad \text{for } \mathbf{H} \in \operatorname{Sym}(n).
\tag{68}
$$

Evaluating the first-order derivatives of the tensor exponential and the scalar volumetric function at the origin yields

$$
D_{\mathbf{Y}}(\mathrm{e}^{2\mathbf{Y}})_{\mathbf{Y}=\mathbf{0}}\,[\mathbf{Z}] = 2\mathbf{Z}, \quad D_s(\mathrm{e}^{2s} - \mathrm{e}^{-2s})\,|_{s=0}\,[v] = 4v.
\tag{69}
$$

Consequently, the directional derivative of $\mathbf{A}$ at the natural state along the increment $\mathbf{H}$ takes the form of

$$
D\,\mathbf{A}(\mathbf{0})\,[\mathbf{H}] = \operatorname{dev}(2\mathbf{Z}) + 4\alpha v\mathbf{I} = 2\operatorname{dev}\mathbf{H} + 4\alpha\,[\operatorname{tr}\mathbf{H}]\,\mathbf{I}.
\tag{70}
$$

By applying the product rule and utilizing the fact that $\mathbf{A}(\mathbf{0}) = \mathbf{0}$, we obtain

$$
\begin{aligned}
D\,\widehat{\boldsymbol{\sigma}}(\mathbf{0})\,[\mathbf{H}] &= D\,K(\mathbf{0})\,[\mathbf{H}]\,\mathbf{A}(\mathbf{0}) + K(\mathbf{0})\,D\,\mathbf{A}(\mathbf{0})\,[\mathbf{H}] \\
&= \mu\,[2\operatorname{dev}\mathbf{H} + 4\alpha\,[\operatorname{tr}\mathbf{H}]\,\mathbf{I}] \\
&= 2\mu\operatorname{dev}\mathbf{H} + \kappa\,[\operatorname{tr}\mathbf{H}]\,\mathbf{I}.
\end{aligned}
\tag{71}
$$

Indeed, the above deviatoric decomposition can be equivalently recast into the standard Lamé form, given by

$$
D\,\widehat{\boldsymbol{\sigma}}(\mathbf{0})\,[\mathbf{H}] = 2\mu\mathbf{H} + \lambda\,[\operatorname{tr}\mathbf{H}]\,\mathbf{I}, \quad \lambda = \kappa - \frac{2\mu}{n}.
\tag{72}
$$

Since the deviatoric and spherical subspaces are orthogonal to one another, the incremental work additively decouples into shear and volumetric contributions. For any arbitrary non-zero increment, these two parts cannot vanish simultaneously. Hence, the tangent stiffness tensor evaluated at the natural state is strictly positive definite:

$$D\widehat{\boldsymbol{\sigma}}(\mathbf{0})[\mathbf{H}] : \mathbf{H} = 2\mu\,\|\mathrm{dev}\,\mathbf{H}\|^2 + \kappa\,[\mathrm{tr}\,\mathbf{H}]^2 > 0, \quad \forall\,\mathbf{0} \neq \mathbf{H} \in \mathrm{Sym}\,(n). \tag{73}$$

To link this linearization of the logarithmic strain to classical small-deformation theory, we consider a one-parameter deformation starting from the natural state as $\mathbf{F}(\epsilon) = \mathbf{I} + \epsilon\mathbf{G}$ and $\boldsymbol{\varepsilon} = \mathrm{sym}\,\mathbf{G}$. Substituting this into the left Cauchy–Green tensor and retaining only the first-order terms, we arrive at

$$\mathbf{B}(\epsilon) = \mathbf{F}(\epsilon)\,\mathbf{F}(\epsilon)^{\mathrm{T}} = \mathbf{I} + 2\epsilon\boldsymbol{\varepsilon} + O(\epsilon^2). \tag{74}$$

Furthermore, to first order, both the left stretch and the logarithmic strain tensors are approximated as

$$\mathbf{V}(\epsilon) = \mathbf{B}(\epsilon)^{1/2} = \mathbf{I} + \epsilon\boldsymbol{\varepsilon} + O(\epsilon^2), \quad \mathbf{X}(\epsilon) = \log\mathbf{V}(\epsilon) = \epsilon\boldsymbol{\varepsilon} + O(\epsilon^2). \tag{75}$$

Expanding the stress around the stress-free natural state $\widehat{\boldsymbol{\sigma}}(\mathbf{0}) = \mathbf{0}$, and substituting the asymptotic expression for $\mathbf{X}(\epsilon)$, it follows that

$$\begin{aligned}\widehat{\boldsymbol{\sigma}}(\mathbf{X}(\epsilon)) &= \widehat{\boldsymbol{\sigma}}(\mathbf{0}) + D\widehat{\boldsymbol{\sigma}}(\mathbf{0})[\mathbf{X}] + O(\|\mathbf{X}\|^2) \\ &= D\widehat{\boldsymbol{\sigma}}(\mathbf{0})[\epsilon\boldsymbol{\varepsilon} + O(\epsilon^2)] + O(\|\mathbf{X}(\epsilon)\|^2) = \epsilon D\widehat{\boldsymbol{\sigma}}(\mathbf{0})[\boldsymbol{\varepsilon}] + O(\epsilon^2).\end{aligned} \tag{76}$$

By invoking the previously derived Lamé representation for the directional derivative, the true Cauchy stress up to the first order is ultimately given by

$$\boldsymbol{\sigma}(\mathbf{F}(\epsilon)) = \widehat{\boldsymbol{\sigma}}(\mathbf{X}(\epsilon)) = \epsilon\,[2\mu\boldsymbol{\varepsilon} + \lambda\,[\mathrm{tr}\,\boldsymbol{\varepsilon}]\,\mathbf{I}] + O(\epsilon^2). \tag{77}$$

This strictly confirms that the proposed finite strain formulation recovers the classical generalized Hooke's law in the small strain limit.

8. Full true-stress-true-strain monotonicity

This section proves the most crucial global property of the main theorem. Given an arbitrary state $\mathbf{X} \in \mathrm{Sym}\,(n)$ and an arbitrary nonzero increment $\mathbf{H} \in \mathrm{Sym}\,(n)$, we still define $s = \mathrm{tr}\,\mathbf{X}$, $\mathbf{Y} = \mathrm{dev}\,\mathbf{X}$, $v = \mathrm{tr}\,\mathbf{H}$, $\mathbf{Z} = \mathrm{dev}\,\mathbf{H}$, so that

$$D\,s(\mathbf{X})[\mathbf{H}] = v, \quad D\,\mathbf{Y}(\mathbf{X})[\mathbf{H}] = \mathbf{Z}. \tag{78}$$

The tensor $\mathbf{Y}$ and the increment $\mathbf{Z}$ generally do not commute. Hence, the derivative of the tensor exponential $\mathrm{e}^{2\mathbf{Y}}$ cannot simply be written as $2\mathrm{e}^{2\mathbf{Y}}\mathbf{Z}$. It is necessary to use the full Fréchet derivative integral formula, namely

$$\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}] = D_{\mathbf{Y}}(\mathrm{e}^{2\mathbf{Y}})[\mathbf{Z}] = 2\int_0^1 \mathrm{e}^{2[1-\theta]\mathbf{Y}}\,\mathbf{Z}\,\mathrm{e}^{2\theta\mathbf{Y}}\,\mathrm{d}\theta. \tag{79}$$

Since $\mathbf{Y}$ and $\mathbf{Z}$ are symmetric, it follows from transposing the integral expression and performing a variable substitution ($\theta \to 1-\theta$) that $\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]$ remains a symmetric tensor, more specifically,

$$\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]^{\mathrm{T}} = 2\int_0^1 \mathrm{e}^{2\theta\mathbf{Y}}\,\mathbf{Z}\,\mathrm{e}^{2[1-\theta]\mathbf{Y}}\,\mathrm{d}\theta = 2\int_0^1 \mathrm{e}^{2[1-\theta]\mathbf{Y}}\,\mathbf{Z}\,\mathrm{e}^{2\theta\mathbf{Y}}\,\mathrm{d}\theta = \mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]. \tag{80}$$

Using the cyclic invariance of the trace, we combine the exponential factors into $\mathrm{e}^{2\mathbf{Y}}$ to obtain the trace of the tensor exponential derivative as

$$\begin{aligned}\mathrm{tr}\,(\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]) &= 2\int_0^1 \mathrm{tr}\,(\mathrm{e}^{2[1-\theta]\mathbf{Y}}\,\mathbf{Z}\,\mathrm{e}^{2\theta\mathbf{Y}})\,\mathrm{d}\theta = 2\int_0^1 \mathrm{tr}\,(\mathrm{e}^{2\theta\mathbf{Y}}\mathrm{e}^{2[1-\theta]\mathbf{Y}}\,\mathbf{Z})\,\mathrm{d}\theta \\ &= 2\int_0^1 \mathrm{tr}\,(\mathrm{e}^{2\mathbf{Y}}\,\mathbf{Z})\,\mathrm{d}\theta = 2\,\mathrm{e}^{2\mathbf{Y}} : \mathbf{Z}.\end{aligned} \tag{81}$$

We then take the directional derivative of Q. As $\mathbf{Z}$ is trace-free, $\mathrm{e}^{2\mathbf{Y}} : \mathbf{Z}$ can equivalently be written as $\mathrm{dev}\,(\mathrm{e}^{2\mathbf{Y}}) : \mathbf{Z}$, which means that

$$\begin{aligned}D\,Q\,(\mathbf{X})\,[\mathbf{H}] &= \mathrm{tr}\,(\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]) + 2\alpha\,[\mathrm{e}^{2s} - \mathrm{e}^{-2s}]\,v \\ &= 2\,\mathrm{e}^{2\mathbf{Y}} : \mathbf{Z} + 2\alpha\,[\mathrm{e}^{2s} - \mathrm{e}^{-2s}]\,v = 2\,\mathrm{dev}\,(\mathrm{e}^{2\mathbf{Y}}) : \mathbf{Z} + 2\alpha\,[\mathrm{e}^{2s} - \mathrm{e}^{-2s}]\,v.\end{aligned} \tag{82}$$

For simplicity, let us introduce the scalars u and t. Here, u is the projection of the stress tensor factor $\mathbf{A}$ onto the direction $\mathbf{H}$, while t measures the spherical stiffness weight at the current volumetric state, which is given by

$$u = \mathbf{A}\,(\mathbf{X}) : \mathbf{H}, \quad t = \alpha\,[\mathrm{e}^{2s} + \mathrm{e}^{-2s}]. \tag{83}$$

Utilizing the orthogonality between the deviatoric and spherical tensors, u can be expanded as the sum of the distortional contribution and the volumetric contribution, such that

$$u = \left[\mathrm{dev}\,(\mathrm{e}^{2\mathbf{Y}}) + \alpha\,[\mathrm{e}^{2s} - \mathrm{e}^{-2s}]\,\mathbf{I}\right] : [\mathbf{Z} + \frac{v}{n}\mathbf{I}] = \mathrm{dev}\,(\mathrm{e}^{2\mathbf{Y}}) : \mathbf{Z} + \alpha\,[\mathrm{e}^{2s} - \mathrm{e}^{-2s}]\,v, \tag{84}$$

which immediately leads to

$$D\,Q\,(\mathbf{X})\,[\mathbf{H}] = 2u. \tag{85}$$

It can be seen that the directional derivative of Q is exactly equal to $2u$. This relationship is key to closing the subsequent quadratic formula. To proceed, we take the derivative of the tensor factor $\mathbf{A}\,(\mathbf{X})$ and find

$$D\,\mathbf{A}\,(\mathbf{X})\,[\mathbf{H}] = \mathrm{dev}\,(\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]) + 2\alpha\,[\mathrm{e}^{2s} + \mathrm{e}^{-2s}]\,v\mathbf{I} = \mathrm{dev}\,(\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]) + 2tv\mathbf{I}. \tag{86}$$

By performing the Frobenius inner product to $D\,\mathbf{A}\,(\mathbf{X})\,[\mathbf{H}]$ and $\mathbf{H}$, we obtain

$$D\,\mathbf{A}\,(\mathbf{X})\,[\mathbf{H}] : \mathbf{H} = \mathrm{dev}\,(\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]) : \mathbf{Z} + 2tv\mathbf{I} : \frac{v}{n}\mathbf{I} = \mathcal{E}_{\mathbf{Y}}[\mathbf{Z}] : \mathbf{Z} + 2tv^2. \tag{87}$$

Meanwhile, differentiating the scalar factor $K\,(\mathbf{X})$ and making use of $D\,Q\,(\mathbf{X})\,[\mathbf{H}] = 2u$ results in

$$D\,K\,(\mathbf{X})\,[\mathbf{H}] = K\,(\mathbf{X})\left[cD\,Q\,(\mathbf{X})\,[\mathbf{H}] - D\,s\,(\mathbf{X})\,[\mathbf{H}]\right] = K\,(\mathbf{X})\,[2cu - v]. \tag{88}$$

Then, by applying the product rule to $\widehat{\boldsymbol{\sigma}}(\mathbf{X}) = K(\mathbf{X})\,\mathbf{A}(\mathbf{X})$, we arrive at

$$\begin{aligned} D\,\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}] &= D\,K(\mathbf{X})[\mathbf{H}]\,\mathbf{A}(\mathbf{X}) + K(\mathbf{X})\,D\,\mathbf{A}(\mathbf{X})[\mathbf{H}] \\ &= K(\mathbf{X})\,[[2cu - v]\,\mathbf{A}(\mathbf{X}) + D\,\mathbf{A}(\mathbf{X})[\mathbf{H}]]. \end{aligned} \tag{89}$$

Finally, by taking the inner product of $D\,\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}]$ with $\mathbf{H}$ and dividing by $K(\mathbf{X}) > 0$, the complete quadratic form reads

$$\begin{aligned} K(\mathbf{X})^{-1} D\,\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}] : \mathbf{H} &= [2cu - v]\,\mathbf{A}(\mathbf{X}) : \mathbf{H} + D\,\mathbf{A}(\mathbf{X})[\mathbf{H}] : \mathbf{H} \\ &= [2cu - v]\,u + \mathcal{E}_{\mathbf{Y}}[\mathbf{Z}] : \mathbf{Z} + 2tv^2 \\ &= \mathcal{E}_{\mathbf{Y}}[\mathbf{Z}] : \mathbf{Z} + 2cu^2 - uv + 2tv^2. \end{aligned} \tag{90}$$

The only terms in this expression where the sign might change are the mixed terms involving u and v. Nevertheless, note that the following identity holds

$$2cu^2 - uv + 2tv^2 = 2c\,[u - \frac{v}{4c}]^2 + [2t - \frac{1}{8c}]\,v^2. \tag{91}$$

Thereby, the complete quadratic form can be reformulated as

$$K(\mathbf{X})^{-1} D\,\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}] : \mathbf{H} = \mathcal{E}_{\mathbf{Y}}[\mathbf{Z}] : \mathbf{Z} + 2c\,[u - \frac{v}{4c}]^2 + [2t - \frac{1}{8c}]\,v^2. \tag{92}$$

Given that $K(\mathbf{X}) > 0$, it suffices to prove that the right-hand side is strictly positive for every nonzero $\mathbf{H}$. We will show that, for any $\mathbf{Z} \neq \mathbf{0}$, the first term is strictly positive. To this end, we perform an orthogonal spectral decomposition of $\mathbf{Y}$ and transform $\mathbf{Z}$ into the same eigen-basis, namely

$$\mathbf{Y} = \mathbf{R}\,\mathrm{diag}\,(y_1, \ldots, y_n)\,\mathbf{R}^{\mathrm{T}}, \quad \widetilde{\mathbf{Z}} = \mathbf{R}^{\mathrm{T}}\mathbf{Z}\mathbf{R}. \tag{93}$$

The Fréchet derivative of the tensor exponential is expressed in terms of first-order divided differences. By taking the continuous limit at repeated eigenvalues, this formula also covers the case of repeated principal values, that is

$$[\mathbf{R}^{\mathrm{T}}\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]\,\mathbf{R}]_{ij} = f^{[1]}(y_i, y_j)\,\widetilde{Z}_{ij}, \quad f^{[1]}(a,b) = \begin{cases} \dfrac{\mathrm{e}^{2a} - \mathrm{e}^{2b}}{a - b}, & a \neq b, \\ 2\,\mathrm{e}^{2a}, & a = b. \end{cases} \tag{94}$$

As the scalar exponential function is strictly increasing, all divided differences are strictly positive. Consequently, as long as $\mathbf{Z} \neq \mathbf{0}$, the deviatoric term is strictly positive, more specifically,

$$\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}] : \mathbf{Z} = 2\sum_{i=1}^{n} \mathrm{e}^{2y_i}\,\widetilde{Z}_{ii}^2 + 2\sum_{1\leqslant i<j\leqslant n} \frac{\mathrm{e}^{2y_i} - \mathrm{e}^{2y_j}}{y_i - y_j}\widetilde{Z}_{ij}^2 \quad \Longrightarrow \quad \mathcal{E}_{\mathbf{Y}}[\mathbf{Z}] : \mathbf{Z} > 0 \quad \text{if} \;\; \mathbf{Z} \neq \mathbf{0}. \tag{95}$$

Furthermore, using $\alpha c = 1/32$, the last coefficient in the complete quadratic form simplifies to

$$2t - \frac{1}{8c} = 2\alpha\,[\mathrm{e}^{2s} + \mathrm{e}^{-2s}] - \frac{1}{8c} = 2\alpha\,[\mathrm{e}^{2s} + \mathrm{e}^{-2s}] - 4\alpha = 2\alpha\,[\mathrm{e}^{2s} + \mathrm{e}^{-2s} - 2] = 8\alpha \sinh^2 s \geqslant 0. \tag{96}$$

We see that this coefficient is allowed to degenerate to zero at $s = 0$, but this does not imply that the complete quadratic form degenerates. Therefore, we prove strict positive definiteness by considering all possible nonzero increments $\mathbf{H}$.

Case 1: $\mathbf{Z} \neq \mathbf{0}$.

From Eq. (95), the following inequality holds strictly:

$$\mathcal{E}_{\mathbf{Y}}\,[\mathbf{Z}] : \mathbf{Z} > 0.$$

Since the other two terms are nonnegative, the complete quadratic form is then strictly positive.

Case 2: $\mathbf{Z} = \mathbf{0}$ *and* $s \neq 0$.

The nonzero increment $\mathbf{H}$ can only be a purely spherical increment, hence $v \neq 0$. In this case, the coefficient of the last term is strictly positive, so the complete quadratic form remains strictly positive, which is expressed as

$$\mathbf{H} = \frac{v}{n}\mathbf{I}, \quad v \neq 0, \quad 2t - \frac{1}{8c} = 8\alpha \sinh^2 s > 0 \quad \Longrightarrow \quad [2t - \frac{1}{8c}]\, v^2 > 0.$$

Case 3: $\mathbf{Z} = \mathbf{0}$ *and* $s = 0$.

At this point, the coefficient of the last term is zero. However, the second term is strictly positive, thereby ensuring the positive definiteness of the complete quadratic form, that is

$$\mathbf{H} = \frac{v}{n}\mathbf{I}, \quad v \neq 0, \quad u = 0 \quad \Longrightarrow \quad 2c\,[u - \frac{v}{4c}]^2 = \frac{v^2}{8c} > 0.$$

With this, we have completed the proof that the local tangent is strictly point-wise positive definite. Finally, we should also note that it implies global strict monotonicity. Consider two different logarithmic strain states, and define the incremental direction as the difference between them, namely

$$\mathbf{H} = \mathbf{X}_1 - \mathbf{X}_0 \neq \mathbf{0}, \quad \forall\, \mathbf{X}_0 \neq \mathbf{X}_1. \tag{97}$$

Applying the fundamental theorem of calculus to the line segment connecting $\mathbf{X}_0$ and $\mathbf{X}_1$ yields

$$\widehat{\boldsymbol{\sigma}}\,(\mathbf{X}_1) - \widehat{\boldsymbol{\sigma}}\,(\mathbf{X}_0) = \int_0^1 D\,\widehat{\boldsymbol{\sigma}}\,(\mathbf{X}_0 + \tau\mathbf{H})\,[\mathbf{H}]\,\mathrm{d}\tau. \tag{98}$$

Then we take the inner product of the above expression with $\mathbf{H} = \mathbf{X}_1 - \mathbf{X}_0$. As the true stress tangent at every point on the line segment is strictly positive definite, the integral is strictly positive, which is demonstrated as

$$[\widehat{\boldsymbol{\sigma}}\,(\mathbf{X}_1) - \widehat{\boldsymbol{\sigma}}\,(\mathbf{X}_0)] : [\mathbf{X}_1 - \mathbf{X}_0] = \int_0^1 D\,\widehat{\boldsymbol{\sigma}}\,(\mathbf{X}_0 + \tau\mathbf{H})\,[\mathbf{H}] : \mathbf{H}\,\mathrm{d}\tau > 0. \tag{99}$$

Therefore, $\widehat{\boldsymbol{\sigma}}$ is strictly monotonic on the entire $\mathrm{Sym}\,(n)$, and it is injective.

9. Properness and global invertibility of the stress map

Strict monotonicity guarantees injectivity, but it does not, on its own, guarantee that the stress can take on all values in $\mathrm{Sym}\,(n)$. In other words, injectivity alone does not guarantee that any symmetric Cauchy stress can be achieved by a given deformation state. In the following, we will first prove the properness of the stress map, and then use the theory of global inverse maps to prove its surjectivity and global invertibility. To this end, an auxiliary potential function $q(\mathbf{X})$ is introduced as

$$q(\mathbf{X}) = \operatorname{tr}(e^{2\operatorname{dev}\mathbf{X}}) + \alpha\,[e^{2\operatorname{tr}\mathbf{X}} + e^{-2\operatorname{tr}\mathbf{X}}] = \operatorname{tr}(e^{2\mathbf{Y}}) + \alpha\,[e^{2s} + e^{-2s}]. \tag{100}$$

In particular, the internal variable of the original energy $Q(\mathbf{X})$ is precisely the increment of $q(\mathbf{X})$ relative to the value in the natural state, that is

$$Q(\mathbf{X}) = q(\mathbf{X}) - q(\mathbf{0}), \quad q(\mathbf{0}) = n + 2\alpha. \tag{101}$$

9.1. Strict convexity of the auxiliary potential

We first calculate the first-order directional derivative of $q(\mathbf{X})$, which is expressed as

$$\begin{aligned} D\,q(\mathbf{X})\,[\mathbf{H}] &= \operatorname{tr}(\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}]) + 2\alpha\,[e^{2s} - e^{-2s}]\,v = 2e^{2\mathbf{Y}} : \mathbf{Z} + 2\alpha\,[e^{2s} - e^{-2s}]\,v \\ &= 2\operatorname{dev}(e^{2\mathbf{Y}}) : \mathbf{Z} + 2\alpha\,[e^{2s} - e^{-2s}]\operatorname{tr}\mathbf{H} = 2\operatorname{dev}(e^{2\mathbf{Y}}) : \mathbf{H} + 2\alpha\,[e^{2s} - e^{-2s}]\operatorname{tr}\mathbf{H} \\ &= 2\,[\operatorname{dev}(e^{2\mathbf{Y}}) + \alpha\,[e^{2s} - e^{-2s}]\,\mathbf{I}] : \mathbf{H} = 2\mathbf{A}(\mathbf{X}) : \mathbf{H}. \end{aligned} \tag{102}$$

Here, the property of $\mathbf{Z}$ as a trace-free tensor is utilized. At the same time, we find that there is a concise relationship between the gradient of the auxiliary potential and $\mathbf{A}(\mathbf{X})$, given by

$$\nabla q(\mathbf{X}) = 2\mathbf{A}(\mathbf{X}). \tag{103}$$

Differentiating again in the same direction, we obtain

$$D^2\,q(\mathbf{X})\,[\mathbf{H},\mathbf{H}] = 2\mathcal{E}_{\mathbf{Y}}[\mathbf{Z}] : \mathbf{Z} + 4\alpha\,[e^{2s} + e^{-2s}]\,v^2 > 0 \quad \forall\, \mathbf{0} \neq \mathbf{H} \in \operatorname{Sym}(n). \tag{104}$$

If $\mathbf{Z} \neq \mathbf{0}$, then it follows from Eq. (95) that the first term is strictly positive. If $\mathbf{Z} = \mathbf{0}$ and $\mathbf{H} \neq \mathbf{0}$, then $v \neq 0$ must hold, and thus the second term is strictly positive. Hence, $q(\mathbf{X})$ is strictly convex over the entire $\operatorname{Sym}(n)$. Furthermore, since $\mathbf{A}(\mathbf{0}) = \mathbf{0}$, it follows that $\nabla q(\mathbf{0}) = \mathbf{0}$. Therefore, $\mathbf{X} = \mathbf{0}$ is the unique global minimum of $q(\mathbf{X})$, namely

$$Q(\mathbf{X}) = q(\mathbf{X}) - q(\mathbf{0}) \geqslant 0, \quad Q(\mathbf{X}) = 0 \quad \Longleftrightarrow \quad \mathbf{X} = \mathbf{0}. \tag{105}$$

9.2. Supercoercivity of the auxiliary potential

The standard coercivity condition merely states that $q(\mathbf{X}) \to +\infty$ as $\|\mathbf{X}\| \to +\infty$, whereas further analysis requires a stronger conclusion, namely that the growth rate of $q(\mathbf{X})$ exceeds that of any linear function, that is

$$\frac{q(\mathbf{X})}{\|\mathbf{X}\|} \to +\infty \quad \text{as } \|\mathbf{X}\| \to +\infty. \tag{106}$$

To prove this, we continue to use the orthogonal decomposition $\mathbf{X} = \mathbf{Y} + (s/n)\,\mathbf{I}$. Since the deviatoric part and the spherical part are orthogonal, the square of their norm can be exactly decomposed as

$$\|\mathbf{X}\|^2 = \|\mathbf{Y} + \frac{s}{n}\mathbf{I}\|^2 = \|\mathbf{Y}\|^2 + \frac{s^2}{n^2}\,\|\mathbf{I}\|^2 = \|\mathbf{Y}\|^2 + \frac{s^2}{n}. \tag{107}$$

Let y_i be the eigenvalues of $\mathbf{Y}$. If $\mathbf{Y} \neq \mathbf{0}$, then it follows from the fact $\sum_i y_i = 0$ that $M > 0$, which is expressed as

$$\operatorname{tr}\mathbf{Y} = 0 \quad \Longleftrightarrow \quad \textstyle\sum_i y_i = 0 \quad \Longrightarrow \quad M = \max\limits_{1\leqslant i\leqslant n} y_i > 0 \quad \text{for } \mathbf{Y} \neq \mathbf{0}. \tag{108}$$

Let the sum of the positive eigenvalues be denoted by P, and note that there are at most $[n-1]$ positive eigenvalues. Then the following inequality holds

$$P = \sum_{y_i>0} y_i = \sum_{y_i<0} (-y_i) \leqslant [n-1]\, M. \tag{109}$$

By estimating the sums of the squares of the positive and negative eigenvalues separately, we have

$$\sum_{y_i>0} y_i^2 \leqslant M \sum_{y_i>0} y_i = MP \leqslant [n-1]\, M^2, \quad \sum_{y_i<0} y_i^2 \leqslant \left(\sum_{y_i<0} (-y_i)\right)^2 = P^2 \leqslant [n-1]^2 M^2. \tag{110}$$

Combining the two parts, we find the linear lower bound of M on $\|\mathbf{Y}\|$ to be

$$\|\mathbf{Y}\|^2 = \sum_{i=1}^{n} y_i^2 \leqslant [[n-1] + [n-1]^2]\, M^2 = n\,[n-1]\, M^2 \quad \Longrightarrow \quad M \geqslant \frac{\|\mathbf{Y}\|}{\sqrt{n\,[n-1]}}. \tag{111}$$

Hence, as long as the deviatoric strain increases, $\operatorname{tr}(\mathrm{e}^{2\mathbf{Y}})$ grows at least exponentially. On the other hand, as long as the volumetric strain $|s|$ increases, $[\mathrm{e}^{2s} + \mathrm{e}^{-2s}]$ also grows exponentially. They are formulated as

$$\operatorname{tr}(\mathrm{e}^{2\mathbf{Y}}) = \sum_{i=1}^{n} \mathrm{e}^{2y_i} \geqslant \mathrm{e}^{2M} \geqslant \exp\left(\frac{2\|\mathbf{Y}\|}{\sqrt{n\,[n-1]}}\right), \quad \mathrm{e}^{2s} + \mathrm{e}^{-2s} \geqslant \mathrm{e}^{2|s|}. \tag{112}$$

Substituting these two estimates into $q(\mathbf{X})$ yields a global lower bound of

$$q(\mathbf{X}) \geqslant \exp\left(\frac{2\|\mathbf{Y}\|}{\sqrt{n\,[n-1]}}\right) + \alpha \mathrm{e}^{2|s|}. \tag{113}$$

From the norm decomposition, it follows that for any $\mathbf{X}$, at least one of the following two inequalities satisfies

$$\|\mathbf{Y}\| \geqslant \frac{\|\mathbf{X}\|}{\sqrt{2}} \quad \text{or} \quad |s| \geqslant \sqrt{\frac{n}{2}}\,\|\mathbf{X}\|. \tag{114}$$

This is because, if neither of these conditions holds, then we have $\|\mathbf{Y}\|^2 < \|\mathbf{X}\|^2/2$ and $s^2/n < \|\mathbf{X}\|^2/2$, which contradicts the fact that their sum equals $\|\mathbf{X}\|^2$. Therefore, if either of the two conditions mentioned above is satisfied, we arrive at

$$\frac{q(\mathbf{X})}{\|\mathbf{X}\|} \geqslant \frac{1}{\|\mathbf{X}\|} \exp\left(\frac{\sqrt{2}\,\|\mathbf{X}\|}{\sqrt{n\,[n-1]}}\right) \to +\infty \quad \text{or} \quad \frac{q(\mathbf{X})}{\|\mathbf{X}\|} \geqslant \frac{\alpha}{\|\mathbf{X}\|} \mathrm{e}^{\sqrt{2n}\,\|\mathbf{X}\|} \to +\infty. \tag{115}$$

Thus, we have shown the supercoercivity of $q(\mathbf{X})$.

9.3. *Growth of the stress and properness*

Since $q(\mathbf{X})$ is a convex function, its tangent plane at any point lies on or below the graph of the function. By choosing the natural state $\mathbf{X} = \mathbf{0}$ as the reference point, we obtain

$$q(\mathbf{0}) \geqslant q(\mathbf{X}) + \nabla q(\mathbf{X}) : [\mathbf{0} - \mathbf{X}]. \tag{116}$$

Using $\nabla q(\mathbf{X}) = 2\mathbf{A}(\mathbf{X})$ to simplify this inequality, we have

$$2\mathbf{A}(\mathbf{X}) : \mathbf{X} = \nabla q(\mathbf{X}) : \mathbf{X} \geqslant q(\mathbf{X}) - q(\mathbf{0}). \tag{117}$$

Then, using the Cauchy–Schwarz inequality, we can transform the directional projection estimate into a norm estimate for $\|\mathbf{A}(\mathbf{X})\|$ as

$$2\,\|\mathbf{A}(\mathbf{X})\|\,\|\mathbf{X}\| \geqslant 2\mathbf{A}(\mathbf{X}) : \mathbf{X} \geqslant q(\mathbf{X}) - q(\mathbf{0}). \tag{118}$$

For $\mathbf{X} \neq \mathbf{0}$, by the supercoercivity of $q(\mathbf{X})$, we know that $\|\mathbf{A}(\mathbf{X})\|$ tends to infinity as $\|\mathbf{X}\|$ tends to infinity, namely

$$\|\mathbf{A}(\mathbf{X})\| \geqslant \frac{q(\mathbf{X}) - q(\mathbf{0})}{2\,\|\mathbf{X}\|} \quad \text{and} \quad \|\mathbf{A}(\mathbf{X})\| \to +\infty \quad \text{as} \;\; \|\mathbf{X}\| \to +\infty. \tag{119}$$

At the same time, we also need to evaluate the exponential factor $\mathrm{e}^{cQ(\mathbf{X})-s}$ in the stress. To this end, we consider the following trace estimate as

$$|s| = |\mathbf{I} : \mathbf{X}| \leqslant \|\mathbf{I}\|\,\|\mathbf{X}\| = \sqrt{n}\,\|\mathbf{X}\|, \tag{120}$$

which naturally leads to

$$cQ(\mathbf{X}) - s = c\,[q(\mathbf{X}) - q(\mathbf{0})] - s \geqslant cq(\mathbf{X}) - cq(\mathbf{0}) - \sqrt{n}\,\|\mathbf{X}\| = \|\mathbf{X}\|\,[c\frac{q(\mathbf{X})}{\|\mathbf{X}\|} - \sqrt{n}] - cq(\mathbf{0}). \tag{121}$$

Furthermore, by virtue of the supercoercivity of $q(\mathbf{X})$, we attain

$$cQ(\mathbf{X}) - s \to +\infty \quad \text{as} \;\; \|\mathbf{X}\| \to +\infty. \tag{122}$$

Notably, $\|\widehat{\boldsymbol{\sigma}}(\mathbf{X})\|$ is the product of $K(\mathbf{X})$ and $\|\mathbf{A}(\mathbf{X})\|$, both of which grow without bound along unbounded strain sequences. As a direct consequence, it naturally follows that

$$\|\widehat{\boldsymbol{\sigma}}(\mathbf{X})\| = K(\mathbf{X})\,\|\mathbf{A}(\mathbf{X})\| = \mu\,\mathrm{e}^{cQ(\mathbf{X})-s}\,\|\mathbf{A}(\mathbf{X})\| \to +\infty \quad \text{as} \;\; \|\mathbf{X}\| \to +\infty. \tag{123}$$

From the above relation, it becomes evident that when the strain sequence is unbounded, the corresponding stress sequence must also be unbounded. By definition, a continuous mapping is termed proper if the pre-image of any compact set is compact. In finite-dimensional spaces, this topological requirement is entirely equivalent to the aforementioned coercivity condition. Therefore, combined with the continuity of $\widehat{\boldsymbol{\sigma}}$, it is directly established that $\widehat{\boldsymbol{\sigma}}$ is a proper map. Physically, this precludes the phenomenon of strain escape, wherein a finite stress state corresponds to infinite distortion or volumetric collapse.

9.4. Global inversion

In Section 8, we established that the local tangent of $\widehat{\boldsymbol{\sigma}}$ is strictly point-wise positive definite. This fundamental property immediately ensures that its derivative $D\widehat{\boldsymbol{\sigma}}(\mathbf{X})$ is invertible everywhere. To see this, we note that the strict positive definiteness dictates

$$D\widehat{\boldsymbol{\sigma}}(\mathbf{X})[\mathbf{H}] : \mathbf{H} = 0 \quad \text{only if} \;\; \mathbf{H} = \mathbf{0} \quad \Longrightarrow \quad \ker D\widehat{\boldsymbol{\sigma}}(\mathbf{X}) = \{\mathbf{0}\}. \tag{124}$$

Since the domain and co-domain of $D\widehat{\boldsymbol{\sigma}}(\mathbf{X})$ have the same finite dimension $n\,[n+1]/2$, its injectivity (trivial kernel) guarantees its surjectivity. Consequently, $D\widehat{\boldsymbol{\sigma}}(\mathbf{X})$ is bijective and invertible at every point $\mathbf{X}$. By the inverse function theorem, $\widehat{\boldsymbol{\sigma}}$ is a local C^∞ diffeomorphism. In particular, every local diffeomorphism is an open map, so $\mathcal{S} = \widehat{\boldsymbol{\sigma}}(\mathrm{Sym}(n))$ is an open set in $\mathrm{Sym}(n)$.

Next, we prove that $\mathcal{S}$ is also a closed set. Let $\mathbf{T}_k \in \mathcal{S}$ and suppose that $\mathbf{T}_k \to \mathbf{T} \in \mathrm{Sym}(n)$. Then choose $\mathbf{X}_k$ such that $\widehat{\boldsymbol{\sigma}}(\mathbf{X}_k) = \mathbf{T}_k$. Clearly, the set $C = \{\mathbf{T}\} \cup \{\mathbf{T}_k : k \in \mathbb{N}\}$ is compact. Since $\widehat{\boldsymbol{\sigma}}$ is proper, it follows that $\widehat{\boldsymbol{\sigma}}^{-1}(C)$ is a compact set. Hence $\{\mathbf{X}_k\}$ has a convergent subsequence $\mathbf{X}_{k_j} \to \mathbf{X}$. At the same time, the continuity implies that

$$\widehat{\boldsymbol{\sigma}}(\mathbf{X}) = \widehat{\boldsymbol{\sigma}}(\lim_{j\to\infty} \mathbf{X}_{k_j}) = \lim_{j\to\infty} \widehat{\boldsymbol{\sigma}}(\mathbf{X}_{k_j}) = \lim_{j\to\infty} \mathbf{T}_{k_j} = \mathbf{T}. \tag{125}$$

Therefore, $\mathbf{T} \in \mathcal{S}$, which proves that $\mathcal{S}$ is a closed set. The space $\mathrm{Sym}(n)$ is a real vector space and is therefore connected. Since the stress image set $\mathcal{S}$ is nonempty, open, and closed, it must be equal to the entire $\mathrm{Sym}(n)$, implying that $\widehat{\boldsymbol{\sigma}}$ is surjective. Combining the surjectivity of $\widehat{\boldsymbol{\sigma}}$ with the injectivity implied by strict monotonicity, we immediately see that $\widehat{\boldsymbol{\sigma}}$ is bijective. Its local smooth inverses, supplied by the inverse function theorem, agree on overlaps because of global injectivity. Hence,

$$\widehat{\boldsymbol{\sigma}} : \mathrm{Sym}(n) \to \mathrm{Sym}(n) \;\; \text{is a global } C^\infty \text{ diffeomorphism.} \tag{126}$$

From a mechanical standpoint, this conclusion simultaneously rules out constitutive multivalence and stress inaccessibility: every symmetric true stress corresponds to a unique true strain. Furthermore, the matrix exponential and the principal matrix logarithm are globally smooth inverse maps of each other between $\mathrm{Sym}(n)$ and $\mathrm{Sym}^{++}(n)$, namely

$$\exp : \mathrm{Sym}(n) \to \mathrm{Sym}^{++}(n), \quad \log : \mathrm{Sym}^{++}(n) \to \mathrm{Sym}(n). \tag{127}$$

The primitive constitutive mapping from $\mathbf{V}$ to $\boldsymbol{\sigma}$ is the composition of two global diffeomorphisms

$$\boldsymbol{\sigma}(\mathbf{V}) = \widehat{\boldsymbol{\sigma}}(\log \mathbf{V}) = (\widehat{\boldsymbol{\sigma}} \circ \log)(\mathbf{V}), \tag{128}$$

which is a global diffeomorphism itself, that is

$$\boldsymbol{\sigma} : \mathrm{Sym}^{++}(n) \to \mathrm{Sym}(n) \;\; \text{is a global } C^\infty \text{ diffeomorphism.} \tag{129}$$

Its inverse mapping has a clear two-step structure: first, the logarithmic strain is uniquely determined from a given stress, and then the positive-definite left stretch Tensor is recovered using the matrix exponential:

$$\boldsymbol{\sigma}^{-1}(\mathbf{T}) = \exp(\widehat{\boldsymbol{\sigma}}^{-1}(\mathbf{T})), \quad \mathbf{T} \in \mathrm{Sym}(n). \tag{130}$$

Finally, we emphasize that the above conclusion concerns the physically relevant map $\mathbf{V} \mapsto \boldsymbol{\sigma}(\mathbf{V})$. The mapping $\mathbf{F} \mapsto \boldsymbol{\sigma}(\mathbf{F})$ cannot be injective. This is because, for any $\mathbf{R} \in \mathrm{SO}(n)$, the deformation gradient $\mathbf{F}$ and $\mathbf{FR}$ have the same left stretch tensor $\mathbf{V}$, and thus produce the same Cauchy stress in isotropic materials.

10. Conclusions

We have proposed an idealized compressible isotropic hyperelastic energy density suitable for both two-dimensional and three-dimensional problems. The energy is smooth and finite over the entire $\mathrm{GL}^+(n)$ and is objective and isotropic, always non-negative and only in a purely rotational state vanishing. The model contains only two independent material parameters, which correspond to an arbitrarily given positive shear modulus and bulk modulus at the natural state, respectively. By constructing a jointly convex extension of the deformation gradient and its determinant, this paper proves the polyconvexity of this energy and thereby achieves rank-one convexity. This result ensures that the model has an appropriate variational structure. However, since polyconvexity alone is not sufficient to guarantee that the Cauchy stress increases monotonically with true strain, we further prove that the self-adjoint symmetric part of the complete logarithmic stress tangent is strictly positive definite everywhere. Strict monotonicity ensures that different logarithmic strain states cannot produce the same Cauchy stress. Combining the supercoercivity of the auxiliary potential with the properness of the stress mapping, we further prove that both the logarithmic strain-Cauchy stress mapping and the left stretch-Cauchy stress mapping are globally smooth diffeomorphisms. Accordingly, every symmetric Cauchy stress corresponds to a unique positive-definite left-stretching state. We believe, this contribution sheds light on Truesdell's Hauptproblem and demonstrates that variational stability of energy, constitutive monotonicity under finite strains, and global reversibility of the stress mapping can all be achieved simultaneously within a single, concise model.

Acknowledgements

AJ gratefully acknowledges the support from The Scientific and Technological Research Council of Türkiye (TÜBITAK), grant number 224M144. CL gratefully acknowledges support from the Alexander von Humboldt Foundation.

Declaration of generative AI and AI-assisted technologies

During the preparation of this work, ChatGPT (OpenAI) was used to check the proof of properness and global invertibility of the stress map. The authors carefully verified all the results and take full responsibility for the originality, integrity, and content of this paper.

References

Anand, L., 1979. On H. Hencky's approximate strain-energy function for moderate deformations. Journal of Applied Mechanics 46, 78–82.

Ball, J.M., 1977. Convexity conditions and existence theorems in nonlinear elasticity. Archive for Rational Mechanics and Analysis 63, 337–403.

Ball, J.M., 2002. Some open problems in elasticity, in: Newton, P., Holmes, P., Weinstein, A. (Eds.), Geometry, Mechanics, and Dynamics. Springer, New York, pp. 3–59.

Ciarlet, P.G., 1988. Mathematical Elasticity. Volume I: Three-Dimensional Elasticity. North-Holland, Amsterdam.

Dacorogna, B., 2008. Direct Methods in the Calculus of Variations. 2nd ed., Springer, New York.

d'Agostino, M.V., Holthausen, S., Bernardini, D., Sky, A., Ghiba, I.D., Martin, R.J., Neff, P., 2025. A constitutive condition for idealized isotropic Cauchy elasticity involving the logarithmic strain. Journal of Elasticity 157.

Ghiba, I.D., Martin, R.J., Neff, P., 2018. Rank-one convexity implies polyconvexity in isotropic planar incompressible elasticity. Journal de Mathématiques Pures et Appliquées 116, 88–104.

Ghiba, I.D., Neff, P., Šilhavý, M., 2015. The exponentiated Hencky-logarithmic strain energy. Improvement of planar polyconvexity. International Journal of Non-Linear Mechanics 71, 48–51.

Hartmann, S., Neff, P., 2003. Polyconvexity of generalized polynomial-type hyperelastic strain energy functions for near-incompressibility. International Journal of Solids and Structures 40, 2767–2791.

Hencky, H., 1928. Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen. Zeitschrift für technische Physik 9, 215–220.

Hill, R., 1968. On constitutive inequalities for simple materials—I. Journal of the Mechanics and Physics of Solids 16, 229–242.

Hill, R., 1970. Constitutive inequalities for isotropic elastic solids under finite strain. Proceedings of the Royal Society of London. A 314, 457–472.

Javili, A., dell'Isola, F., Steinmann, P., 2013. Geometrically nonlinear higher-gradient elasticity with energetic boundaries. Journal of the Mechanics and Physics of Solids 61, 2381–2401.

Jog, C.S., Patil, K.D., 2013. Conditions for the onset of elastic and material instabilities in hyperelastic materials. Archive of Applied Mechanics 83, 661–684.

Klein, D.K., Fernández, M., Martin, R.J., Neff, P., Weeger, O., 2022. Polyconvex anisotropic hyperelasticity with neural networks. Journal of the Mechanics and Physics of Solids 159, 104703.

Leblond, J.B., 1992. A constitutive inequality for hyperelastic materials in finite strain. European Journal of Mechanics A/Solids 11, 447–466.

Martin, R.J., Ghiba, I.D., Neff, P., 2018. A non-ellipticity result, or the impossible taming of the logarithmic strain measure. International Journal of Non-Linear Mechanics 102, 147–158. ArXiv:1712.04846.

Morrey, C.B., 1952. Quasi-convexity and the lower semicontinuity of multiple integrals. Pacific Journal of Mathematics 2, 25–53.

Neff, P., 2024. Truesdell's Hauptproblem: on constitutive stability in idealized isotropic nonlinear elasticity and a 500 € challenge. Talk at the IUTAM Symposium "Computational Fracture Mechanics in Multi-Field Problems" in honor of the 70th birthday of Michael Ortiz, Physikzentrum Bad Honnef.

Neff, P., Eidel, B., Martin, R.J., 2016. Geometry of logarithmic strain measures in solid mechanics. Archive for Rational Mechanics and Analysis 222, 507–572.

Neff, P., Ghiba, I.D., Lankeit, J., 2015. The exponentiated Hencky-logarithmic strain energy. Part I: Constitutive issues and rank-one convexity. Journal of Elasticity 121, 143–234.

Neff, P., Holthausen, S., d'Agostino, M.V., Bernardini, D., Sky, A., Ghiba, I.D., Martin, R.J., 2025a. Hypo-elasticity, Cauchy-elasticity, corotational stability and monotonicity in the logarithmic strain. Journal of the Mechanics and Physics of Solids 202, 106074. ArXiv:2409.20051.

Neff, P., Holthausen, S., Korobeynikov, S.N., Ghiba, I.D., Martin, R.J., 2025b. A natural requirement for objective corotational rates—on structure preserving corotational rates. Acta Mechanica ArXiv:2409.19707.

Neff, P., Husemann, N.J., Tchakoutio, A.S.N., Korobeynikov, S.N., Martin, R.J., 2025c. The corotational stability postulate: positive incremental Cauchy stress moduli for diagonal, homogeneous deformations in isotropic nonlinear elasticity. International Journal of Non-Linear Mechanics 174, 105033.

Ogden, R.W., 1970. Compressible isotropic elastic solids under finite strain—constitutive inequalities. The Quarterly Journal of Mechanics and Applied Mathematics 23, 457–468.

Richter, H., 1948. Das isotrope Elastizitätsgesetz. Zeitschrift für Angewandte Mathematik und Mechanik 28, 205–209.

Schek, L., Lewintan, P., Müller, W., Muench, I., Zilian, A., Bordas, S.P.A., Neff, P., Sky, A., 2026. A structure-preserving discretisation of SO(3)-rotation fields for finite Cosserat micropolar elasticity. Computer Methods in Applied Mechanics and Engineering 461, 119249.

Schröder, J., Neff, P., 2003. Invariant formulation of hyperelastic transverse isotropy based on polyconvex free energy functions. International Journal of Solids and Structures 40, 401–445.

Šilhavý, M., 1997. The Mechanics and Thermodynamics of Continuous Media. Springer, Berlin.

Truesdell, C., 1956. Das ungelöste Hauptproblem der endlichen Elastizitätstheorie. Zeitschrift für Angewandte Mathematik und Mechanik 36, 97–103.

Truesdell, C., Noll, W., 2004. The Non-Linear Field Theories of Mechanics. 3rd ed., Springer, Berlin.

Wollner, M.P., Holzapfel, G.A., Neff, P., 2026a. In search of constitutive conditions in isotropic hyperelasticity: polyconvexity versus true-stress-true-strain monotonicity. Journal of the Mechanics and Physics of Solids 209, 106465.

Wollner, M.P., Klein, D.K., Baaser, H., Holzapfel, G.A., Neff, P., 2026b. Concurrent enforcement of polyconvexity and true-stress-true-strain monotonicity in incompressible isotropic hyperelasticity: application to neural network constitutive models. arXiv preprint ArXiv:2605.20031.

Xiao, H., Bruhns, O.T., Meyers, A., 1997. Logarithmic strain, logarithmic spin and logarithmic rate. Acta Mechanica 124, 89–105.

Xie, J., Javili, A., Linder, C., 2025. The kirsch problem in second strain gradient elasticity. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 481, 20250389.

Xie, J., Javili, A., Linder, C., 2026. Mechanics of defects in flexoelectric semiconductors. Journal of the Mechanics and Physics of Solids 215, 106743.