

A local solution of Ericksen's problem

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Abstract

Ericksen's problem asks for all universal deformations of homogeneous incompressible isotropic hyperelastic solids [3]. Such deformations can be maintained in equilibrium, without body forces, for every isotropic strain energy function, while the pressure and surface tractions may depend on the constitutive law. Six classical families are known, but their completeness has remained open for three distinct constant principal stretches with unrestricted principal directions [30, 33]. We prove that every orientation preserving C^3 incompressible local universal deformation in three dimensions belongs locally to one of these families, including their regular Cartesian continuations.

The main step is to prove that, for constant pairwise distinct principal stretches, the plane field orthogonal to one principal direction is locally integrable. Using the principal frame formulation [14, 15], we reduce the universal equilibrium and compatibility equations to a finite differential system. Removing the length scale yields a normalized state map whose fibres of positive dimension induce common local Euclidean similarities of the reference and current configurations, with the same dilation factor. On a hypothetical region where all three principal abnormalities are nonzero, analysis of these similarities and the frame equations excludes ranks zero, one and two. For rank three, we bound the dimension of the Zariski closure of the state image by two, using estimates for the generic fibres of the projection onto the two abnormality ratios.

Real analyticity follows from neo-Hookean equilibrium. It selects a fixed principal label whose orthogonal plane field is integrable throughout a neighbourhood, and permits continuation of the classical deformations through critical points of the principal invariant map and removable coordinate singularities. Kafadar's theorem [11] and the earlier classifications [16–18] then establish the completeness of the six classical families and resolve the local form of Ericksen's problem.

Keywords. Universal deformation, Incompressible isotropic elasticity, Ericksen problem

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1 Introduction

Ericksen [3] asked which deformations of a homogeneous incompressible isotropic elastic solid can be maintained in equilibrium, without body forces, for every isotropic strain energy function. These are the universal deformations. The pressure and surface tractions may depend on the constitutive law. Material homogeneity means that this law is independent of position. The classical examples also provide settings for constitutive experiments [24, 25].

Rivlin's work on finite bending and torsion [21–23] preceded Ericksen's systematic classification. For unconstrained compressible isotropic solids, Ericksen [4] proved that every universal deformation is homogeneous. For incompressible solids, his earlier paper [3] gave four inhomogeneous families in addition to the homogeneous family. Singh and Pipkin [26] subsequently found the logarithmic cylindrical family, and Klingbeil and Shield [12] obtained a plane special case. Fosdick [5] observed that a known cylindrical deformation has constant principal invariants despite its inhomogeneity. The six classical families therefore include deformations with constant principal stretches and a spatially varying deformation gradient.

Ericksen's classification left two cases to be considered: repeated principal values and constant principal invariants. The results of Marris and Shiau [16, 17] and Martin and Carlson [18], together with Ericksen's analysis, settle the classification on regular patches where at least one principal invariant varies. Kafadar [11] treated constant repeated principal values. What remains is the case of three distinct constant principal stretches, with no restriction on their principal directions.

Fixing the stretches leaves the principal frame free to vary in space. Kafadar [11] proved that a deformation in this remaining case is homogeneous or logarithmic cylindrical if one principal direction is normal to a local family of surfaces. The condition is equivalent to the vanishing of one of the abnormalities of the principal frame. The missing step is therefore to show that the universal equilibrium and compatibility equations force one of the principal abnormalities to vanish.

Marris [14] studied these equations through the curvatures and abnormalities of the principal frame. He excluded additional solutions when two abnormalities are constant. His 1982 paper [15, Main Theorems 1 and 2] established a homogeneous polynomial relation among the principal value ratios and the frame curvatures and abnormalities, and ruled out additional solutions when the abnormality ratios are constant. The concluding discussion [15, pp. 171–172] isolates the difficulty of obtaining independent differential restrictions in the presence of common polynomial factors.

Other reductions imposed a particular deformation geometry: plane deformations with uniform transverse stretch in Fosdick and Schuler [7], radial symmetry in Fosdick [6], and finite anti-plane shear in Knowles [13]. Hill and Arrigo [10] later reduced a constant invariant problem to coupled nonlinear partial differential equations. An account of these questions within the theory of universal solutions and relations is given by Saccomandi [25].

More recent work has considered anelasticity [9, 31], anisotropy [32], and material inhomogeneity [28, 33]. Goodbrake, Yavari and Goriely [9] formulated the problem geometrically and identified the Lie subgroup symmetries of the six known families. The universal deformations of isotropic Cauchy elasticity and hyperelasticity were shown to coincide by Yavari [29]. Related studies concern implicit constitutive equations [34], the geometry of nonlinear Cauchy elasticity [35], and residual stress [30, 36]. Motaghian and Yavari [20] extended the equivalence of universality constraints to transversely isotropic, orthotropic and monoclinic Cauchy elastic solids.

We prove that every C^3 universal deformation with three distinct constant principal stretches has, near each point, a fixed principal direction normal to a family of surfaces (Theorem 2.1). Kafadar's theorem [11] then gives the homogeneous and logarithmic cylindrical families. Combined with the earlier classifications [3, 16–18], this establishes the local C^3 classification into the six classical families (Theorem 2.2).

The proof starts from the principal frame equations [14, 15]. We adjoin the derivatives needed to close these equations and use their weighted homogeneity to remove the length scale. This gives a normalized state map on a hypothetical region where all three abnormalities are nonzero. Its fibres determine common local Euclidean similarities of the reference and current configurations, with the same dilation factor (Proposition 3.4). Ranks zero and one yield at least two independent infinitesimal similarities; rank two yields one. Their classification and the frame equations exclude all three ranks.

At rank three, the normalized state distinguishes nearby points. We estimate the dimension of its Zariski closure by projection onto the two abnormality ratios. Their image cannot be a point. Over the whole ratio plane the generic fibre is finite; over a ratio curve it has dimension at most one. In either case the state image has dimension at most two (Theorem 8.1), contradicting rank three. The fibre estimates use a weighted projective completion. Normal compatibility equations exclude a boundary point when every incident arc is forced to remain at infinity. On the remaining boundary family, the induced differential equations have no algebraic solutions.

Equilibrium for a neo-Hookean material gives a properly elliptic system, so analytic interior regularity [19, Theorem 6.8.1] applies to the C^3 deformation. Analyticity selects a fixed principal label with an integrable orthogonal plane field throughout a neighbourhood. It also continues the classical models through critical points of the principal invariant map and removable coordinate singularities (Proposition 9.6). This includes their regular Cartesian continuations. Direct equilibrium calculations for arbitrary strain energies prove the converse.

Organization of the paper. Section 2 states the main results, records the six classical models, and derives the geometric equations. Section 3 constructs the normalized state map and identifies its fibres with common similarities. Section 4 excludes ranks zero and one; Sections 5–7 exclude rank two. Section 8 proves the algebraic dimension bound, and Section 9 completes the classification, continuation and converse arguments. Section 10 discusses global realizations and the classification of material preferred directions. Appendices A–E contain the supporting geometric and algebraic arguments. The polynomial coefficient matrix for the linear equations is given in Supplement S8.

2 Universal deformations and the geometric equations

2.1 Universal deformations and the main results

Let $\Omega \subset \mathbb{R}^3$ be a Euclidean domain and let $\chi : \Omega \rightarrow \mathbb{R}^3$ be an orientation preserving C^3 local diffeomorphism. Write

$$F = D\chi, \quad \det F = 1, \quad B = FF^\top, \quad c = B^{-1}, \quad I_1 = \operatorname{tr} B, \quad I_2 = \operatorname{tr} c. \quad (2.1)$$

Let g denote the spatial Euclidean metric and vol_g its oriented volume form. Here B is the left Cauchy–Green tensor and c is the spatial metric induced by the inverse deformation, called the Cauchy deformation tensor in [11]. Spatial tensors are defined on each local inverse chart. Their invariants are pulled back to Ω when a statement concerns a connected reference domain. The principal values of B are its eigenvalues and are the squares of the principal stretches. Marris and Shiau use the term “proper values” for eigenvalues [16]. The principal values of c are their reciprocals. Positivity, constancy and pairwise distinctness are equivalent for these two triples of values and for the principal stretches.

The deformation is *universal* if, for every sufficiently smooth strain energy $W(I_1, I_2)$, there is a local pressure p_W such that

$$T_W = -p_W I + 2W_1 B - 2W_2 c, \quad \operatorname{div} T_W = 0. \quad (2.2)$$

Body forces are absent. The pressure and the required surface traction may depend on W . Here $W_i = \partial W / \partial I_i$, and all derivatives in (2.2) are spatial.

Following the principal frame formulation in [14, 15], choose on a chart with distinct principal values an oriented orthonormal principal frame (e_1, e_2, e_3) with dual coframe $(\alpha_1, \alpha_2, \alpha_3)$. Write $h_i = e_i \cdot \text{curl } e_i$ for the abnormality of e_i [14, p. 112]. The identity $\alpha_i \wedge d\alpha_i = h_i \text{vol}_g$ shows that $h_i = 0$ precisely when the two dimensional plane field orthogonal to e_i is integrable. Equivalently, e_i is normal to a local family of surfaces. All references to integrability below concern this plane field.

Theorem 2.1 (A principal direction normal to surfaces). *Let a C^3 local universal deformation have constant, positive, pairwise distinct principal values. Every spatial point has a connected ball neighbourhood on which a fixed principal label i satisfies*

$$h_i \equiv 0.$$

Equivalently, that principal direction is normal to a local family of surfaces.

This theorem supplies Kafadar's geometric hypothesis. Its proof occupies Sections 2–8 and is concluded in Section 9. Together with the classical classifications it gives the following theorem. Here $\mathcal{F}_0$ denotes the homogeneous family and $\mathcal{F}_5$ the logarithmic cylindrical family. All six models are defined in the next subsection.

Theorem 2.2 (Local classification). *Every C^3 , orientation preserving, incompressible local universal deformation in three dimensions is locally one of $\mathcal{F}_0, \dots, \mathcal{F}_5$ with the parameter restrictions in Section 2.2, including their regular Cartesian continuations. Conversely, each listed model is locally universal wherever it is a regular deformation.*

2.2 The classical deformations

The models $\mathcal{F}_0, \dots, \mathcal{F}_4$ are those of Ericksen [3]. The logarithmic cylindrical model $\mathcal{F}_5$ is due to Singh and Pipkin [26]; Klingbeil and Shield [12] obtained a plane special case. We use the conventional mechanical descriptions in [33, §1]. Uppercase cylindrical coordinates (R, Θ, Z) refer to the reference configuration, and lowercase coordinates (r, θ, z) to the current configuration. Angles are local coordinates on a sector. Fixed Euclidean coordinate changes and additive coordinate constants are understood. The parameter restrictions below impose the signed Jacobian $\det D\chi = 1$.

$\mathcal{F}_0$. Homogeneous deformations:

$$\chi(X) = LX + b_0, \quad L \in \text{SL}(3, \mathbb{R}). \quad (2.3)$$

$\mathcal{F}_1$. Bending, stretching and shearing of a rectangular block:

$$r^2 = 2\alpha X + d, \quad \theta = \beta Y + \theta_0, \quad z = (\alpha\beta)^{-1}Z + \gamma Y + z_0, \quad \alpha\beta \neq 0. \quad (2.4)$$

The displayed cylindrical chart requires $r > 0$.

$\mathcal{F}_2$. Straightening, stretching and shearing of a sector of a cylindrical shell:

$$\begin{aligned} x &= \frac{\alpha}{2}R^2 + x_0, & y &= \beta\Theta + y_0, \\ z &= (\alpha\beta)^{-1}Z + \gamma\Theta + z_0, & \alpha\beta &\neq 0, \quad R > 0. \end{aligned} \quad (2.5)$$

$\mathcal{F}_3$. Inflation, bending, torsion, extension and shearing of an annular wedge:

$$\begin{aligned} r^2 &= AR^2 + B_0, & \theta &= k\Theta + \delta Z + \theta_0, \\ z &= \kappa\Theta + \gamma Z + z_0, & A(k\gamma - \delta\kappa) &= 1, \end{aligned} \quad R, r > 0. \quad (2.6)$$

All parameters in this model are constants; A is its radial coefficient.

$\mathcal{F}_4$. Inflation and inversion of a spherical shell, including reversal of the radial direction with the compensating angular orientation:

$$\begin{aligned}\chi(P + Rn) &= Q_0 + r(R)Qn, & r^3 &= \varepsilon R^3 + B_0, \\ \varepsilon &= \det Q, & Q &\in \text{O}(3), \quad |n| = 1.\end{aligned}\tag{2.7}$$

Both radii are positive in this expression.

$\mathcal{F}_5$. Inflation, bending, extension and azimuthal shearing of an annular wedge, with constant invariants (the logarithmic cylindrical family):

$$\begin{aligned}r &= \alpha R, & \theta &= \beta \log R + k\Theta + \theta_0, \\ z &= \gamma Z + z_0, & \alpha > 0, \quad \alpha^2 k \gamma &= 1, \quad R > 0.\end{aligned}\tag{2.8}$$

Its plane deformation gradient in orthonormal cylindrical bases has the constant block $\alpha \begin{pmatrix} 1 & 0 \\ \beta & k \end{pmatrix}$. Consequently the principal stretches are constant although the Cartesian gradient usually is not. Either planar value may coincide with the axial value.

The families overlap at special parameters. At an interior axis or centre, a model means its regular Cartesian continuation when one exists. The continuations are described in Section 9.5.

The remainder of this section develops the geometric equations needed to prove Theorem 2.1. The principal values are fixed, while the principal frame may vary.

2.3 Equilibrium for constant principal values

When the principal invariants are constant, W_1, W_2 are spatial constants. Taking $W = I_1/2$ and $W = I_2/2$ gives the necessary conditions

$$d(\text{div } B)^b = d(\text{div } c)^b = 0.\tag{2.9}$$

Conversely, linear combinations of the local pressure potentials in (2.9) give equilibrium for every strain energy W . Thus these conditions are also sufficient. The inverse deformation $f = \chi^{-1}$ satisfies $c = f^* g_{\text{ref}}$. Conversely, a positive flat metric c of determinant one has an orientation preserving local Euclidean development f , with $\det Df = 1$. Thus the local problem with constant invariants is

$$c > 0, \quad \det c = 1, \quad \text{spec}_g c \text{ constant}, \quad \text{Riem}(c) = 0, \tag{2.9}.\tag{2.10}$$

Flatness is a compatibility equation independent of the two pressure equations.

2.4 Interior analyticity

Universality includes equilibrium for the neo-Hookean energy. The following interior regularity result therefore applies.

Lemma 2.3. *Every C^3 incompressible local diffeomorphism in equilibrium for the neo-Hookean energy is real analytic. In particular, a C^3 universal deformation and each of its local inverses are real analytic.*

Proof. Use reference Cartesian coordinates X_A , put $A_\chi = \text{cof } F$, and select the energy $W(F) = |F|^2/2$. The spatial pressure p_W has pullback $p_W \circ \chi$ to the reference domain. The Piola identity gives

$$\Delta \chi_i = (A_\chi)_{iA} \partial_A (p_W \circ \chi), \quad \det F = 1, \quad \nabla (p_W \circ \chi) = F^\top \Delta \chi.\tag{2.11}$$

Since $\chi \in C^3$, the last expression is C^1 and $p_W \circ \chi$ is C^2 . Applying the reference Laplacian to $\det F = 1$ and substituting the first equation gives

$$(\mathbf{A}_\chi^\top \mathbf{A}_\chi)_{AB} \partial_{AB}(p_W \circ \chi) + (\mathbf{A}_\chi)_{iA} (\partial_A (\mathbf{A}_\chi)_{iB}) \partial_B (p_W \circ \chi) + \sum_C D^2 \det(F) [\partial_C F, \partial_C F] = 0. \quad (2.12)$$

The three equations for χ and (2.12) form a second-order nonlinear analytic system for $\chi, p_W \circ \chi$, with coefficients polynomial in their first and second derivatives.

The principal symbol of its linearization has the triangular form

$$L(\xi) = \begin{pmatrix} |\xi|^2 I_3 & 0 \\ K(\xi) & \xi^\top \mathbf{A}_\chi^\top \mathbf{A}_\chi \xi \end{pmatrix}, \quad \det L(\xi) = |\xi|^6 |\mathbf{A}_\chi \xi|^2 > 0 \quad (\xi \neq 0).$$

The row K consists of the second-order variations of χ in the pressure equation. For independent real ξ, η , each of the positive quadratic factors in $\det L(\xi + z\eta)$ has one root in each half-plane. Counting multiplicity gives four roots in each half-plane, so the system is properly elliptic. Take equation weights zero and unknown weights two. The solution $(\chi, p_W \circ \chi)$ is C^2 , and the equations are polynomial in its derivatives. Morrey's interior regularity theorem [19, Theorem 6.8.1, pp. 277–278] first gives smoothness and, in its analytic case, real analyticity of $\chi, p_W \circ \chi$. The analytic inverse function theorem makes each local inverse analytic, and composition with it gives analyticity of the spatial pressure p_W . $\square$

On the reference domain the invariants are represented by $|F|^2$ and $|\mathbf{A}_\chi|^2$, so they are analytic. On a connected domain, constancy on a nonempty open subset extends to the whole domain by the analytic identity theorem.

2.5 The equations in a principal frame

We use the principal frame formulation of Marris [14, 15]. We now restrict to constant, positive, pairwise distinct principal values. Normalize the eigenvalues of B by a common constant factor to $(1, a, b)$, where

$$a, b > 0, \quad (a - 1)(b - 1)(a - b) \neq 0. \quad (2.13)$$

The physical eigenvalues of B , with product one, are $(ab)^{-1/3}(1, a, b)$. Constant rescaling does not affect flatness or closedness of tensor divergence, so the metric weight vectors are $(1, 1, 1)$ and $(1, 1/a, 1/b)$. Write λ_i for the principal values of the physical metric $c = B^{-1}$; thus

$$(\lambda_1, \lambda_2, \lambda_3) = (ab)^{1/3}(1, a^{-1}, b^{-1}), \quad c = \sum_i \lambda_i \alpha_i^2.$$

All orderings of the principal values are allowed. The parameters a, b are fixed ratios of principal values.

Choose the oriented principal frame and its dual coframe as in Section 2.1, and put $P_i = e_i \otimes e_i$. Define the frame coefficients and directional derivatives by

$$p_{ij} = e_j \cdot \operatorname{curl} e_i, \quad h_i = p_{ii}, \quad D_i = e_i \cdot \nabla. \quad (2.14)$$

The matrix whose rows are $(1, 1, 1)$, $(1, a, b)$ and $(1, 1/a, 1/b)$ is invertible on (2.13). Since $\sum P_i = I$, the two pressure equations are equivalent to

$$\operatorname{div} P_i = \nabla \phi_i, \quad \sum_i \phi_i = 0. \quad (2.15)$$

Indeed, its determinant is a nonzero Vandermonde factor divided by ab . For a unit vector field e ,

$$\operatorname{div}(e \otimes e) = (\operatorname{div} e)e - e \times \operatorname{curl} e.$$

Its component perpendicular to e_i therefore gives

$$\begin{aligned}\text{curl } e_i &= h_i e_i + e_i \times \nabla \phi_i, & \vartheta_i &= e^{\phi_i} \alpha_i, \\ \kappa_i &= e^{2\phi_i} h_i, & d\vartheta_i &= \kappa_i \vartheta_j \wedge \vartheta_k\end{aligned}\tag{2.16}$$

for cyclic (i, j, k) , with $\alpha_i \wedge d\alpha_i = h_i \text{vol}_g$.

To combine the pressure equations with flatness, we express both systems in the same principal frame [15]. Let ϵ_{ijk} be the alternating symbol. The structure coefficients and the Levi-Civita connection of a constant diagonal frame metric of weights w are

$$C_{ij}^k = -\sum_{\ell} \epsilon_{ij\ell} p_{k\ell}, \quad \Gamma_{ij}^k(w) = \frac{w_k C_{ij}^k - w_i C_{jk}^i + w_j C_{ki}^j}{2w_k}.\tag{2.17}$$

With the first lower index indicating differentiation, the curvature is

$$R_{kij}^{\ell}(w) = D_i \Gamma_{jk}^{\ell} - D_j \Gamma_{ik}^{\ell} + \sum_m (\Gamma_{jk}^m \Gamma_{im}^{\ell} - \Gamma_{ik}^m \Gamma_{jm}^{\ell} - C_{ij}^m \Gamma_{mk}^{\ell}).\tag{2.18}$$

For w constant define

$$A_i(w) = (w_i - w_j) p_{jk} - (w_i - w_k) p_{kj} \quad ((i, j, k) \text{ cyclic}).$$

These are the frame components of $\text{div}(\sum w_i P_i)$, and its curl components are

$$D_j A_k(w) - D_k A_j(w) + \sum_{\ell} A_{\ell}(w) p_{\ell i}.\tag{2.19}$$

2.6 Finite closure of the frame equations

The curvature expressions (2.18) and the pressure expressions (2.19) are linear in the first derivatives of p_{ij} , with quadratic terms in the coefficients themselves. Their combined linear system therefore determines which derivatives must be retained as independent state coordinates. Commutator identities then determine the additional derivatives needed for closure.

The curvature and pressure equations determine the derivatives of the nine frame coefficients up to six functions. Two prolongations determine their further derivatives up to two additional functions. The resulting system has seventeen coordinates. The construction and residual equations are given in Appendix A.2.

The six first derivative coordinates are

$$(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5) = (D_2 p_{23}, D_3 p_{13}, D_3 p_{22}, D_3 p_{23}, D_3 p_{31}, D_3 p_{32}).\tag{2.20}$$

The two remaining coordinates are $u_0 = D_3 \tau_0$ and $u_1 = D_3 \tau_2$. The state is

$$S = (p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23}, p_{31}, p_{32}, p_{33}, \tau_0, \dots, \tau_5, u_0, u_1).\tag{2.21}$$

We write $\mathbf{p} = (p_{ij})$, $\boldsymbol{\tau} = (\tau_0, \dots, \tau_5)^{\top}$ and $\mathbf{u} = (u_0, u_1)^{\top}$ for the three coordinate blocks, with the matrix entries ordered by rows as in (2.21).

The corresponding coordinate components of its three fields are denoted by $\mathcal{Q}_i^{p_{rs}}$, $\mathcal{Q}_i^{\tau_r}$ and $\mathcal{Q}_i^{u_r}$. They are the rational functions defined by the three successive linear systems in the appendix. The denominators are powers of

$$P_{\text{mat}} = ab(a-1)(b-1)(a-b).\tag{2.22}$$

They are nonzero throughout the principal value domain (2.13).

Proposition 2.4 (Finite differential system). *Every solution of (2.10) satisfies $D_i S = Q_i(S)$. With coordinate weights $(d_1, \dots, d_{17}) = (1^9, 2^6, 3^2)$, the component Q_i^ℓ has weight $d_\ell + 1$. The three cubic residuals, the twenty-one quartic residuals and the bracket defects, together with all their iterated derivatives under the Q_i , vanish on every solution.*

Proof. The first linear system is obtained from the two curvature tensors and the two pressure equations. The next two systems impose the commutator identities for the frame coefficients and the six coordinates τ_r . Their determinants are nonzero on (2.13); see (A.4), (A.6) and (A.7). Substitution into the unselected rows gives the stated residuals. Their weights follow from the curvature equations and the fact that directional differentiation increases weight by one. $\square$

Thus the state at one point, together with an initial principal frame, determines its evolution along each principal curve by an ordinary differential equation. Agreement between these evolutions in different directions is imposed by the residual equations and the bracket identities.

For a nonzero signed length scale ℓ , write

$$S^{[\ell]} = (\ell p, \ell^2 \tau, \ell^3 u). \quad (2.23)$$

Its coordinates are denoted by $p_{ij}^{[\ell]}$, $\tau_r^{[\ell]}$ and $u_r^{[\ell]}$. Coordinates without a superscript refer to the unscaled state.

Let $\mathcal{J}_{\text{raw}}$ be the ideal in $\mathbb{Q}[a, b, P_{\text{mat}}^{-1}][S]$ generated by the cubic and quartic residuals, the components of

$$B_{ij} = Q_i(Q_j) - Q_j(Q_i) - \sum_k C_{ij}^k Q_k, \quad (2.24)$$

and all their iterated derivatives under Q_i . Thus $Q_i \mathcal{J}_{\text{raw}} \subset \mathcal{J}_{\text{raw}}$. The coefficient ring is Noetherian, so this ideal has a finite generating family.

2.7 Reconstruction and linearization

Proposition 2.5. *Fix an admissible triple of principal values. Suppose a local real state set is invariant under the local flows of the three fields Q_i . On it, suppose the curvature expressions (2.18) vanish for $w = (1, 1, 1)$ and $w = (1, 1/a, 1/b)$, the pressure expressions (2.19) vanish for $w = (1, 1/a, 1/b)$ and $w = (1, a, b)$, and the bracket defects (2.24) vanish. Then each point of that set has a local frame realization and a corresponding incompressible universal deformation. It suffices that the state set be the real zero set of a finitely generated ideal preserved by all three derivations and containing these relations.*

Proof. Write $\Gamma_i(S) = (\Gamma_{ij}^k(1, 1, 1))_{k,j}$, so $\Gamma_i \in \mathfrak{so}(3)$, and set $\Gamma(v, S) = \sum_i v_i \Gamma_i(S)$. Let $\mathbf{E} \in \text{SO}(3)$ be the matrix whose columns are the principal vectors in Cartesian coordinates. On the ambient space $\mathbb{R}^{17} \times \text{SO}(3)$ define

$$L_i = Q_i + (\mathbf{E} \mapsto \mathbf{E} \Gamma_i), \quad \mathcal{L}_a = \sum_i \mathbf{E}_{ai} L_i.$$

The bracket identities and Euclidean flatness give $[L_i, L_j] = \sum C_{ij}^k L_k$. The torsion-free and metric compatibility identities

$$\Gamma_{ij}^k - \Gamma_{ji}^k = C_{ij}^k, \quad \Gamma_{ij}^k = -\Gamma_{ik}^j$$

then give $[\mathcal{L}_a, \mathcal{L}_b] = 0$ along the invariant state set by direct expansion. Composing their ambient flows from a point of this set with parameters x_a yields $S(x)$, $\mathbf{E}(x)$ and $e_i = \sum_a \mathbf{E}_{ai} \partial_{x_a}$. Since $\sum_a \mathbf{E}_{ai} \mathcal{L}_a = L_i$, one has $D_i S = Q_i$ and $D_i \mathbf{E} = \mathbf{E} \Gamma_i$. Its coframe satisfies $*d\alpha_i = \sum_j p_{ij} \alpha_j$, as required.

The second metric is positive and flat, and hence develops locally into Euclidean space. Restoring the constant determinant normalization gives the required inverse deformation; (2.9) supplies the pressure for every energy.

For the ideal formulation, let $\mathbf{q}$ be a finite vector of generators. Differential preservation gives $\mathcal{Q}_i \mathbf{q} = A_i(\mathbf{S}) \mathbf{q}$. Along each flow, $\mathbf{q}$ solves a homogeneous linear ordinary differential equation, so its zero set is invariant. The lifted flows preserve this zero set, where the Cartesian fields $\mathcal{L}_a$ commute by the same bracket identity. $\square$

Lemma 2.6 (Linearized reconstruction). *At a state of a solution, tangent vectors annihilating the differentials of $\mathcal{J}_{\text{raw}}$ correspond to germs satisfying the linearized frame equations and the equations for the two metrics, with initial frames and development origins fixed. At order m , states satisfying $\mathcal{J}_{\text{raw}}$ modulo ϵ^{m+1} correspond to formal frame realizations over $\mathbb{R}[\epsilon]/(\epsilon^{m+1})$.*

Proof. For first order work over $\mathbb{R}[\epsilon]/(\epsilon^2)$. A state $\mathbf{S} + \epsilon v$ annihilates every generator of $\mathcal{J}_{\text{raw}}$ precisely when the relations vanish at $\mathbf{S}$ and their differentials annihilate v . The lift and Cartesian combinations in the proof of Proposition 2.5 are algebraic operations on the state, frame, and their derivatives. The torsion, curvature, and bracket identities consequently make those Cartesian fields commute also modulo ϵ^2 . Their flows are constructed by solving the ordinary real flow and its inhomogeneous linear variational equations. Commutation supplies compatibility in all physical directions and therefore a germ of a linearized frame realization. Developing the second flat metric gives the linearized deformation. Conversely, differentiating a germ of the original equations annihilates every relation in $\mathcal{J}_{\text{raw}}$. Initial translations and rotations are the development gauges. Replacing ϵ^2 by ϵ^{m+1} and solving successive linear variational equations proves the order- m statement. $\square$

3 Normalized states and common similarities

On a solution with $h_1 h_2 h_3 \neq 0$, weighted homogeneity removes one scale and the residual equations determine five further coordinates of $\mathbf{S}$. The resulting eleven coordinates define the normalized state map $\mathbf{X}$.

3.1 The normalized state

Put

$$\Delta_a = a(b-1), \quad \Delta_b = b(a-1), \quad \Delta_{ab} = a-b.$$

The cubic residuals in Appendix B.1 are linear in the coordinates τ_r .

Define

$$\mu_1 = \Delta_{ab} \left(\frac{h_3}{\Delta_b} - \frac{h_2}{\Delta_a} \right), \quad \mu_2 = h_1 - \frac{\Delta_{ab} h_3}{\Delta_b}, \quad \mu_3 = -h_1 - \frac{\Delta_{ab} h_2}{\Delta_a}. \quad (3.1)$$

In the cubic coefficient matrix, the columns belonging to (τ_4, τ_5, τ_0) are respectively $(\mu_1, 0, 0)^\top$, $(0, \mu_2, 0)^\top$, $(0, 0, \mu_3)^\top$. Consequently these three coordinates have a unique rational reconstruction wherever $\mu_1 \mu_2 \mu_3 \neq 0$.

The remaining two coordinates are recovered by the combined left inverse (B.4). Its product with the coefficient matrix is $7\Delta_b h_1 h_2 \text{Id}_2$, where Id_2 is the two by two identity matrix. The construction is given in Appendix B.1.

Normalize the state coordinates by their weights. Write $\hat{p}_{ij} = p_{ij}/h_1$, $\hat{\tau}_j = \tau_j/h_1^2$, and $\hat{u}_j = u_j/h_1^3$, and put $\hat{h}_i = h_i/h_1$. Thus $\hat{p}_{ii} = \hat{h}_i$ and $\hat{p}_{11} = \hat{h}_1 = 1$. The normalized coefficient matrix is $\hat{\mathbf{p}} = (\hat{p}_{ij})$. The retained coordinate map is

$$\mathbf{X} = (\hat{p}_{12}, \hat{p}_{13}, \hat{p}_{21}, \hat{p}_{22}, \hat{p}_{23}, \hat{p}_{31}, \hat{p}_{32}, \hat{p}_{33}, \hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3). \quad (3.2)$$

These are eight normalized frame coefficients and the three retained derivative coordinates $\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3$. The map itself is smooth wherever $h_1 \neq 0$. On the domain

$$h_1 h_2 h_3 \mu_1 \mu_2 \mu_3 \neq 0 \quad (3.3)$$

the preceding triangular reconstruction defines all normalized coordinates $\hat{S}_\ell = S_\ell / h_1^{d_\ell}$ as rational functions of $\mathbf{X}$. The spatial scale is h_1 .

Let $E_i = D_i / h_1$ and define

$$s_i(\mathbf{X}) = \mathcal{Q}_i^{p_{11}}(\hat{\mathbf{S}}(\mathbf{X})).$$

For each retained coordinate $\hat{S}_\ell$ in (3.2), the corresponding component of $\mathcal{V}_i$ is

$$\mathcal{Q}_i^\ell(\hat{\mathbf{S}}(\mathbf{X})) - d_\ell \hat{S}_\ell(\mathbf{X}) s_i(\mathbf{X}). \quad (3.4)$$

Weighted homogeneity gives, on every solution,

$$E_i \mathbf{X} = \mathcal{V}_i(\mathbf{X}), \quad E_i \log |h_1| = s_i(\mathbf{X}), \quad [E_i, E_j] = \sum_k \tilde{C}_{ij}^k(\mathbf{X}) E_k, \quad (3.5)$$

where $\tilde{C}_{ij}^k = C_{ij}^k / h_1 - s_i \delta_{jk} + s_j \delta_{ik}$.

Figure 1 shows these coordinate maps. The isometry states in Section 6 augment $\mathbf{S}$ with prescribed motion data. The rank three reduction in Section 8 uses an invertible rational change of the retained coordinates.

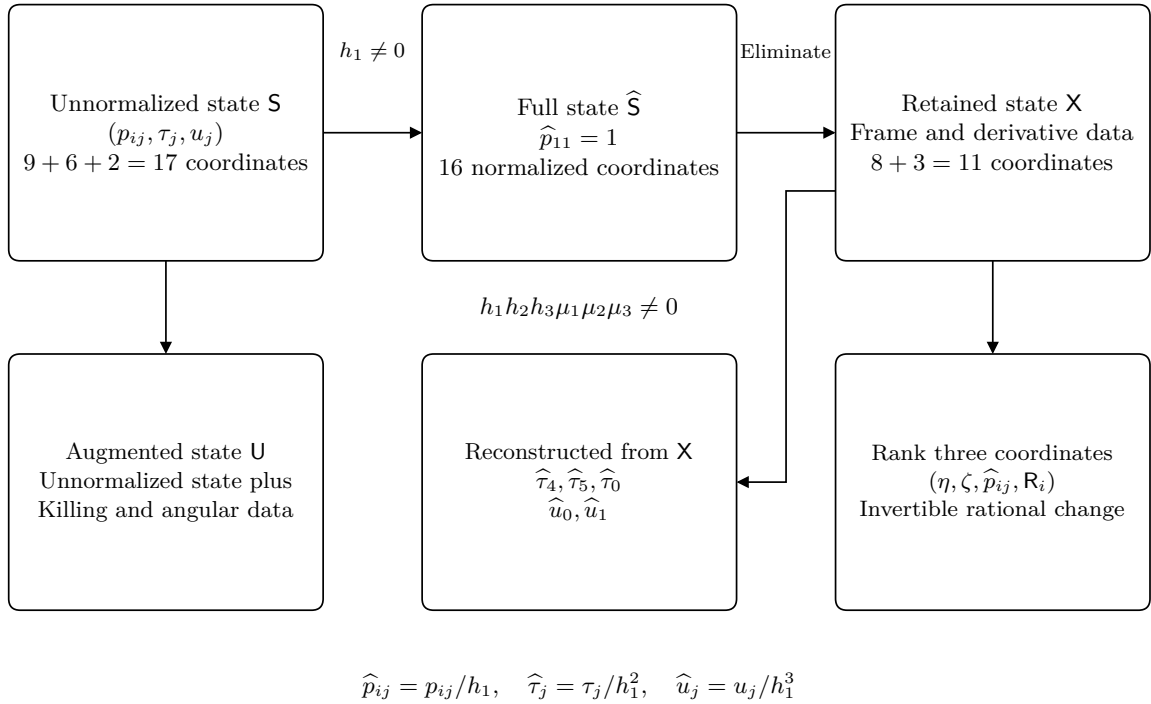


Figure 1: The state coordinates under weighted normalization, reconstruction and the rational change in Section 8. The augmented state $\mathbf{U}$ incorporates Killing and angular data.

3.2 Nonconstancy of the ratios

The frame equations imply that the two ratios of abnormalities cannot both be constant.

Lemma 3.1. *For principal values satisfying (2.13), a solution on an open set with $h_1 h_2 h_3 \neq 0$ cannot have both $\hat{h}_2$ and $\hat{h}_3$ constant.*

Proof. Use the normalized abnormalities $\widehat{h}_2, \widehat{h}_3$. Put $\alpha_{ij} = 1 - \lambda_j/\lambda_i$. The first derivative equations give

$$D_i h_i = -2h_i \operatorname{div} e_i, \quad \operatorname{div} e_i = p_{jk} - p_{kj}.$$

Constant ratios then imply $D_i h_j = -2h_j \operatorname{div} e_i$ for every pair i, j . Substitution into the transverse diagonal-derivative equations yields

$$p_{13} = R_1 p_{31}, \quad p_{21} = R_2 p_{12}, \quad p_{32} = R_3 p_{23},$$

where

$$R_1 = -\frac{\alpha_{21}}{\alpha_{23}\widehat{h}_3}, \quad R_2 = -\frac{\alpha_{32}\widehat{h}_2}{\alpha_{31}}, \quad R_3 = -\frac{\alpha_{13}\widehat{h}_3}{\alpha_{12}\widehat{h}_2}, \quad R_1 R_2 R_3 = 1.$$

For example, if u denotes the free transverse derivative, the equations $D_2 h_3 = u - 2p_{31}h_3$ and $D_2 h_1 = 2p_{13}h_1 + (\alpha_{23}/\alpha_{21})u$ give the first relation. The other two follow by cyclic relabeling.

Differentiate the three undivided opposite-pair relations in directions D_2, D_3, D_1 , respectively, and insert (A.3). After division by h_1^2 the coefficients of $(\widehat{p}_{31}^2, \widehat{p}_{12}^2, \widehat{p}_{23}^2)$ form

$$\begin{pmatrix} R_1(R_1 + 1) & -\frac{\alpha_{31}}{\alpha_{21}}R_2 & -\frac{\alpha_{13}}{\alpha_{23}}R_1 R_3 \\ -\frac{\alpha_{21}}{\alpha_{23}}R_1 R_2 & R_2(R_2 + 1) & -\frac{\alpha_{12}}{\alpha_{32}}R_3 \\ -\frac{\alpha_{23}}{\alpha_{13}}R_1 & -\frac{\alpha_{32}}{\alpha_{12}}R_2 R_3 & R_3(R_3 + 1) \end{pmatrix}.$$

The remaining term depends only on the fixed principal values and $\widehat{h}_2, \widehat{h}_3$. The two identities $R_1 R_2 R_3 = 1$ and

$$\frac{\alpha_{12}\alpha_{23}\alpha_{31}}{\alpha_{21}\alpha_{32}\alpha_{13}} = -1$$

reduce its determinant to

$$(1 + R_1)(1 + R_2)(1 + R_3) + 2 - R_3(1 + R_1) - R_1(1 + R_2) - R_2(1 + R_3) = 4.$$

All three squares are therefore constant. Real continuity makes $\widehat{p}_{31}, \widehat{p}_{12}, \widehat{p}_{23}$ constant. Hence $\mathbf{p} = h_1 M$ for a constant real matrix M .

Put $v_i = -2(M_{jk} - M_{kj})$, $\sigma = \sum_i v_i^2$, and $u = 1/h_1$. Then

$$|\nabla u|^2 = \sigma, \quad \Delta u = \frac{\sigma}{2u}. \quad (3.6)$$

If $K = \operatorname{Hess} u$ and $V = \nabla u$, differentiating the first identity gives $KV = 0$ and $\nabla_V K = -K^2$. Taking traces and differentiating again yields

$$\operatorname{tr} K^2 = \frac{\sigma^2}{2u^2}, \quad \operatorname{tr} K^3 = \frac{\sigma^3}{2u^3}.$$

For $\sigma > 0$, one eigenvalue is zero. If the remaining two are l, m , then

$$l + m = \frac{\sigma}{2u}, \quad lm = -\frac{\sigma^2}{8u^2}, \quad l^3 + m^3 = \frac{5\sigma^3}{16u^3},$$

contradicting the preceding cube trace. Thus $\sigma = 0$ and h_1 is constant.

The constant matrix $\mathbf{p} = N$ is symmetric since each $\operatorname{div} e_i = 0$. Its Euclidean Ricci tensor, obtained directly from (2.17), is

$$2N^2 - (\operatorname{tr} N)N + \left(\frac{(\operatorname{tr} N)^2}{2} - \operatorname{tr} N^2 \right) I.$$

Diagonalizing the real symmetric N shows that flatness permits only $N = 0$ or $N = k(I - n \otimes n)$, with n having constant unit components in the principal frame. Indeed, if ℓ_i are the eigenvalues of N , their equations are $\ell_i^2 = (\ell_j - \ell_k)^2$, whose real solutions are $(0, k, k)$ and their permutations. In the latter case

$$U_i := \operatorname{div} P_i = kn_i(e_i \times n), \quad \operatorname{curl} U_i = NU_i = kU_i.$$

The first equality uses $\operatorname{div} e_i = 0$; the second uses constant frame components and $U_i \perp n$. Projector universality implies $n_i(e_i \times n) = 0$ for every i . Some n_i is nonzero, making $n = \pm e_i$ and $h_i = 0$, a contradiction. The case $N = 0$ is immediate. $\square$

3.3 Density of the normalization domain

Lemma 3.2. *On any solution chart with $h_1 h_2 h_3 \neq 0$, each set $\{\mu_i = 0\}$ has empty interior. Consequently the domain (3.3) is open and dense.*

Proof. For $\mu_1 = 0$, differentiate $h_3 - (\Delta_b/\Delta_a)h_2 = 0$ in the original frame equations. On a nonempty subdomain the resulting algebraic relations reduce to two cubics in $R = (p_{12}/h_2)^2$. The reduction and its cases are given in Appendix B.3. The short combination (B.11) and its three Laurent roots at infinity show that their resultant is a nonzero polynomial in $z = h_1/h_2 = \hat{h}_2^{-1}$; see (B.14). Thus z is locally constant, contrary to Lemma 3.1. In the parametrization $d = 1 - \Delta_b/\Delta_a = (b - a)/[a(b - 1)]$, the exceptional principal value relation is $ad - a - 1 = 0$. At that value the same combination is strictly positive and yields the contradiction directly. Cyclic relabeling proves the assertion for μ_2 and μ_3 . The three zero sets are closed with empty interior, so their complement is open and dense. $\square$

3.4 Propagation of the state rank

The rank of the normalized state map is

$$\rho = \operatorname{rank} dX \in \{0, 1, 2, 3\} \tag{3.7}$$

Lemma 3.2 gives connected charts satisfying (3.3) in every open region with $h_1 h_2 h_3 \neq 0$. On each such chart the normalized equations propagate the rank.

Theorem 3.3. *The rank $\rho = \operatorname{rank} dX$ is constant on every connected solution chart satisfying (3.3), and belongs to $\{0, 1, 2, 3\}$.*

Proof. Let $\mathcal{A} = [\mathcal{V}_1(X), \mathcal{V}_2(X), \mathcal{V}_3(X)]$. Since E_i is a physical frame, this matrix has rank $\operatorname{rank} dX$. Applying the bracket identity in (3.5) to X gives

$$E_j \mathcal{A} = (D\mathcal{V}_j)(X) \mathcal{A} + \mathcal{A} \mathcal{C}_j, \quad (\mathcal{C}_j)_{ki} = \tilde{C}_{ji}^k(X).$$

Along an E_j curve this is a left-right linear matrix equation. Its solution is $\mathcal{A}(t) = L(t)\mathcal{A}(0)R(t)$, where

$$L' = (D\mathcal{V}_j)(X(t))L, \quad R' = R\mathcal{C}_j(X(t)), \quad L(0) = I, \quad R(0) = I.$$

Both fundamental matrices are invertible, so the rank is unchanged. Compositions of the three short frame flows reach a neighborhood of each point; the inverse function theorem applies because their three initial velocities form a basis. Hence the rank is locally constant, and connectedness proves the assertion. $\square$

3.5 The similarities determined by the fibres

The constant rank theorem gives fibres of dimension $3 - \rho$. The equations (3.5) give these fibres a further geometric meaning: points with the same state have the same normalized frame equations, so their frames can be identified by local transformations. Since both flat metrics are recovered from that frame with the same scale factor, these transformations act as common Euclidean similarities.

Proposition 3.4. *If a solution chart satisfying (3.3) has constant state rank ρ , its normalized frame has a local symmetry algebra of dimension $3 - \rho$. Every such symmetry is a common Euclidean similarity of the two flat metrics, with the same constant dilation factor.*

Proof. On the local fiber product $\{(x, y) : X(x) = X(y)\}$, the fields $(E_i(x), E_i(y))$ are tangent, by (3.5). Their brackets close because both sides have the same $\tilde{C}_{ij}^k(X)$. Frobenius leaves project diffeomorphically to each factor: their projections are the frame E_i . Their graphs therefore give local diffeomorphisms preserving X and all E_i , with initial value freely chosen in the $(3 - \rho)$ -dimensional fibre of X . Composition and inversion preserve these conditions, yielding a local transformation group.

Its infinitesimal fields satisfy $ZX = 0$, $[Z, E_i] = 0$. Such a field is determined by its value at one point, by uniqueness along frame flows, so the group has exactly dimension $3 - \rho$ and its orbits are the fibres of X . The coframe dual to E_i is $\hat{\alpha}_i = h_1 \alpha_i$. Hence

$$g = h_1^{-2} \sum_i \hat{\alpha}_i^2, \quad c = h_1^{-2} \sum_i \lambda_i \hat{\alpha}_i^2. \quad (3.8)$$

The symmetry preserves $\hat{\alpha}_i$, and

$$E_i(Z \log |h_1|) = Z s_i(X) = 0.$$

Thus $k_Z = Z \log |h_1|$ is constant and $\mathcal{L}_Z g = -2k_Z g$, $\mathcal{L}_Z c = -2k_Z c$. In either Euclidean development the homothety equation, differentiated once and combined with its cyclic permutations, gives $\partial_i \partial_j Z_k = 0$. Consequently $Z(x) = \sigma x + \Omega x + t$, with $\Omega^\top = -\Omega$ and $\sigma = -k_Z$, the same in both configurations. $\square$

The rank now determines the available geometric information. Ranks zero and one give at least two independent common similarity generators; rank two gives one. We use these generators in Sections 4–7. Rank three gives discrete fibres and leads to the algebraic dimension argument in Section 8. Figure 2 shows how these exclusions establish the geometric hypothesis used in Section 9.

4 States of rank zero and one

The fibres of a state of rank one are surfaces. By Proposition 3.4, they are the orbits of a two dimensional algebra of similarities common to the two flat metrics. We first classify these algebras and then show that their orbit equations force an integrable principal plane field. This will exclude ranks zero and one on the domain (3.3). In Euclidean developments, write $F_f = Df$ and $C_f = F_f^\top F_f$ for the matrix of c .

4.1 Two dimensional algebras of common similarities

At a point of a rank one chart, evaluation identifies the common similarity algebra with the tangent plane to the state fibre. Its two generators are therefore independent after the chart is restricted. The two Euclidean developments realize the same Lie algebra, with the same dilation character, and the developing map intertwines their actions.

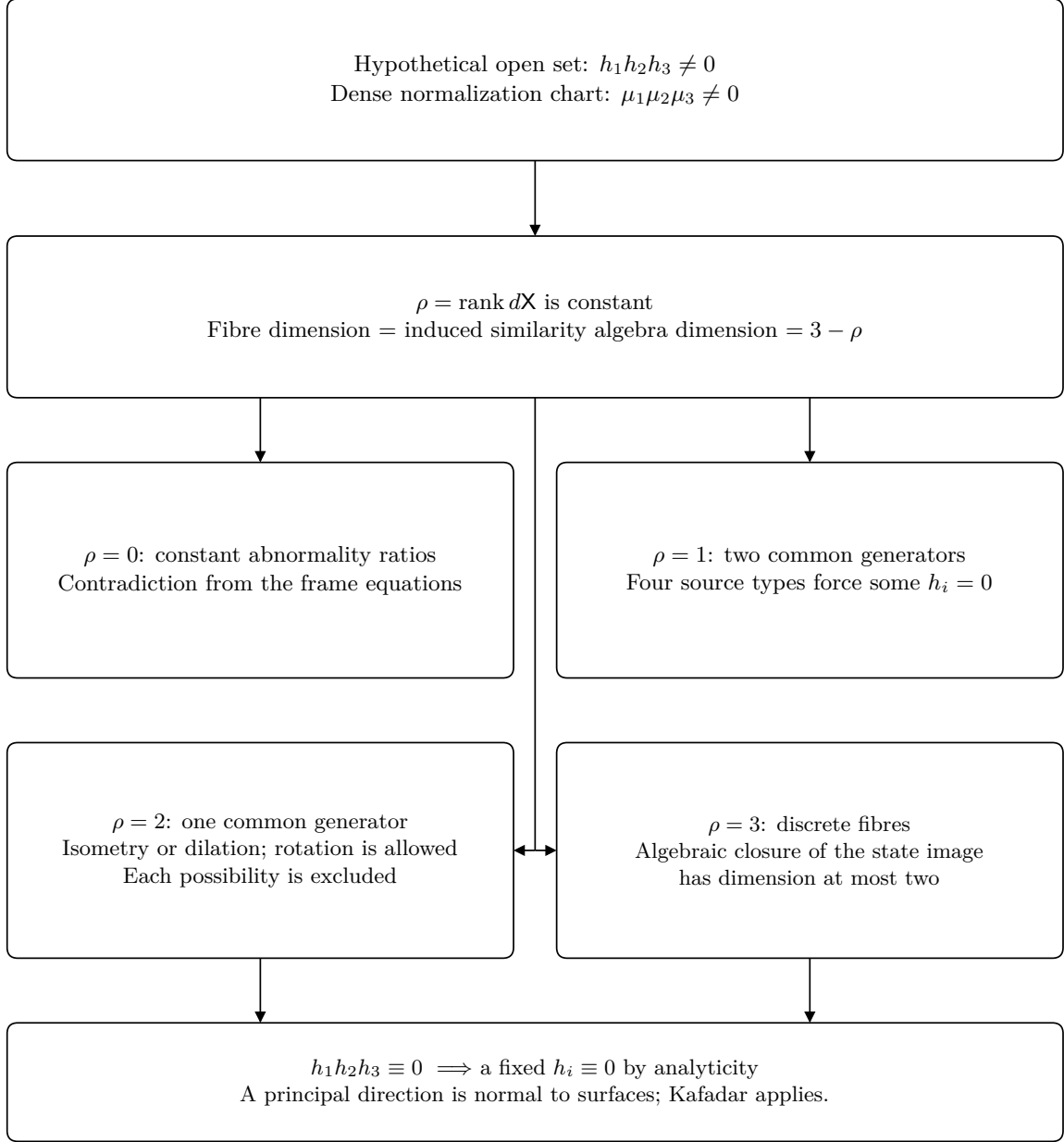


Figure 2: The four ranks of the normalized state map on a region with three nonzero abnormalities. Proposition 3.4 identifies the fibre dimension with the dimension of the induced similarity algebra. Analyticity then selects one integrable principal plane field.

Write $D = x \cdot \partial_x$, $R_z = -y \partial_x + x \partial_y$, and $T_z = \partial_z$. Up to a Euclidean change of coordinates and a change of generators, the list is

$$\langle T_x, T_y \rangle, \quad \langle R_z, T_z \rangle, \quad \langle D + \omega R_z, T_z \rangle, \quad \langle D, R_z \rangle. \quad (4.1)$$

The rotation projection of a two dimensional algebra has dimension at most one: two independent rotation vectors have a cross product outside their plane. If its dilation character vanishes, a nonabelian two dimensional isometry algebra would make a skew rotation act on a translation line by a nonzero real eigenvalue, which is impossible. The algebra is abelian; its rotational part is either zero, giving two translations, or one dimensional, giving a rotation and an axial translation after removing the perpendicular center and subtracting the axial translation.

If the dilation character is nonzero, its kernel is one dimensional. A kernel generator with rotational part forces commutation in the rotational projection. Center the generator with nonzero scale; its linear part is $I + \Omega$, which is invertible over the reals. A commuting kernel translation would solve $(I + \Omega)t = 0$, so vanishes. This gives $\langle D, R_z \rangle$ after subtracting a rotation. A translational kernel must be a real eigenline of $I + \Omega$, hence lies on the rotation axis, giving $\langle D + \omega R_z, T_z \rangle$. The same classification applies in the second Euclidean development with the common dilation character.

For the Killing types, the target algebra is abelian and consists of two translations or a rotation and an axial translation. When the dilation character is nonzero, the Lie bracket distinguishes the translation ideal from the commuting rotation ideal. The target algebra has the corresponding type, with an independent angular rate.

4.2 Two common Killing fields

Lemma 4.1. *If a two dimensional algebra of common similarities has zero dilation character, at least one principal abnormality vanishes on a nonempty open subset.*

Proof. There are two source orbit types in (4.1). For two translations, choose their orbit planes to be $z = \text{constant}$. The distinct principal values imply that each labelled principal projector is translation invariant, so $P_i = P_i(z)$. Set $q_i = (P_i)_{zz}$ and $v_i = ((P_i)_{xz}, (P_i)_{yz})$. The projector pressure equations and the orthogonality of the projectors give

$$v_i'' = 0, \quad |v_i|^2 = q_i(1 - q_i), \quad -v_i \cdot v_j = q_i q_j =: C_{ij}, \quad \sum_i q_i = 1. \quad (4.2)$$

If all q_i are positive, then $q_i^2 = C_{ij}C_{ik}/C_{jk}$ and $q_i = q_i^2 + C_{ij} + C_{ik}$ show that q_i is rational in z . The monic equation $q_i^2 - q_i + |v_i|^2 = 0$ makes it a polynomial of degree at most one. Its quadratic coefficient gives $(q_i')^2 + |v_i'|^2 = 0$. Consequently every P_i is constant. Here $P_i = (v_i, q_i)(v_i, q_i)^\top / q_i$.

If some q_i vanishes on an interval, its principal direction lies in an orbit plane. If two vanish, the remaining direction is axial. Otherwise the orbit metric has one fixed relative eigenvalue λ_i and another eigenvalue m . Its flat Gauss equation gives $(\lambda_i - m)^2(\theta')^2 = 0$, where θ is the angle of the horizontal principal direction. A nonconstant θ would force $m = \lambda_i$ and a constant mixing angle for the other two directions. Their affine off-diagonal components would then have constant norm, and hence be constant, contrary to rotation. Thus the horizontal principal direction is constant as well. These alternatives exhaust an interval after restriction; in each case some h_i vanishes.

For a rotation and an axial translation, use cylindrical coordinates (r, ϕ, z) with $r > 0$. Write $q_i = (P_i)_{rr}$ and $(U_i, V_i) = ((P_i)_{r\phi}, (P_i)_{rz})$ in the physical frame. The projector pressure equations integrate to

$$U_i = \beta_{i,0} + \beta_{i,-2}/r^2, \quad V_i = \beta_{i,1}r + \beta_{i,-1}/r. \quad (4.3)$$

The coefficients are real constants. If all q_i are positive, the same rational reconstruction and monic equation make them Laurent polynomials. In $q_i^2 - q_i + U_i^2 + V_i^2 = 0$, an extreme nonzero power in the squared terms has a positive sum of squares as coefficient. Thus all components are constant and (4.3) gives $V_i = 0$, which would place all three principal directions in the radial–azimuthal plane.

If the three q_i are not simultaneously positive on any subinterval, one vanishes on a subinterval. Restrict so that each remaining q_j is either positive or identically zero, and develop the cylindrical orbit metric into the target Euclidean space. The target orbits are planar or cylindrical. The planar case gives an axial principal direction. In the cylindrical case, constant volume makes the squared target radius affine in r^2 . The fixed orbit eigenvalue then gives either an axial or azimuthal principal direction, or the relation $V_j = \kappa r U_j$ for the other two projectors. In the last alternative, their two-plane projection identity and (4.3) force $U_j = V_j = 0$, contrary to $q_j q_k > 0$. The orbit metric calculation, including the case in which its two relative eigenvalues coincide, is given in Appendix B.5. Every remaining principal direction found above has $h_i = 0$. $\square$

4.3 Two common similarities with nonzero scale

Lemma 4.2. *If a two dimensional algebra of common similarities has nonzero dilation character, at least one principal abnormality vanishes on a nonempty open subset.*

Proof. Consider first the nonabelian algebra $\langle D + \omega R_z, \partial_z \rangle$. Its translation ideal is preserved by the developing map, and its target generators can be written as $D + \tilde{\omega} R_Z$ and $\kappa \partial_Z$, with $\kappa \neq 0$. Equivariance therefore gives

$$f(x, y, z) = (f_\perp(x, y), \kappa z + \varphi(x, y)). \quad (4.4)$$

The planar Jacobian $J_\perp = \det Df_\perp$ is a nonzero constant. Both φ and $\tilde{\varphi} = \varphi \circ f_\perp^{-1}$ are harmonic. Indeed, $\operatorname{div} C_f = F_f^\top \Delta f$ because the trace is constant. Its axial component is $\kappa \Delta \varphi$, independent of z , and closedness makes it constant. Equivariance gives $(D + \omega R_z)\varphi = \varphi$, so $[\Delta, D + \omega R_z] = 2\Delta$ forces this constant to vanish. The inverse divergence identity (B.21) gives the same conclusion for $\tilde{\varphi}$ in the target plane.

If φ is locally constant, the axial direction is principal. Otherwise take its harmonic conjugate and use the resulting complex potential as a local coordinate. Equivariance makes this potential a power of $x + iy$ with exponent $(1 + i\omega)^{-1}$. The two trace constraints and constant planar Jacobian then give (B.22)–(B.23). For $\omega \neq 0$, equality of mixed derivatives would give a finite relation between distinct complex exponentials with a nonzero extreme coefficient. Their linear independence excludes this case. The identities and coefficient comparison are derived in Appendix B.6. If $\omega = 0$, the source height is affine and the target height has constant gradient norm. Harmonicity and the Bochner identity make the latter affine. Their constant gradients fix one row of $Df_\perp$; its constant determinant and norm fix the other row. Thus f is affine. The nonabelian type always has some $h_i = 0$.

For the abelian algebra $\langle D, R_z \rangle$, centering the target homothety removes its translation part. A commuting target Killing field is a nonzero rotation about the same axis. In spherical source coordinates, the general equivariant map is

$$f = r(U(\theta) \cos \Psi, U(\theta) \sin \Psi, V(\theta)), \quad \Psi = k\phi + \omega \log r + \psi(\theta), \quad U > 0, \quad (4.5)$$

where $k \neq 0$. Constant volume gives

$$\det F_f = J = \frac{kU(U'V - UV')}{\sin \theta} \neq 0. \quad (4.6)$$

Set $x = \cos \theta$, $\mathcal{N} = U^2 + V^2$, $y = V/\sqrt{\mathcal{N}}$ and $\widetilde{\mathcal{N}}(y) = 1/\mathcal{N}(x)$. The pressure equations for C_f and C_f^{-1} , together with (4.6), yield

$$\begin{aligned} (1 - x^2)\mathcal{N}'' - 2x\mathcal{N}' + 6\mathcal{N} &= A_0, \\ y' &= (J/k)\mathcal{N}^{-3/2}, \quad Jk \neq 0, \\ (1 - y^2)\widetilde{\mathcal{N}}'' - 2y\widetilde{\mathcal{N}}' + 6\widetilde{\mathcal{N}} &= B_0, \quad \widetilde{\mathcal{N}}(y(x)) = \mathcal{N}(x)^{-1}, \end{aligned} \quad (4.7)$$

with real constants A_0, B_0 ; the derivation is in Appendix B.8. Lemma B.1 gives either a constant $\mathcal{N}$ or the quadratic alternative (B.35). In both cases, the remaining angular pressure equation and the constant trace reduce (4.5) to the logarithmic cylindrical map (B.38); see Appendix B.10. Its axial direction is principal, and the other two principal directions make constant angles with the radial direction. Their abnormalities all vanish. $\square$

Theorem 4.3. *A solution chart satisfying (3.3) has neither state rank zero nor state rank one.*

Proof. Theorem 3.3 makes the state rank constant. Rank zero makes $\widehat{h}_2, \widehat{h}_3$ constant, contrary to Lemma 3.1. At rank one, Proposition 3.4 gives a two dimensional common similarity algebra. The list (4.1) is exhausted by Lemmas 4.1 and 4.2, both of which contradict $h_1 h_2 h_3 \neq 0$. $\square$

Thus a positive dimensional state fibre on (3.3) has dimension one, and its symmetry algebra is generated by a single common similarity. The next three sections exclude this rank two case.

5 Similarity reduction

At rank two, the state fibres are the orbits of one common similarity. Its Euclidean representations determine the deformation along each orbit, leaving data on a transverse surface. We derive the equations for these data and the normal forms used in Sections 6 and 7.

5.1 The common generator

Proposition 5.1 (One common similarity). *If $\text{rank } d\mathbf{X} = 2$, there is locally a one dimensional Lie algebra of vector fields Z satisfying*

$$[Z, E_i] = 0, \quad Z\mathbf{X} = 0. \quad (5.1)$$

Evaluation identifies this algebra with $\ker d\mathbf{X}$. A nonzero generator is nowhere zero on a sufficiently small chart and is a homothety of both metrics with the same constant factor.

Proof. Apply Proposition 3.4 with $\rho = 2$. The evaluation isomorphism makes a nonzero generator nowhere zero after restriction. If $k = Z \log |h_1|$, the common homothety equations are

$$\mathcal{L}_Z g = -2kg, \quad \mathcal{L}_Z c = -2kc. \quad (5.2)$$

Thus the two homothety factors agree. $\square$

In the two Euclidean developments, the representations Z_s, Z_t are intertwined by f :

$$Df(x)Z_s(x) = Z_t(f(x)). \quad (5.3)$$

Differentiating the Euclidean homothety equation $\partial_i Z_j + \partial_j Z_i = 2\sigma\delta_{ij}$ and taking cyclic permutations gives $\partial_i \partial_j Z_k = 0$. Hence

$$\begin{aligned} Z_s(x) &= \sigma_s x + \Omega_s x + t_s, & Z_t(y) &= \sigma_t y + \Omega_t y + t_t, \\ \Omega_s^\top &= -\Omega_s, & \Omega_t^\top &= -\Omega_t. \end{aligned} \quad (5.4)$$

The fixed trace gives

$$0 = Z \operatorname{tr}(g^{-1}C_f) = 2(\sigma_t - \sigma_s) \operatorname{tr}(g^{-1}C_f), \quad (5.5)$$

so $\sigma_s = \sigma_t$. The common orbit parameter therefore fixes the relative speeds of the two affine fields.

For the transverse equations, write the fixed invariants as

$$\mathcal{I}_1 = \operatorname{tr} C_f, \quad \mathcal{I}_2 = \operatorname{tr} \operatorname{adj} C_f, \quad J = \det Df.$$

In physical incompressible coordinates $(\mathcal{I}_1, \mathcal{I}_2, J) = (I_2, I_1, 1)$. A constant metric normalization may give a different nonzero signed Jacobian J in a developing chart.

5.2 Equilibrium under inversion

Exchanging the Euclidean developments replaces f by its inverse and the principal values by their reciprocals. The following lemma shows that the same interchange preserves the hypotheses of the classification.

Lemma 5.2 (Local inverse equivalence). *Let f be a local diffeomorphism whose matrix C_f has constant, positive, pairwise distinct principal values. The closedness conditions*

$$d(\operatorname{div} C_f)^\flat = 0, \quad d(\operatorname{div} C_f^{-1})^\flat = 0 \quad (5.6)$$

hold if and only if they hold for f^{-1} . Each labelled projector pressure is transported as the same scalar function, and each labelled principal abnormality has the same zero set in the two configurations.

Proof. Put $F_{f^{-1}} = D(f^{-1})$ and $\overline{C}_f = F_{f^{-1}}^\top F_{f^{-1}}$. Since the traces and Jacobian are constant, differentiation gives

$$\begin{aligned} \operatorname{div}_x C_f &= F_f^\top \Delta_x f, \\ \operatorname{div}_x C_f^{-1} &= (\Delta_y f^{-1}) \circ f, \\ \operatorname{div}_y \overline{C}_f &= F_{f^{-1}}^\top \Delta_y f^{-1}, \\ \operatorname{div}_y \overline{C}_f^{-1} &= (\Delta_x f) \circ f^{-1}. \end{aligned} \quad (5.7)$$

The first identity follows by differentiating $(C_f)_{ij} = \sum_a (F_f)_{ai} (F_f)_{aj}$ and commuting mixed derivatives. For the second, write $(C_f^{-1})_{ij} = \sum_a (F_{f^{-1}})_{ia} (F_{f^{-1}})_{ja}$ at $f(x)$ and use the chain rule. Differentiation of the two factors gives $\Delta_y (f^{-1})_i$ and $-(C_f^{-1} \nabla_x \log |J|)_i$, respectively; the latter vanishes. The last two identities follow by interchanging f and f^{-1} .

On a simply connected chart, choose pressure potentials $\operatorname{div} C_f = \nabla p_{C_f}$ and $\operatorname{div} C_f^{-1} = \nabla p_{C_f^{-1}}$. Equation (5.7) yields

$$\operatorname{div}_y \overline{C}_f = \nabla_y (p_{C_f^{-1}} \circ f^{-1}), \quad \operatorname{div}_y \overline{C}_f^{-1} = \nabla_y (p_{C_f} \circ f^{-1}). \quad (5.8)$$

The same argument applied to the inverse proves equivalence.

Write $C_f = \sum_i \lambda_i P_i$ and $\operatorname{div} P_i = \nabla \phi_i$, with $\sum_i \phi_i = 0$. The inverse projectors $\overline{P}_i$ have principal values λ_i^{-1} . The coefficient matrix with rows $(1, 1, 1)$, $(\lambda_1, \lambda_2, \lambda_3)$ and $(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})$ is invertible. Thus (5.8) and $\operatorname{div} \sum_i \overline{P}_i = 0$ imply

$$\operatorname{div}_y \overline{P}_i = \nabla_y (\phi_i \circ f^{-1}). \quad (5.9)$$

For unit principal vectors, put $\bar{e}_i = F_f e_i / \sqrt{\lambda_i}$. Then $f^* \bar{\alpha}_i = \sqrt{\lambda_i} \alpha_i$, and hence

$$f^* (\bar{\alpha}_i \wedge d\bar{\alpha}_i) = \lambda_i \alpha_i \wedge d\alpha_i, \quad \bar{h}_i \circ f = \frac{\lambda_i}{J} h_i. \quad (5.10)$$

This preserves the zero sets for either sign of J . $\square$

5.3 Normal forms of the generator pair

When the common scale is zero, each generator is a Euclidean Killing field. In cylindrical coordinates, its three normal forms are

$$\begin{aligned} \text{T translation : } & p \partial_z, \quad p \neq 0, \\ \text{R rotation : } & \omega \partial_\theta, \quad \omega \neq 0, \\ \text{S screw motion : } & \omega \partial_\theta + p \partial_z, \quad \omega p \neq 0. \end{aligned} \tag{5.11}$$

A pair lists the source generator first and the target generator second. Lemma 5.2 permits their interchange.

Proposition 5.3 (Normal forms of one generator). *Up to separate fixed Euclidean coordinate changes and a common nonzero rescaling of Z , a rank two chart has either one of the six Killing pairs*

$$(T, T), \quad (T, R), \quad (T, S), \quad (R, R), \quad (R, S), \quad (S, S), \tag{5.12}$$

or a pair with nonzero common scale

$$(D + \omega_s R, D' + \omega_t R'), \quad \omega_s, \omega_t \in \mathbb{R}. \tag{5.13}$$

Here $D = x \cdot \partial_x$, $D' = y \cdot \partial_y$, $R = \partial_\theta$ and $R' = \partial_\Theta$. The two angular rates may vanish independently. The speeds and screw pitches are subject to the intertwining identity (5.3).

Proof. For a Killing field with nonzero rotational part, decompose the translation into axial and perpendicular components. The skew map is invertible on the perpendicular plane, so shifting the axis removes the perpendicular component. The remaining axial component gives a rotation or a screw motion; a zero rotational part gives a translation. Both generators are nonzero because Df is invertible. Inversion reduces their ordered pairs to (5.12).

If $\sigma \neq 0$, the matrix $\sigma I + \Omega$ is invertible over the reals. A unique change of origin removes the translation. Rescaling the common generator by σ^{-1} and aligning the axes gives (5.13). $\square$

Table 1: The six Killing pairs on the branch $h_1 h_2 h_3 \neq 0$. For the two cylindrical alternatives, all principal abnormalities vanish.

Pair	Argument	Consequence	Proof location
(T, T)	Finite Hessian choices on transverse slices	$\rho \leq 1$	Lemmas 6.6 and 6.8
(T, R)	Algebraic reconstruction and finite transverse fibres	$\rho \leq 1$	Lemmas 6.6 and 6.10
(T, S)	Algebraic reconstruction and finite transverse fibres	$\rho \leq 1$	Lemmas 6.6 and 6.10
(R, R)	Compact continuation and local dimension bounds	Cylindrical; $h_i = 0$	Sections 6.3 and 6.4
(R, S)	Continuation and the Killing field norm estimate	No solution germ	Equation (6.6) and Lemma 6.5
(S, S)	Radial limits and the pitch relation	Cylindrical; $h_i = 0$	Section 6.5

The common parameter fixes the relative speeds, but does not equate the angular rates. For the logarithmic cylindrical map (B.38), $f_*(D + \omega_s R) = D' + (\omega + k\omega_s)R'$. All angular coordinates below are local lifts.

5.4 Equations on a transverse surface

Let Φ_s^t, Φ_t^t be the affine flows, and choose a source surface $x_0(u, v)$ transverse to Z_s . Equivariance is equivalent to

$$f(\Phi_s^t x_0(u, v)) = \Phi_t^t y_0(u, v). \quad (5.14)$$

Indeed, both sides solve the target orbit equation with the same initial value. Conversely, any surface y_0 for which the resulting map is a local diffeomorphism defines an equivariant map. At $t = 0$, its derivative is determined by

$$F_{f,0}[Z_s(x_0), \partial_u x_0, \partial_v x_0] = [Z_t(y_0), \partial_u y_0, \partial_v y_0]. \quad (5.15)$$

The equal scale factors cancel along the flows, so

$$Df(\Phi_s^t x_0) = e^{t\Omega_t} F_{f,0} e^{-t\Omega_s}. \quad (5.16)$$

Consequently constant principal values are equivalent to the three constraints on the transverse surface

$$\text{tr}(F_{f,0}^\top F_{f,0}) = \mathcal{I}_1, \quad \text{tr adj}(F_{f,0}^\top F_{f,0}) = \mathcal{I}_2, \quad \det F_{f,0} = J. \quad (5.17)$$

Flatness follows from the representation (5.14).

For equilibrium, let M be the covariant tensor associated with $g^{-1}c$ or its inverse; in Euclidean coordinates its matrix is C_f or C_f^{-1} . A constant homothety preserves the connection. The homothety weights of M and of the inverse metric in its divergence cancel, giving

$$\mathcal{L}_Z((\text{div}_g M)^b) = 0. \quad (5.18)$$

In flow coordinates, write this invariant one-form as

$$(\text{div}_g M)^b = a_t(u, v) dt + a_u(u, v) du + a_v(u, v) dv.$$

Closedness is equivalent to

$$\partial_u a_t = 0, \quad \partial_v a_t = 0, \quad \partial_u a_v - \partial_v a_u = 0. \quad (5.19)$$

On a small connected transverse disk, the pressure therefore has the form

$$p_M(t, u, v) = K_M t + \psi_M(u, v) + \text{constant}, \quad (5.20)$$

where K_M is a real constant. There is one such system for each of the two tensors. For individual projectors, the same argument gives

$$Z\phi_i = K_i, \quad \sum_i K_i = 0. \quad (5.21)$$

Equation (5.9) transports these constants to the inverse configuration.

Thus (5.17) and (5.19) give the kinematic and equilibrium equations on the transverse surface. The next section treats the Killing pairs; Section 7 treats (5.13).

6 Common Euclidean isometries

We now consider the pairs of common Euclidean isometries in Section 5. Throughout, rank means the rank of $\mathbf{X}$ in (3.2), and $C_f, \mathcal{I}_1, \mathcal{I}_2, J$ have the meaning fixed there. We also write $j = \det C_f = J^2 > 0$. The constant principal values λ_i are positive and pairwise distinct. The universal deformation equations are

$$\text{curl div } C_f = 0, \quad \text{curl div } C_f^{-1} = 0. \quad (6.1)$$

The pressures are local potentials. When an angular coordinate is used, we lift it to a real coordinate on the universal cover of the complement of the axis. The matrices Ω_s, Ω_t retain the meaning in (5.4). Their axial vectors are defined by $\Omega_s x = \Omega_s^{\text{axial}} \times x$ and $\Omega_t y = \Omega_t^{\text{axial}} \times y$; in principal components they are expressed in the corresponding principal frames.

A common Killing field makes the state constant along its orbits. The two transverse variables can still carry a state of rank two; the problem is therefore to constrain this transverse dependence. For pairs containing a translation, the orbit equations leave one scalar parameter with finite reconstruction fibres. For rotating sources, the natural length is the distance from the axis. Bounding the state at this scale permits continuation and radial limits. The two-rotation problem then becomes a dimension comparison on a compact state set, while the normalized pitches of two screw motions tend to zero at large radius.

Theorem 6.1 (Exclusion of a common isometry at rank two). *For every triple of positive, pairwise distinct, constant principal values of the strain tensor, no rank two state arising from a solution on an open set where $h_1 h_2 h_3 \neq 0$ admits a nonzero common Euclidean Killing field.*

6.1 Reconstruction and continuation

Continuation and radial limits require a state set closed under limits of the frame coefficients. We therefore include the common Killing field, its angular data and the longitudinal pressure data in the state of Section 2. Differentiating their algebraic constraints gives an invariant closed set on which the same reconstruction applies.

We use $\mathbf{S}$ and the fields $\mathcal{Q}_i$ from (2.21) and (2.23). Write $w_i^Z = \alpha_i(Z_s)$ for the principal components of the common Killing field. The augmented derivations B_i are defined on these coordinates by the first two relations below. The third relation is the algebraic constraint that the state is constant along the common orbit:

$$B_i \mathbf{S} = \mathcal{Q}_i(\mathbf{S}), \quad B_i w_l^Z = \sum_k w_k^Z C_{ki}^l(\mathbf{S}), \quad \sum_i w_i^Z \mathcal{Q}_i(\mathbf{S}) = 0. \quad (6.2)$$

Their coefficients have only the fixed spectral denominators.

Definition 6.2 (State with fixed isometry data). Let $\mathbf{U}$ consist of $\mathbf{S}$, the principal components of the common Killing field, and the source and target angular vectors. Transporting the latter by the two metric connections extends (6.2) to polynomial derivations B_i of $\mathbb{R}[\mathbf{U}]$. The principal values and isometry data are fixed, so every spectral denominator is a nonzero constant.

Let $\mathcal{J}_0$ be the ideal of the original compatibility and bracket equations, the orbit relation in (6.2), and the equations identifying the angular vectors, their prescribed norms and pitches, and the derivatives of the pressure potentials along the common orbit. The angular and pressure equations are written explicitly in (D.16) and (E.6), with the physical target angular vector obtained by the metric conversion specified there. Define

$$\mathcal{J} = \langle B_{i_1} \cdots B_{i_m} q : q \in \mathcal{J}_0, m \geq 0, i_\ell \in \{1, 2, 3\} \rangle \subset \mathbb{R}[\mathbf{U}]. \quad (6.3)$$

Write $\mathcal{N}$ for its real zero set, with a subscript specifying the isometry pair. Fix source and target frames and base values to remove the development constants, including the constant target phase and axial translation when the target is axial.

Noetherianity of $\mathbb{R}[\mathbf{U}]$ gives finitely many generators of $\mathcal{J}$. Thus $\mathcal{N}$ is a closed real algebraic set and is invariant under the ambient polynomial state flows wherever they are defined. The augmented bracket identity is

$$\left([B_i, B_j] - \sum_k C_{ij}^k B_k \right) w_l^Z = \sum_k w_k^Z \mathcal{Q}_k(C_{ij}^l) = 0, \quad (6.4)$$

where the last equality is the weight one part of (6.2).

Lemma 6.3 (Reconstruction with fixed isometry data). *The state set in Definition 6.2 has local realizations satisfying the prescribed common isometry and pressure equations. After the development constants are fixed, the kernel of the differential of its defining ideal identifies with the space of formal linearized germs with the same fixed data.*

Proof. Lift the augmented fields to the source orthonormal frame bundle. The displayed bracket identity, torsion and Euclidean flatness make their Cartesian combinations commute. Their flows reconstruct the first metric, and development of the second flat metric reconstructs f , as in Proposition 2.5. The appended equations reconstruct a common Killing field; its value and covariant derivative determine its Euclidean translation and angular parts. The fixed data therefore impose the prescribed isometry pair. The closed pressure forms reconstruct the local potentials.

The same calculation over $\mathbb{R}[\epsilon]/(\epsilon^2)$ includes the appended equations and all their derivatives. The commuting Cartesian flows and their variational equations reconstruct a formal linearized germ of the full equivariance system. The converse follows by differentiating that system. This is Lemma 2.6 applied to the augmented system. $\square$

The connection and its two retained jets have weights one, two and three. Multiplication by the corresponding powers of $\mu = |Z_s|$ removes their physical length scale. The angular equations constrain these dimensionless quantities when the common field is transverse to the rotation axis.

Lemma 6.4 (Transverse compactness). *Fix positive, pairwise distinct principal values, bounded angular data, and bounded longitudinal pressure data. Put $\mu = |Z_s|$, $\nu = Z_s/\mu$. If the source angular vector Ω_s^{axial} satisfies $|\Omega_s^{\text{axial}} \times \nu| \geq d > 0$, then*

$$(\mu \mathbf{p}, \mu^2 \boldsymbol{\tau}, \mu^3 \mathbf{u}, \nu)$$

is bounded on the augmented real state set.

Proof. Suppose an unbounded sequence satisfies the bounds on the data and the stated transversality. The common-field and compatibility equations give

$$|\mu^2 \boldsymbol{\tau}| \leq C(1 + |\mu \mathbf{p}|)^2, \quad |\mu^3 \mathbf{u}| \leq C(1 + |\mu \mathbf{p}|)^3. \quad (6.5)$$

Hence, after passing to a subsequence, $|\mu \mathbf{p}| \rightarrow \infty$. The unit directions and bounded data have limits. Normalize by the inverse size ε of these frame coefficients. Away from principal directions the angular data leave at most one homogeneous connection direction; a nonzero quadratic compatibility equation excludes it. This also covers directions with just one zero principal component.

At a principal direction the homogeneous limit is the plane (D.15). If the normal distance to this plane is $O(|\varepsilon|)$, the first-order angular forcing in the unprojected equations cannot be cancelled by their quadratic remainder. Its source components are nonzero by transversality, giving a contradiction.

If the normal distance divided by $|\varepsilon|$ tends to infinity, the quadratic normal equations apply. At either extreme principal direction their first form is definite. At the middle direction it has two null lines. Along either line, the next orders first determine the moving normal centre and then give a strictly positive fourth-order remainder. When the coefficient determining that centre tends to zero, the equivalent system in the derived normal variables gives incompatible limits for its quadratic and linear components. The two projective endpoints are treated in the charts where their nonzero connection coordinate is normalized to one. The normal equations and their fourth-order terms are written in Appendix D.3. After passage to a subsequence, the ratio of the normal distance to $|\varepsilon|$ is bounded or tends to infinity. The first case contradicts the angular forcing; the second contradicts the normal equations just described. $\square$

Lemma 6.5 (Continuation away from the axis). *A solution germ with a rotating source satisfying the augmented equations extends along every finite path in the complement of its source axis. It therefore extends to the universal cover of that complement.*

Proof. Write $Z_s = \omega_e z \times x + p e_z$. On an off-axis compact set,

$$|Z_s|^2 = \omega^2 r^2 + p^2, \quad |\Omega_s^{\text{axial}} \times Z_s|/|Z_s| = \omega^2 r/(\omega^2 r^2 + p^2)^{1/2} > 0$$

uniformly. Lemma 6.4 bounds all unnormalized states there. Along a finite path integrate (6.2), both frame connections, and

$$df = \bar{\mathbf{E}} \operatorname{diag}(\sqrt{\lambda_i}) \mathbf{E}^\top dx.$$

Frames remain in compact orthogonal groups and f remains bounded by the fixed largest singular value times path length. The analytic right-hand sides have only the positive principal values and their differences. A finite first obstruction therefore has a Cauchy limit and the ODE extends through it. The differential ideal is preserved, and reconstruction realizes every continued state as a local germ. $\square$

Continuation immediately excludes a source rotation paired with a target screw motion. Keep the common flow parameter and write

$$Z_s = \omega_s e_z \times x, \quad Z_t = \omega_t e_z \times y + p e_z, \quad \omega_s \omega_t p \neq 0.$$

Let $M = \sqrt{\lambda_{\max}}$. Equivariance gives at every point

$$|p| \leq |Z_t(f(x))| = |Df Z_s| \leq M |\omega_s| r. \quad (6.6)$$

Hence $r \geq r_* := |p|/(M |\omega_s|) > 0$. Starting at any germ, continue along a fixed radial segment to $r_*/2$. This finite segment is entirely off axis, so Lemma 6.5 applies. At its endpoint, (6.6) would give $|p| \leq |p|/2$. Thus a rotation cannot be paired with a screw motion, even off the region where $h_1 h_2 h_3 \neq 0$.

6.2 Pairs containing a translation

For the three pairs with a translating source, the equivariant map leaves a scalar coordinate transverse to the common orbits. We shall prove that the remaining reconstruction data have finite fibres over that coordinate. The following lemmas connect this algebraic property to the rank of the differential state and provide a criterion for finiteness of an algebraic projection.

Lemma 6.6 (Finite fibres and state rank). *Let v be a local coordinate on a solution chart. Suppose that, for each fixed v outside a finite set, all remaining scalar reconstruction variables take values in finite algebraic sets. Suppose the matrices of both metrics in an adapted orthonormal frame $(\tilde{e}_i)$, that frame's connection coefficients, and the derivatives $\tilde{D}_i v$ are reconstructed from these variables and v , where $\tilde{D}_i = \tilde{e}_i \cdot \nabla$. Then the state of their principal frame has rank at most one.*

Proof. Shrink to coordinates (v, w, z) avoiding the finite exceptional levels. On a connected (w, z) -slice, each reconstruction variable is continuous and has finite image, hence is constant. Its dependence on v is smooth, as follows by restriction to a fixed transverse section. The distinct principal values allow the eigenframe to be chosen smoothly as a function of the reconstructed matrix. The adapted connection and $\tilde{D}_i v$ depend only on v by hypothesis. For any scalar $a(v)$ its frame derivative is $a'(v) \tilde{D}_i v$, again a function of v . This closes the induction for the eigenframe connection and all successive frame derivatives. The normalized invariant state therefore depends only on v . $\square$

Lemma 6.7 (Projection to an affine line). *Let V be a constructible complex algebraic set and b a regular function on it. If no algebraic curve $\mathcal{C} \subset \bar{V}$ with $\mathcal{C} \cap V$ dense in $\mathcal{C}$ has a place with $b = \infty$, then $b(V)$ is finite.*

Proof. An infinite constructible subset of the affine line contains a nonempty Zariski open set. A component dominating that line contains an algebraic curve on which b is nonconstant. On its smooth projective normalization, the resulting nonconstant rational function has a pole. A local uniformizer gives a Puiseux branch with a pole. $\square$

We first consider two translations.

In Cartesian coordinates, the equivariant map is

$$f(x, y, z) = (f_{\perp}(x, y), \kappa z + v(x, y)), \quad \kappa \neq 0. \quad (6.7)$$

Put $G = (Df_{\perp})^{\top} Df_{\perp}$, $d_0 = \det G > 0$, $K = \kappa^2$, and $v = \nabla v$. The constant full determinant makes d_0 constant. In the source splitting,

$$C_f = \begin{pmatrix} G + vv^{\top} & \kappa v \\ \kappa v^{\top} & K \end{pmatrix}, \quad C_f^{-1} = \begin{pmatrix} G^{-1} & -G^{-1}v/\kappa \\ -v^{\top}G^{-1}/\kappa & (1 + v^{\top}G^{-1}v)/K \end{pmatrix}. \quad (6.8)$$

The axial components of (6.1) and then its planar component give, for arbitrary real constants P, Q ,

$$\Delta v = P, \quad \operatorname{div}(G^{-1}\nabla v) = Q, \quad \operatorname{curl} \operatorname{div} G = 0. \quad (6.9)$$

Indeed $\operatorname{div}(vv^{\top}) = \nabla(|v|^2/2) + P\nabla v$ is a gradient. The two trace conditions imply, for constants E, d ,

$$\operatorname{tr} G = E - |v|^2, \quad v^{\top}G^{-1}v = (K|v|^2 + d)/d_0.$$

On $\tau = |v| > 0$ take $n = v/\tau$ and its oriented quarter-turn t . The metric reconstruction is

$$G = \begin{pmatrix} A & G_{nt} \\ G_{nt} & C_0 \end{pmatrix}_{n,t}, \quad C_0 = K + d/\tau^2, \quad A = E - \tau^2 - C_0, \quad G_{nt}^2 = AC_0 - d_0. \quad (6.10)$$

Here G_{nt} may have either sign.

If $v = 0$ on an open set, the fixed axial direction is principal and its corresponding h_i is zero. If τ is a positive constant, the Bochner identity gives

$$0 = \Delta|\nabla v|^2 = 2|D^2v|^2,$$

so v is affine and (6.10) makes G constant. The principal frame is consequently constant, giving $h_i = 0$. Otherwise restrict to $d\tau \neq 0$. The rational function G_{nt}^2 has leading term $-K\tau^2$ at infinity and is not identically zero. Its finitely many zero levels therefore cannot contain a solution coordinate patch; on a smaller chart $G_{nt} \neq 0$.

Lemma 6.8 (Translation fibres). *At each fixed positive value of τ , apart from finitely many levels, the Hessian D^2v takes at most two values in the frame (n, t) . This holds for every real P, Q and for both reconstruction signs.*

Proof. The second equation in (6.9) places the two independent Hessian entries on an affine line. Its two derivatives determine the third derivatives through a matrix with strictly positive determinant, including when $d = 0$. Substitution in the remaining planar equation gives a quadratic polynomial on that line. In the normalization of Appendix D.1, its leading coefficient is a nonzero rational function of τ , for every pressure pair and either reconstruction sign. It vanishes at only finitely many levels, which proves the claim. $\square$

Lemma 6.8 supplies the finite Hessian choices needed in Lemma 6.6; the metric reconstruction already supplies the finite choices of G_{nt} . Write $a_H = D^2v(n, n)$ and $b_H = D^2v(n, t)$, the Hessian entries used in Appendix D.1. In the frame (n, t, e_z) ,

$$C_f = \begin{pmatrix} A + \tau^2 & G_{nt} & \kappa\tau \\ G_{nt} & C_0 & 0 \\ \kappa\tau & 0 & K \end{pmatrix},$$

$$\nabla_n n = (b_H/\tau)t, \quad \nabla_t n = ((P - a_H)/\tau)t, \quad \nabla_{e_z} n = 0.$$

The Hessian entries also give $n\tau = a_H$, $t\tau = b_H$ and $e_z\tau = 0$. Thus both the adapted connection and the frame derivatives of the coordinate τ are reconstructed from the same finite data. On each connected fixed- τ slice those data are constant, so Lemma 6.6 shows that the state of the principal frame has rank at most one, contradicting Theorem 4.3.

For a translation paired with a rotation or screw motion, the target angular speed is nonzero.

Write the common generators as $Z_s = a\partial_z$ and $Z_t = \omega\partial_\Theta + b\partial_Z$, with $a\omega \neq 0$. Divide both generators by ω and use the common length scale $|a/\omega|$ in the source and target. After choosing the source axis orientation, the generators become ∂_z and $\partial_\Theta + p\partial_Z$. These common spatial scalings and rigid coordinate changes preserve the singular values of Df . The equivariant map is

$$f = (R \cos(z + \psi), R \sin(z + \psi), pz + z_f), \quad (6.11)$$

where p is any real number. Thus $p = 0$ pairs the source translation with a target rotation, and $p \neq 0$ pairs it with a target screw motion. The open image of a deformation cannot lie in the target axis. On an off-axis patch put

$$q = R^2/2, \quad v = z_f - p\psi, \quad v = 2q + p^2, \quad Q = v - p^2 > 0.$$

The Jacobian identity $\det Df = -\text{Jac}(q, v)$ makes (q, v) local coordinates in the source plane. If G is the Euclidean plane metric in these coordinates, then

$$\begin{aligned} g &= G + dz^2, \quad C_f = K + v(dz + a)^2, \\ K &= \text{diag}(Q^{-1}, Q/v), \\ a &= d\psi + (p/v)dv, \quad \det G = j^{-1}. \end{aligned} \quad (6.12)$$

The equilibrium and spectral equations are

$$\begin{aligned} \text{tr}[G^{-1}(K + vaa^\top)] &= \mathcal{I}_1 - v, \\ \text{tr}(GK^{-1}) &= \mathcal{I}_2/j - (1 + va^\top K^{-1}a)/v, \end{aligned} \quad (6.13)$$

$$\begin{aligned} \text{div}(vG^{-1}a) &= K_{C_f}, \\ -\text{div}(K^{-1}a) &= K_{C_f^{-1}}, \end{aligned} \quad (6.14)$$

$$\begin{aligned} d \text{div}_G(K + vaa^\top) &= 0, \\ d \text{div}_G(vK) &= 0. \end{aligned} \quad (6.15)$$

Here $K_{C_f}, K_{C_f^{-1}} \in \mathbb{R}$.

Lemma 6.9 (A harmonic axial coordinate). *Every deformation satisfying (6.13)–(6.15) satisfies $\Delta_G z_f = 0$.*

Proof. The product rule, phase closedness, and the forward vertical equation give

$$\text{div}_G(vaa^\top) = K_{C_f}a + \frac{v}{2}d(a^\top G^{-1}a) + v\iota_{a^\#}da.$$

Subtracting $v^{-1} \operatorname{div}_G(vK)$ and using the first trace equation yields

$$(d \operatorname{div}_G(K + vaa^\top) - v^{-1} d \operatorname{div}_G(vK))_{qv} = -\frac{4}{v} \Delta_G z_f.$$

Here $dz_f = pa + K_{22}dv$ and $\det G$ is constant. The full product identity is (D.5). Both horizontal equations vanish, so z_f is harmonic. $\square$

The two trace equations reconstruct G algebraically from (q, a) . Set $b = \sqrt{v}K^{-1/2}a$, $\mathcal{N}_b = b^\top b$ and $\widehat{G} = K^{-1/2}GK^{-1/2}$. The trace and determinant of $\widehat{G}$ are fixed functions of $(v, \mathcal{N}_b)$ and its entry in the b direction is fixed as well; its off-diagonal entry has two signs. If $a = 0$ on an open set, the vertical direction is principal. If that off-diagonal entry vanishes on an open set, the horizontal direction perpendicular to a is principal. In either case the corresponding h_i vanishes: a horizontal, z -independent unit vector has vertical curl. This contradicts $h_1 h_2 h_3 \neq 0$.

On the remaining chart the metric and phase are reconstructed from four signed scalar variables $\mathcal{N}_b, L, T, U$, defined in (D.6)–(D.7). The equations and their finite-fibre consequence are proved in Appendix D.2.

The chart with $U = 0$ has a separate elementary reduction. Closedness of the phase and inverse equilibrium imply that $w = \psi_v$ is constant when $p \neq 0$. For $p = 0$, a nonconstant w would make (v, w) independent coordinates, whereas harmonicity and the trace equations impose a nonzero polynomial relation between them. Thus w is constant in either case, and the reconstructed metrics and connections depend only on v . Lemma 6.6 excludes this branch.

Lemma 6.10 (Translation and rotation fibres). *Fix the principal values, the pitch p , and the real constants $K_{C_f}, K_{C_f^{-1}}$. For a solution on the reconstruction chart (D.6)–(D.7), all the variables $\mathcal{N}_b, L, T, U$ have finite fibres at each fixed v , apart from finitely many levels of v .*

Proof. Closedness of the phase, vertical equilibrium, and harmonicity give a first-order system for the four reconstruction variables. On the zero set of its determinant, its compatibility condition and the two quadratic reconstruction relations leave finitely many choices over each v , apart from finitely many levels.

Where the determinant is nonzero, solving the system gives two Frobenius equations. Their algebraic solution set has no place with $\mathcal{N}_b = \infty$, except at finitely many v levels, as proved in Appendix D.2.1. The leading pressure relations there are nonzero for every positive, pairwise distinct triple of principal values and every real pressure pair. Lemma 6.7 therefore makes the $\mathcal{N}_b$ fibre finite.

Continuity on a fixed- v slice of the solution now implies $\partial_v \mathcal{N}_b = 0$. Together with the solved system this gives a nonzero algebraic equation for L . Its nonvanishing on the exceptional spectral locus $\mathcal{I}_2^2 = 4j\mathcal{I}_1$ is included in (D.14). Thus L has finite fibres away from finitely many further levels, and (D.6)–(D.7) determine finitely many T, U . $\square$

Because $v_q = 2$, v is a local coordinate and no finite set of exceptional levels contains an open solution. The two reconstruction signs and the finite choices of $\mathcal{N}_b, L, T, U$ determine both metric matrices, their adapted connections and the frame derivatives of v . The same descent as for two translations therefore applies: continuity makes these choices constant on a connected fixed- v slice, and frame differentiation preserves dependence on v alone. Lemma 6.6 reduces both (T, R) and (T, S) to invariant rank at most one. This contradicts the result of Section 4 on a region where $h_1 h_2 h_3 \neq 0$.

6.3 Two rotations and cylindrical states

For two rotations the orbit equations leave the radius and the axial coordinate. The powers of the radius in the state equations suggest using $r\partial_r$ and $r\partial_z$ on a compact normalized state set. Their bracket will constrain the possible two dimensional orbits. Normalize the source angular speed to one and let $k \neq 0$ be the target speed. Radius normalization replaces the physical state by $(r\mathbf{p}, r^2\boldsymbol{\tau}, r^3\mathbf{u})$ and the common field value by $\mathbf{w}^Z/r$, leaving the angular vectors unchanged. Identify these coordinates with $\mathbf{U}$ at radius one and set

$$N = \{\mathbf{U} \in \mathcal{N}_{(\mathbf{R}, \mathbf{R})} : |\mathbf{w}^Z|^2 = 1\}.$$

Here the angular norms are $|\Omega_s^{\text{axial}}|^2 = 1$, $|\Omega_t^{\text{axial}}|^2 = k^2$, both pitches vanish, and the pressure data are fixed. In particular,

$$\Omega_s^{\text{axial}} \cdot \mathbf{w}^Z = 0, \quad \Omega_t^{\text{axial}} \cdot \text{diag}(\sqrt{\lambda_i})\mathbf{w}^Z = 0.$$

The full zero set includes states with $h_i = 0$ and the principal directions on the boundaries of the coordinate charts. It is closed and semialgebraic. Since $|\Omega_s^{\text{axial}} \times \mathbf{w}^Z| = 1$, Lemma 6.4 bounds all seventeen state coordinates; the field value and angular vectors have fixed norms. Thus N is compact. Its normalized physical fields

$$\mathcal{R} = r\partial_r, \quad \mathcal{T} = r\partial_z$$

are analytic in a neighbourhood of N by weighted homogeneity and preserve N . Compactness therefore makes both state flows complete. They satisfy

$$[\mathcal{R}, \mathcal{T}] = \mathcal{T}. \tag{6.16}$$

The source angular axis and $Z_s/|Z_s|$ recover the physical cylindrical frame. Thus a constant augmented state determines a constant tensor in that frame.

The relation (6.16) has two uses. It propagates vanishing of the axial field, which gives the cylindrical family below, and it prevents two independent fields from filling a compact surface. To use the latter argument we shall first show that the cylindrical states are isolated; deleting them will then leave a compact invariant set on which both fields are independent.

Lemma 6.11 (Cylindrical deformations). *If $\mathcal{T} = 0$ at a $(\mathbf{R}, \mathbf{R})$ solution state, its continued germ is*

$$f(r, \theta, z) = (\alpha r \cos(k\theta + \beta \log r + \theta_0), \alpha r \sin(k\theta + \beta \log r + \theta_0), \gamma z + z_0). \tag{6.17}$$

Proof. The bracket transports the zero value along radial flow, and axial flow fixes it. Thus the full tensor is independent of z on the whole meridional half-plane. The pushforward of ∂_z is a Euclidean Killing field commuting with the target rotation, hence $a\partial_\Theta + \gamma\partial_Z$. Since R/r is bounded above and below by positive singular-value bounds, $R \rightarrow \infty$ as $r \rightarrow \infty$; the bounded length of $Df\partial_z$ forces $a = 0$. Invertibility gives $\gamma \neq 0$. We may consequently write $R = R(r)$, $\psi = \psi(r)$, $z_f = \gamma z + d(r)$. The Jacobian gives $R^2 = Jr^2/(k\gamma) + B_0$, and the two-sided R/r bounds as $r \rightarrow 0$ give $B_0 = 0$, hence $R = \alpha r$. The quantities $r\psi'$ and d' are bounded by the fixed principal values. The axial and angular components of (6.1) integrate to

$$(C_f)_{rz} = L_z r/2 + D_z/r, \quad r\psi' = \beta + D_\theta/r^2.$$

The bounds at 0 and ∞ force $L_z = D_z = D_\theta = 0$. Thus $d' = 0$ and $\psi = \beta \log r + \theta_0$. $\square$

If $\mathcal{R}, \mathcal{T}$ are dependent at a point, they either give the cylindrical deformation directly or a dilation about a centre $D = r\partial_r + (z - z_c)\partial_z$ annihilating the full state there. Because D commutes with both physical frame fields, its action on the state satisfies a homogeneous

linear transport equation. It vanishes everywhere on the continued half-plane. The two metrics therefore share the independent dilation and rotation. The result for two common symmetries leading to (B.38) in Section 4 gives (6.17). Conversely its normalized state is constant. Thus dependence occurs exactly at cylindrical states; both fields vanish there. For fixed data these states form a finite set: γ^2 is one of the three principal values, and the transverse determinant and trace determine α^2 and β^2 . Only finitely many principal frame signs remain after the development constants are fixed.

The following linearization and second order obstruction exclude accumulation of non-cylindrical states at cylindrical states.

Lemma 6.12 (Linearization at a cylindrical deformation). *Normalize a reference cylinder by $R = r$, $\psi = \omega \log r$, $z_f = \gamma z$, and put $E = \gamma^2$ and $\Delta = E^2 - E(1 + \omega^2 + k^2) + k^2 \neq 0$. With the pressure data and development constants fixed, the linearized system has five modes (A_0, A_1, h, b, v) , described in Appendix D.4. The radial linearization is semisimple, with weights*

$$(-2, -1, -1, +1, +1). \quad (6.18)$$

For $\omega \neq 0$, the leading coefficient of every nonconstant real analytic arc in N through the cylinder satisfies $A_0 = A_1 = 0$ and

$$(E - k^2)h^2 = 0, \quad \mathcal{P}(b, v) = 0. \quad (6.19)$$

Here $\mathcal{P}$ is the homogeneous quadratic form defined in Appendix D.4; it is nonzero for every admissible triple of principal values, including $E = k^2$.

Proof. The linearized spectral equations first separate the axial variation from the two planar variations. The angular equilibrium equations then restrict their dependence on z , and the horizontal equations give the two radial axial modes. Fixing the pressure data removes the remaining quadratic axial variation. At zero twist the two pressure equations also remove the two additional phase modes. The integrations are given in Appendix D.4. Lemma 6.3 identifies these full linearized germs with the Jacobian kernel of the differential ideal. Independent Jacobian rows therefore define a five dimensional analytic manifold containing N ; extending the state fields to this manifold leaves their values on N and their linearizations unchanged. The radial weights follow by scaling the five physical variations.

By semialgebraic curve selection [1, Theorem 2.5.5 and Proposition 8.1.13], the nonzero directions in the real tangent cone of N are represented by leading coefficients of analytic arcs, up to positive scalar multiples. The quadratic obstruction concerns these coefficients.

For the quadratic assertion, apply analytic curve selection to the real algebraic state set and collect the first nonzero order of the resulting arc. In its reconstruction the first nonzero order satisfies the linearized system and twice that order satisfies the second order compatibility. Intermediate orders are homogeneous linear solutions and do not alter the obstruction. The coefficient identity (D.22), together with $\Delta \neq 0$ and $\omega \neq 0$, proves that $\mathcal{P}$ is nonzero also on the zero set of its first coefficient. $\square$

Lemma 6.13 (Isolation of cylindrical states). *For fixed principal values and pressure data, each cylindrical state is isolated in N .*

Proof. Write U_c for the cylindrical state and first suppose $\omega \neq 0$. We exclude convergent nonconstant radial orbits, then apply closedness and invariance to all nearby points of N .

There is no nonconstant forward radial orbit in N converging to this cylinder. In the nonresonant case (6.19) leaves no stable tangent direction. In the resonant case only h remains in the stable tangent cone. Thus $|U - U_c| \asymp |h|$, $h' = -h + O(h^2)$, and $h(t) = c_0 e^{-t}(1 + o(1))$ with $c_0 \neq 0$. The linear axial action sends A_1 to A_0 and vanishes on (h, b, v) , so $\mathcal{T}(U(t)) = o(e^{-t})$. On the other hand (6.16) gives, for $q(t) = \mathcal{T}(U(t))$,

$$q' = (D\mathcal{R}(U(t)) + I)q. \quad (6.20)$$

The least symmetric eigenvalue of the limiting diagonal matrix is -1 , and its error is $O(e^{-t})$. A nonzero solution therefore satisfies $|q(t)| \geq ce^{-t}$, a contradiction. Hence $\mathcal{T} = 0$ and Lemma 6.11 gives the cylinder itself.

We pass from convergent radial orbits to a neighbourhood of the cylinder. For a hyperbolic field straighten its stable and unstable manifolds [2, Theorem 1.1 and Remark 6] to the coordinate axes (s, u) . In a small product block, $d|s|^2/dt \leq -c|s|^2$ and $d|u|^2/dt \geq c|u|^2$. If an invariant closed set with no nonconstant stable orbit had points $(s_n, u_n) \rightarrow 0$ with $s_n \neq 0$, flow backwards to $|s| = \delta$. The exit points have $u_n^* \rightarrow 0$, and a subsequential limit is a nonzero point of its stable manifold in the closed set, a contradiction. Consequently N lies locally in the two dimensional unstable manifold. On that manifold $\mathcal{R} = u + O(|u|^2)$ and the convergent analytic map

$$L(u) = \lim_{t \rightarrow -\infty} e^{-t} \Phi_t(u)$$

linearizes the field. Convergence follows from the bound $|e^{-t} O(\Phi_t(u)^2)| \leq C|u|^2 e^{(1-2\epsilon)t}$. The image $L(N)$ is a radial cone, and every radial segment lifts to an analytic arc in N . Hence (6.19) gives $L(N) \subset \{\mathcal{P} = 0\}$, a union of at most two real lines. Two independent state fields cannot act on such a set. Every nearby point is therefore a cylinder, and the finite set of cylinders is locally isolated.

At zero twist the fixed angular pressures vanish. The phase equation on the full meridional half-plane is

$$\operatorname{div}_{r,z}(r^3 A_{\text{ell}}(r, z) \nabla \psi) = 0, \quad \lambda I \leq A_{\text{ell}} \leq \Lambda I, \quad |\nabla \psi| \leq C/r. \quad (6.21)$$

Here one can take $A_{\text{ell}} = (R/r)^2 I$ from the forward equation. Lift $\psi(r, z)$ to $U(x, z) = \psi(|x|, z)$ on $\mathbb{R}^4 \times \mathbb{R}$. The axis has vanishing flux $O(\epsilon^2)$, so the lift is an entire weak uniformly elliptic solution. Integrating the gradient along the three-segment path through radius $\max(1, r, |z|)$ gives $|\psi| \leq C(1 + |\log r| + \log(2 + |z|))$. Its mean-square growth on five dimensional balls is $O(1 + \log^2 R)$. The interior elliptic oscillation estimate [8, Theorems 8.17 and 8.22], after rescaling to the unit ball, gives

$$\operatorname{osc}_{B_\ell} U \leq C(\ell/R)^\alpha (1 + \log R) \rightarrow 0.$$

Thus ψ is constant. Constancy of the principal values makes $R = \alpha r$; the Jacobian makes $z_f = \gamma z + d(r)$; constant trace gives constant d' , and the remaining curl equation gives $d' = 0$. The zero twist endpoint is therefore isolated as well. $\square$

Lemma 6.14 (Pointwise dimension lower bound). *Every noncylindrical point of N has local real dimension at least three.*

Proof. The finite cylindrical set is closed, and Lemma 6.13 makes it open in N . Its complement N^* is therefore compact and both open and closed in N . Cylindrical states are fixed by both flows, so N^* is invariant and the two complete fields are independent there. If an invariant compact semialgebraic subset $A \subset N^*$ had dimension two, an affine transverse section Σ and the two state flows identify it locally with $(-\epsilon, \epsilon)^2 \times (A \cap \Sigma)$. Semialgebraic curve selection forces the transverse factor to be discrete; otherwise a three dimensional immersed piece would occur. Thus each point has a full two dimensional orbit neighbourhood, and A is a compact smooth surface without boundary. Define its positive area form by $\omega(\mathcal{R}, \mathcal{T}) = 1$. Equation (6.16) gives $\mathcal{L}_{\mathcal{R}} \omega = -\omega$. Stokes' theorem gives $0 = \int_A d(\iota_{\mathcal{R}} \omega) = -\int_A \omega$, a contradiction. If only the local dimension at some point were two, take the semialgebraic two dimensional local-dimension stratum, one of its connected components, and its closure in N^* . The flows preserve local dimension and each connected component, hence also its closure. Since N^* is closed and compact and semialgebraic closure does not increase dimension, this produces just such a compact A . This proves the local dimension bound at each point. $\square$

6.4 Connection fibres for two rotations

The preceding lower bound concerns the augmented state set, including its higher derivatives. We now project that set to the two dimensional surface of rotation directions and bound its connection fibres. At the directions where the fibre is finite this gives an upper bound of two; those directions are consequently absent from any noncylindrical solution germ. The resulting restriction on the direction image supplies an additional differential equation for the remaining fibres.

Choose the direction $\nu = Z_s/|Z_s|$ of the common field with all principal components nonzero. Singular value decomposition in the two transverse planes gives the physical chart

$$F_f = \begin{pmatrix} s & 0 & 0 \\ 0 & t & 0 \\ p & q & \gamma \end{pmatrix}, \quad s, t, \gamma > 0, \quad pq(s^2 - t^2) \neq 0. \quad (6.22)$$

The identity

$$\det[\nu, C_f^{-1}\nu, C_f^{-2}\nu] = \frac{pq(s^2 - t^2)}{s^4 t^4 \gamma^2}$$

shows that this is a coordinate chart for the stated direction condition.

Lemma 6.15 (A finite fibre direction for every triple of principal values). *For each triple of positive, pairwise distinct principal values there is a rotation direction of the common field whose affine connection fibre is finite. The set E of directions where this fails is a proper algebraic subset of the direction surface and has real dimension at most one.*

Proof. The exact Hessian and third derivative systems in Appendix D.5 leave three necessary homogeneous quadrics in the two angular amplitudes and one Hessian entry. Near an extreme principal direction, two consequences are a binary cubic and quartic with no common projective root for any admissible triple of principal values; Appendix D.6 proves this nonvanishing. Keeping the principal values fixed changes the chart parameters only at second order and leaves these leading equations unchanged. Thus the fibre is finite at a nearby direction for every fixed triple of principal values. Closedness of projective projection gives the assertions concerning E . $\square$

At a good direction the affine connection fibre is finite, since its homogeneous system has no projective root. Higher states are recovered uniquely by the common field equations on the reconstruction chart. Thus the augmented state has local dimension at most two, and Lemma 6.14 forbids noncylindrical points there. Every remaining noncylindrical solution germ has direction image in E and direction rank at most one. Consequently

$$d(\nu^\top C_f \nu) \wedge d(\nu^\top C_f^2 \nu) = 0. \quad (6.23)$$

The two scalars are local direction coordinates by the nonzero spectral Vandermonde and $\nu_1 \nu_2 \nu_3$. Since $\mu = |Z_s|$, differentiation in Cartesian coordinates gives

$$\mu \partial_{x_a} \nu = \Omega_s^{\text{axial}} \times \partial_{x_a} - \nu \{ \nu \cdot (\Omega_s^{\text{axial}} \times \partial_{x_a}) \}.$$

Equation (6.23) holds on an open set and may therefore be added to the differential ideal.

Lemma 6.16 (Connection fibres on the exceptional directions). *After (6.23) and all its derivatives are included in the differential ideal, every affine connection fibre over a direction in E has real dimension at most one.*

Proof. Appendix D.7 eliminates the remaining Hessian entry from the direction equation and an axial pressure equation. The resulting binary cubic is nonzero for every point of (6.22): the positive coefficient identity given there prevents its two extreme coefficients from

vanishing simultaneously. There are therefore at most three projective angular ratios. For each ratio the remaining entry is either determined linearly or satisfies a nonzero quadratic. The point with both angular amplitudes zero is excluded by the same quadratic coefficient. Hence the projective part at infinity is finite, and an affine component has dimension at most one. $\square$

The good directions contribute dimension at most $2 + 0$, and the bad directions at most $1 + 1$. Semialgebraic triviality and the dimension formula for products [1, Theorem 9.3.2 and Proposition 2.8.5] give the upper bound two everywhere, contradicting Lemma 6.14. Finally the principal-direction chart is dense in every solution chart where $h_1 h_2 h_3 \neq 0$: at $w_i^Z = 0$ the coframe identity is

$$|\nabla w_i^Z|^2 = K_i^2 + h_i^2 |\mathbf{w}^Z|^2 > 0.$$

Each such zero set has empty interior. Their finite union cannot cover a solution chart. Thus $h_1 h_2 h_3 = 0$ in the two-rotation case.

The augmented states are analytic. On a connected analytic eigenframe neighbourhood, as justified by Lemma 2.3, $h_1 h_2 h_3 \equiv 0$ implies some fixed $h_i \equiv 0$. Since $\alpha_i \wedge d\alpha_i = h_i \text{vol}_g$, Kafadar's theorem on a principal direction normal to surfaces, followed by his classification with a constant principal direction, applies [11]. Analyticity supplies its Taylor hypothesis. Its alternatives are affine maps and (6.17). An affine map with distinct principal stretches cannot intertwine nonzero rotations: $F_f A_s = A_t F_f$ with skew A_s, A_t implies $[F_f^\top F_f, A_s] = 0$, hence $A_s = 0$. Thus only the known cylindrical family remains in $(\mathbb{R}, \mathbb{R})$.

6.5 Two screw motions

Normalize the source angular speed to one and write

$$Z_s = \partial_\theta + p\partial_z, \quad Z_t = k\partial_\Theta + q\partial_Z, \quad pqk \neq 0.$$

We prove that every such local deformation with constant, pairwise distinct principal stretches is (6.17) with $\gamma = q/p$.

At large radius the normalized pitches tend to zero, so continuation and compactness reduce limits of these states to the two-rotation classification. We identify the limiting axial stretch from the pitch ratio, then use the resulting cylinder to study the radial and axial fields near the limit.

By Lemma 6.5 continue the solution germ to all $r > 0$ and all axial positions on the angular cover. Normalize the augmented states by $\mu = \sqrt{r^2 + p^2}$. On $r \geq r_0 > 0$, the transverse quantity r/μ has a positive lower bound. Lemma 6.4 bounds the weighted state, and the two orthonormal frames range over compact groups. Thus the augmented states have convergent subsequences. Rescaling source and target by the same μ preserves the principal values, k and the longitudinal pressure data. Its normalized pitches $p/\mu, q/\mu$ tend to zero. Every limiting state has a local realization as a $(\mathbb{R}, \mathbb{R})$ germ by the closed-ideal realization argument, and the preceding classification identifies it as a cylinder.

For fixed principal values, k and longitudinal pressure data, the cylindrical state set is finite after the target phase and axial translation gauges have been removed. The quotient half-plane (r, w) , where $w = z - p\theta$, is contractible, so choose a continuous ordered eigenframe on it. Compactness first gives uniform convergence to the finite cylinder set: otherwise an escaping sequence away from that set would have a noncylindrical limiting state. Each exterior tail $[R, \infty) \times \mathbb{R}$ is connected. For small disjoint neighbourhoods of the finitely many cylinders its image must lie in just one; nested tails select the same cylinder. Thus for a single augmented cylindrical state $\mathcal{U}_c$,

$$\sup_{r \geq R, w \in \mathbb{R}} |\mathcal{U}(r, w) - \mathcal{U}_c| \longrightarrow 0. \quad (6.24)$$

This controls every point of a full axial orbit at each large radius, as well as every point on its subsequent outward radial trajectories. These are the two orbit hypotheses needed in Lemma 6.17. At sufficiently large radius the target cylindrical frame is well defined, since $k^2 R_{\text{target}}^2 + q^2 \geq \lambda_{\min}(r^2 + p^2)$.

Let $M = \sqrt{\lambda_{\max}}$. The bound $|Df| \leq M$ gives a continuous radial trace on the source axis. Traces from different finite covering angles agree, since their connecting arc has length $O(r)$. Equivariance therefore gives an axis trace of the form

$$\lim_{r \downarrow 0} f(r, \theta, z) = \text{Rot}_{kz/p} v_{\perp} + (q/p) z e_Z + z_0 e_Z. \quad (6.25)$$

The transverse vector $v_{\perp}$ is arbitrary. Radial integration yields the global bound

$$|f_3(r, \theta, z) - (q/p)z - z_0| \leq Mr. \quad (6.26)$$

If the limiting cylinder has axial stretch γ_{∞} , (6.24) implies $\partial_z f_3 \rightarrow \gamma_{\infty}$ uniformly in z . Fix a sufficiently large r such that $|\partial_z f_3 - \gamma_{\infty}| \leq \delta$ for every z . Integrating over an axial interval of length L and using (6.26) gives

$$|\gamma_{\infty} - q/p| \leq \delta + 2Mr/L.$$

Let $L \rightarrow \infty$ and then $\delta \rightarrow 0$. Hence

$$\gamma_{\infty} = q/p. \quad (6.27)$$

There is therefore an exact reference cylinder matching the prescribed principal values, k , pressures, and pitch ratio.

We linearize about the cylinder with this axial stretch. Variation of its normalized pitch leaves the five two-rotation modes unchanged.

Normalize the state by r and put $e = p/r$, so $\mathcal{R}e = -e$. The ratio $\mu/r = \sqrt{1 + e^2}$ is analytic and nonzero near $e = 0$. After a fixed target length normalization the matched reference cylinder is $R = r$, $\psi = \omega \log r$, $z_f = \gamma z$, $\gamma = q/p$. The exact reference cylinder intertwines the normalized screw fields for every pitch e . Its derivative with respect to e is therefore an exact reference variation. After subtracting it, equivariance linearizes to the two-rotation system, since $\gamma e_Z - Df_c e_Z = 0$. The reference pressure potentials have zero axial derivative, so the angular pressure equations are unchanged. Thus the five modes of Lemma 6.12 remain, with one independent pitch mode. The corresponding equations, including the zero twist endpoint, are given in Appendix D.8.

The full linearized system consequently has six coordinates

$$(A_0, A_1, h, b, v, \delta e).$$

Lemma 6.3 applies to the fixed common screw equations, the pressure data and their differential closure. After the derivative of the exact matched cylinder is subtracted, the augmented variational reconstruction is precisely the (R, R) system above together with the independent pitch variation. Its six dimensional space is therefore the Jacobian kernel of the differential ideal. Independent Jacobian rows define an analytic manifold of dimension six containing $\mathcal{N}_{(S,S)}$.

The radial scaling acts on these six coordinates with weights

$$(-2, -1, -1, +1, +1, -1). \quad (6.28)$$

The exact matched reference curve is an eigenvector in the last direction. The five old modes give an invariant complementary hyperplane and their weights are already diagonal. Thus the linearization is semisimple. The radial and axial state fields can be analytically extended to this containing manifold, preserving their values on the invariant state set. On the invariant state set, $[\mathcal{R}, \mathcal{T}] = \mathcal{T}$.

The weights show that any point of the invariant state set whose forward radial orbit stays near this cylinder and converges to it lies in its local stable manifold. The bracket then constrains the axial field on that stable manifold. The next lemma uses boundedness of the entire axial orbit to prove that the axial field vanishes.

Lemma 6.17 (A bounded axial orbit in the stable set is constant). *Suppose a real analytic radial field X has a semisimple equilibrium with weights $(-2, -1, -1, -1, +1, +1)$, and $T(0) = 0$ is an analytic field on an invariant subset satisfying $[X, T] = T$ there. Suppose complete two-sided T orbits stay in one sufficiently small neighbourhood, and every point on them has a forward X orbit staying there and tending to zero. Then T vanishes on these orbits.*

Proof. All these points lie in the four-dimensional local stable manifold [2, Theorem 1.1 and Remark 6]. There are analytic coordinates $(x, a) \in \mathbb{R}^3 \times \mathbb{R}$ on it in which

$$x' = -x, \quad a' = -2a + Q_2(x), \quad (6.29)$$

with Q_2 a homogeneous quadratic form. The only resonant terms are quadratic terms in the a equation. The analytic normalization and its convergence are proved in Appendix D.8.

The integral curves of T in the invariant subset lie in the stable manifold because every one of their points has the prescribed radial limit. Along a radial trajectory $y(t)$ in the invariant subset, put $q(t) = T(y(t))$. The bracket identity gives

$$q' = (DX(y(t)) + I)q.$$

In (6.29), $q'_x = 0$. Since $T(y(t)) \rightarrow T(0) = 0$, $q_x = 0$. Then $q'_a = -q_a$. Write $T_a(x, a) = L(x) + \ell a + O(|(x, a)|^2)$ and insert the explicit radial trajectory. Multiplying by e^t and taking the limit gives the exact identity on the invariant subset

$$T(x) = 0, \quad T(a) = L(x). \quad (6.30)$$

Along a complete axial orbit this says $x(s) = x_0$, $a(s) = a_0 + sL(x_0)$ for all real s . Boundedness in one coordinate neighbourhood forces $L(x_0) = 0$. $\square$

Apply the lemma to the whole sufficiently far exterior tail in (6.24), after passing from μ normalization to the analytic r normalization above. The pitch coordinate $e = p/r$ tends to zero on every forward radial trajectory, and the remaining coordinates converge uniformly to the matched cylinder. Taking the tail sufficiently far out puts all these trajectories, and every fixed-radius full axial orbit, in one coordinate neighbourhood of that equilibrium. The hypotheses of Lemma 6.17 are thus satisfied on the invariant state set. Its conclusion is that the axial state field vanishes on the tail.

The source angular vector and the common field recover the physical cylindrical frame for every $r > 0$. Thus $\mathcal{T} = 0$ makes the physical tensor independent of z ; the bracket and ODE uniqueness continue this identity to the entire off-axis germ. The pushforward of ∂_z commutes with the target screw and is therefore $a\partial_\Theta + \gamma\partial_Z$. The fixed singular-value bound and $R_{\text{target}} \rightarrow \infty$ force $a = 0$. Then

$$f_*\partial_\theta = k\partial_\Theta + (q - p\gamma)\partial_Z.$$

Its norm is at most Mr , so letting $r \rightarrow 0$ gives $q = p\gamma$. We now have both a pure-rotation pair and a common axial translation. The calculation in Lemma 6.11 gives exactly (6.17).

The parameter range of the cylindrical deformations is as follows. Choose an axial eigenvalue $\gamma^2 = \lambda_i$, and write the other principal stretches as $0 < u < v$. Set $K = |k|$. The cylindrical gradient is

$$\begin{pmatrix} \alpha & 0 & 0 \\ \alpha\beta & \alpha k & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

Its planar trace and determinant give

$$\frac{u}{v} \leq K \leq \frac{v}{u}, \quad \alpha^2 = \frac{uv}{K}, \quad \beta^2 = (K - u/v)(v/u - K). \quad (6.31)$$

Positive orientation requires $k\gamma > 0$ and $J = \alpha^2 k\gamma$ is the specified Jacobian. For two screw motions one must additionally have $\gamma = q/p$.

Conversely, every parameter choice in (6.31) gives a universal deformation. Both C_f and C_f^{-1} have constant cylindrical components and zero axial cross components. For any such tensor Q ,

$$\operatorname{div} Q = \left(\frac{Q_{rr} - Q_{\theta\theta}}{r}, \frac{2Q_{r\theta}}{r}, 0 \right) = \nabla \{ (Q_{rr} - Q_{\theta\theta}) \log r + 2Q_{r\theta} \theta \}.$$

Thus both equations (6.1) hold on an angular cover. The principal vectors are constant mixtures of e_r, e_θ and the axial direction, so all three quantities h_i vanish.

Proof of Theorem 6.1. (T, T), (T, R) and (T, S) reduce to state rank at most one, contradicting Section 4; (R, S) admits no germ. The (R, R) and (S, S) arguments give a cylindrical deformation with $h_i = 0$; the screw case also requires the pitch relation $q = p\gamma$. $\square$

7 Common dilations and rotations

We now treat the common similarities whose scale is nonzero. A single rescaling of their common parameter makes that scale equal to one. Since $I + \Omega$ is invertible for every real skew matrix Ω , each generator has a unique centre. Translating these centres gives

$$Z_s(x) = x + \Omega_s x, \quad Z_t(y) = y + \Omega_t y, \quad Df Z_s = Z_t \circ f. \quad (7.1)$$

Write $\Omega_s x = \omega_s e_z \times x$ after a source rotation, and use an independent target rotation to put Ω_t in its corresponding axial form. Here $\omega_s, \omega_t \in \mathbb{R}$.

Theorem 7.1. *Let f be a C^3 local universal deformation with fixed, positive, pairwise distinct principal stretches. If it satisfies (7.1), then*

$$h_1 h_2 h_3 \equiv 0.$$

The proof passes from the local deformation to its continuation on punctured space. An estimate at the distance from the source axis provides this continuation and identifies the possible axial limits. Cylindrical limits give an additional common rotation; homogeneous limits give regular points of the axis. In the latter case the pressure flux through a sphere bounds the longitudinal pressure constants, and a strictly increasing scalar along a principal curve gives the final contradiction.

7.1 An estimate at the cylindrical scale

We use the state S and the fields Q_i of (2.21), with the weighted scaling (2.23). The homothety identity $\mathcal{L}_{Z_s} \alpha_i = \alpha_i$ determines the derivative of every state coordinate along Z_s by its weight. The fixed angular speeds give bounded components of the source and target angular vectors in the principal frames. The three longitudinal projector pressures K_i are constant, with $\sum_i K_i = 0$. Their expressions as linear functions of the connection are given in (E.5). The fixed data in the estimate below are the principal values, angular speeds and pressure constants.

When both angular speeds vanish, the transverse angular term used in the estimate below is zero. We first use the homothety equations directly in this case.

Lemma 7.2 (Pure dilation on both sides). *If $Df(x)x = f(x)$, then $h_1h_2h_3 \equiv 0$ for every triple of positive, pairwise distinct principal values.*

Proof. Suppose $h_1h_2h_3 \neq 0$ on an open set, and write $D = x \cdot \nabla$ and $w_i^Z = \alpha_i(D)$. If w_i^Z vanished on an open set, Cartan's formula and $\mathcal{L}_D\alpha_i = \alpha_i$ would give $\iota_D d\alpha_i = \alpha_i$. On the contact plane $\ker \alpha_i$, $d\alpha_i$ is nondegenerate and D belongs to that plane. Its contraction would therefore force $D = 0$, impossible on an open set. Thus each zero set has empty interior, and on a smaller chart $w_1^Z w_2^Z w_3^Z \neq 0$.

When both angular vectors vanish, the homothety equations and their directional derivatives imply $h_1h_2h_3 = 0$ on this chart. Indeed, after ordering $0 < b < a < 1$, their reduction in Appendix E.2.1 gives two affine equations for the same nonzero scaled component $w_3^Z h_1$. Their determinant (E.11) never vanishes on the real direction domain. This contradicts the assumed nonvanishing. $\square$

The maps $(r, \theta, z) \mapsto (\alpha r, k\theta, \gamma z)$ are homogeneous of degree one on local angular sectors. They satisfy $h_i = 0$, as required by Lemma 7.2.

For the rest of the proof at least one angular speed is nonzero. By inversion of the deformation if necessary, assume $\omega_s \neq 0$. Write $r = \text{dist}(x, \mathbb{R}e_z)$ and $\mu = |Z_s(x)|$.

The homothety equations naturally control the state at scale μ . Near a noncentral point of the axis, however, r/μ tends to zero. The transverse component of the source angular vector supplies the factor that converts this estimate to the scale r .

Lemma 7.3 (Bound at the cylindrical scale). *There is a constant C , depending only on the distinct principal values, angular speeds and projector pressure constants, such that every state arising from a solution satisfies*

$$r|\mathbf{p}| + r^2|\boldsymbol{\tau}| + r^3|\mathbf{u}| \leq C. \quad (7.2)$$

Proof of Lemma 7.3. Consider first normalization by μ . The angular and pressure equations bound the connection away from principal directions. If the connection were unbounded, a normalized subsequence would therefore approach a principal direction, and its leading state would lie in the cylindrical plane (E.14). Lemma E.1 gives a linear estimate for the normal coordinates at this plane in terms of the inverse connection size. The original angular equations then give the same estimate for the two transverse components of the source angular vector. Equivalently,

$$\mu|\mathbf{p}| |\Omega_s^{\text{axial}} \times Z_s / \mu| \leq C. \quad (7.3)$$

In each principal-direction chart, Lemma E.1 bounds the normal distance by a constant times the inverse connection size. Substitution in the angular equations gives (7.3) on that chart. A finite cover of the direction sphere and the limiting projective circles gives a common constant. The parameters in these equations are the principal values, the moving components of the angular vectors and the projector pressure constants; the angular components are bounded by the prescribed speeds.

Since $\Omega_s^{\text{axial}} = \omega_s e_z$, the Euclidean identity

$$|\Omega_s^{\text{axial}} \times Z_s| = |\omega_s| \sqrt{1 + \omega_s^2} r$$

proves the first term of (7.2). Equation (E.13) and $r/\mu \leq (1 + \omega_s^2)^{-1/2}$ prove the other two terms. The resulting constant depends only on the fixed principal values, angular speeds and projector pressure constants. $\square$

7.2 Continuation and limits at the axis

On a compact subset of the axis complement, (7.2) is an ordinary bound on all physical state coordinates. Together with the original differential equations this prevents a finite endpoint of continuation.

Lemma 7.4. *When $\omega_s \neq 0$, the germ extends as an analytic universal deformation to the universal cover of $\mathbb{R}^3 \setminus \mathbb{R}e_z$. Its state and both Euclidean frames extend there as well.*

Proof. Let $x(\tau)$ be a finite piecewise C^1 path whose image is contained in a compact subset of the axis complement. The radius r is bounded below there. Lemma 7.3 bounds all physical state coordinates. Lift the source principal frame to $\mathbf{E}(\tau) \in SO(3)$ and solve the original state and frame equations along the path:

$$\frac{dS}{d\tau} = \sum_i (\mathbf{E}^\top x')_i \mathcal{Q}_i(S), \quad \frac{d\mathbf{E}}{d\tau} = \mathbf{E} \sum_i (\mathbf{E}^\top x')_i \Gamma_i(S). \quad (7.4)$$

Their right sides are analytic with spectral denominators only. The differential ideal is preserved along these fields, so its constraints hold at each limiting state. The source frame lies in a compact group, the state is bounded, and finite path length gives a Cauchy limit at every finite endpoint. The ordinary existence and uniqueness theorem then continues the solution.

The target frame can be lifted at the same time. If m, M are the smallest and largest fixed principal stretches, $|Df| \leq M$ bounds the change of the target position along the path. Thus the map continues. Its target centre cannot be hit, since $|Z_t \circ f| = |Df Z_s| \geq m|Z_s|$. The two flatness equations and the bracket identities give compatibility around contractible loops. This gives continuation on the universal cover. $\square$

Let v and a_0 be the components of the physical vectors e_r and e_z in the principal frame, and put $w = a_0 \times v$. Define the normalized state

$$\mathbf{U}_{\text{cyl}}^{[r]} = (S^{[r]}, v, a_0), \quad |v| = |a_0| = 1, \quad v \cdot a_0 = 0. \quad (7.5)$$

It lies in a bounded subset of a finite-dimensional space. All of its derivatives on a compact set away from the physical axis are controlled by the original state equations.

To identify the axial limits, we rescale a continued deformation at its distance from the axis. At a noncentral axial point the common homothety becomes a translation in this rescaling, placing the limit under the classification already proved in Section 6.

Lemma 7.5. *Fix $z_0 \neq 0$ and a lifted angle, and let $r \downarrow 0$. Every limit of (7.5) is either the state of a homogeneous deformation or an axis-aligned state of the known cylindrical family. The full α -limit set of the radial orbit is contained in the set of homogeneous states or in a single compact cylindrical equilibrium circle.*

Proof. More generally take points x_n with $|z_n|/r_n \rightarrow \infty$. Translate to x_n , scale both configurations by r_n^{-1} , and subtract the target base value. Equation (7.2) and the analytic state equations give bounds on all jets on every compact subset of the rescaled original-axis complement. A diagonal subsequence produces a limiting deformation.

Multiply the rescaled common generator by r_n/z_n . In the source its expression tends to the nonzero translation e_z ; its linear part and the target linear part both tend to zero. The singular-value bounds keep the target limiting translation nonzero and bounded. The longitudinal pressures are multiplied by the same scalar, so their limits vanish. Hence the limiting deformation has a common translation, along whose orbits the normalized state is constant. Its state rank is therefore at most two. On any open set where $h_1 h_2 h_3 \neq 0$, Theorem 4.3 leaves only rank two, which the common translation excludes by Theorem 6.1. Thus $h_1 h_2 h_3 \equiv 0$. Analyticity fixes a principal label on a neighbourhood, and Kafadar's classification in Proposition 9.1 makes the limit homogeneous or cylindrical.

For an inhomogeneous cylindrical limit the common translation is axial. Its cylinder axis cannot be a line parallel to, but different from, the original axis. Such a line has a neighbourhood at positive distance from the original one; (7.2) provides ordinary unscaled

bounds there, so the limiting strain would extend regularly through its own cylinder axis. The strain tensor of an inhomogeneous cylindrical deformation with constant, pairwise distinct principal stretches has constant nonisotropic coefficients in a rotating cylindrical frame and has no such extension. Thus the two axes coincide.

The norm of the normalized connection at a homogeneous limit is zero. At an aligned cylindrical limit it is one, as in (E.30). The possible axis labels and frame sign choices give finitely many disjoint compact circles. The α -limit set of a bounded continuous radial orbit is the intersection of the nested compact closures of its connected tails, and is connected. It cannot meet both norm levels, or two disjoint cylindrical circles. $\square$

7.3 Rigidity at a cylindrical limit

Suppose the radial limit set lies in a cylindrical equilibrium circle. We will show that the continued germ is invariant under axial rotation. For $D_i^{[r]} = rD_i$, the state fields induced by the radial, angular and axial coordinate directions are

$$\mathcal{R} = \sum_i v_i D_i^{[r]} = r\partial_r, \quad \mathcal{A} = \sum_i w_i D_i^{[r]} = \partial_\theta, \quad \mathcal{T} = \sum_i (a_0)_i D_i^{[r]} = r\partial_z. \quad (7.6)$$

On the invariant state set,

$$[\mathcal{R}, \mathcal{A}] = [\mathcal{A}, \mathcal{T}] = 0, \quad [\mathcal{R}, \mathcal{T}] = \mathcal{T}. \quad (7.7)$$

Near a noncentral axial point put $\epsilon = r/z$. Its exact derivatives are

$$D_i^{[r]}\epsilon = \epsilon v_i - \epsilon^2 (a_0)_i, \quad \mathcal{R}\epsilon = \epsilon, \quad \mathcal{A}\epsilon = 0, \quad \mathcal{T}\epsilon = -\epsilon^2. \quad (7.8)$$

The original common similarity imposes the vector equation

$$\mathcal{T} + \epsilon(\mathcal{R} + \omega_s \mathcal{A}) = 0 \quad (7.9)$$

on all twenty-three coordinates of $\mathbf{U}_{\text{cyl}}^{[r]}$. It also holds on the additional coordinate ϵ , where its component is identically zero.

An orbit approaching the cylindrical circle as $r \rightarrow 0$ can contain only radial modes that remain bounded in negative time. Once these modes are parametrized, the commutation relation $[\mathcal{R}, \mathcal{A}] = 0$ compares their radial growth with the order of vanishing of the angular field. This comparison will force angular invariance.

The reference circle consists of the axis-aligned cylindrical states. Its principal-frame coordinates are given in (E.30). We use the radial linearization on the state set subject to (7.9).

Lemma 7.6 (Radial linearization at cylindrical states). *For each triple of positive, pairwise distinct principal values, the augmented state set near the cylindrical equilibrium circle is contained in a ten dimensional analytic manifold. The radial field extends analytically to this manifold. At the circle its centre is the tangent to the circle, its positive eigenvalue is 1 with semisimple multiplicity three, and its negative eigenvalues are -1 with multiplicity five and -2 with multiplicity one. The angular field vanishes on the circle, and its differential vanishes on the centre and the positive spectral subspace.*

Proof. The linearized constraints in the 23 state coordinates have constant rank fourteen along the circle. Their nine dimensional kernel is invariant under the linearized principal fields, and the Jacobians of all further differential consequences vanish on this kernel. The calculation in Appendix E.4 also gives the radial spectrum and the vanishing of the angular differential on the centre and positive eigenspaces. The additional coordinate ϵ contributes an independent radial eigenvector of weight 1, with zero angular linearization.

The implicit-function theorem now gives a containing analytic manifold near each point of the circle. Projecting the state fields onto its tangent bundle preserves their values on the invariant state set and their restricted linearizations at the circle. To make this construction uniform, choose finitely many constraints whose Jacobian has rank fourteen along the compact circle. Project their values onto the image of their Jacobian at the analytic nearest point of the circle. The resulting rank-fourteen analytic bundle map has surjective differential in the normal directions. Its zero set is the required tubular manifold. $\square$

The positive radial directions approach the circle in negative time. An orbit in the invariant state set whose entire limit set lies on the circle must belong to one of these incoming leaves. Commutation with the radial flow will then force the angular field to vanish.

Lemma 7.7 (Angular rigidity at a cylindrical limit). *Suppose the full negative-time limit set of a radial orbit in the invariant state set is contained in one reference circle (E.30). Then $\mathcal{A} = 0$ at every sufficiently late point in that negative-time tail.*

Proof. Work first near a point $c(p)$ of the circle on the containing manifold from Lemma 7.6. The radial centre is tangent to the circle. Its three positive eigenvalues equal one, and its six remaining eigenvalues have negative real parts. Choose an analytic frame $V(p)$ for the positive space. We construct an analytic parametrization $K(p, y)$, $y \in \mathbb{R}^3$, satisfying

$$K(p, 0) = c(p), \quad D_y K(p, 0) = V(p), \quad \mathcal{R}(K(p, y)) = D_y K(p, y)y. \quad (7.10)$$

Expand in powers of y . At degree $m \geq 2$, the coefficient of K is obtained by applying $(mI - D\mathcal{R}(c(p)))^{-1}$ to an expression in lower coefficients. The possible divisors are $m, m-1, m+1, m+2$, all nonzero. A Jordan block at the negative eigenvalue -1 contributes powers of $(m+1)^{-1}$. On a compact parameter interval the inverses have norm at most C/m . A convergent analytic majorant for the original vector field therefore majorizes the recursion and proves convergence of K . The resulting radial flow is exactly

$$(p, y) \mapsto (p, e^t y). \quad (7.11)$$

Every approaching orbit lies on this parametrized set. In tubular coordinates about its image, the transverse coordinate z satisfies

$$\dot{z} = A(p)z + O((|y| + |z|)|z|), \quad \text{spec } A(p) \subset \{-1, -2\}.$$

Choose a smoothly varying positive definite matrix $Q(p)$ satisfying $A(p)^\top Q(p) + Q(p)A(p) = -I$. After shrinking the tube, the quadratic form satisfies

$$\frac{d}{dt}(z^\top Q(p)z) \leq -cz^\top Q(p)z$$

with one $c > 0$. Unless $z = 0$, it grows by at least the factor $e^{c(t_0-t)}$ for $t < t_0$. An orbit remaining in the tube for its whole negative tail must have $z = 0$. Once on the parametrized set, the exact flow (7.11) makes p constant. Thus an orbit in the invariant state set whose whole limit set lies there has a unique limiting point and is of the form $K(p, e^{t-t_0}y_0)$.

The angular state flow preserves the augmented state set and commutes with its radial flow by (7.7). Since $\mathcal{A}(c(p)) = 0$, it fixes the limiting point and sends an incoming radial orbit into the same incoming leaf. On points of the invariant state set its representation in the coordinates of (7.11) is consequently $(0, a(p, y))$. The angular linearization in Lemma 7.6 gives

$$|a(p, y)| \leq C|y|^2. \quad (7.12)$$

On the radial orbit in the invariant state set, commutation gives the exact identity

$$a(p, e^t y) = e^t a(p, y), \quad t \leq 0. \quad (7.13)$$

Combining (7.13) with (7.12) and letting $t \rightarrow -\infty$ yields $a(p, y) = 0$. $\square$

The vanishing angular state field gives an additional Euclidean symmetry. If $\mathcal{A} = 0$ at a state in the invariant state set, commutation with $\mathcal{R}$ and $\mathcal{T}$ propagates this equality along their state flows. The angular flow itself is constant in state space. These three physical coordinate flows give an open neighbourhood. In particular the components of e_r, e_z in the principal frame are fixed by the angular state flow. Differentiating $\mathbf{E}^\top e_r = v$ and $\mathbf{E}^\top e_z = a_0$ then shows $\partial_\theta \mathbf{E} = J_z \mathbf{E}$, where $J_z x = e_z \times x$. Thus the strain is invariant under the source axial rotation:

$$\mathcal{L}_{R_s} C_f = 0. \quad (7.14)$$

Here $R_s = \partial_\theta$. The push-forward $R_t = f_* R_s$ is a Euclidean Killing field for the target metric, hence $R_t(y) = b + Ky$ with K skew. Since $[Z_s, R_s] = 0$, equation (7.1) implies $[Z_t, R_t] = 0$. Its constant and linear parts give

$$(I + \Omega_t)b = 0, \quad [K, \Omega_t] = 0.$$

The first equation gives $b = 0$; R_t is nonzero because Df is invertible and R_s is nonzero off the source axis. Hence $K \neq 0$ and the additional symmetry is a rotation on both sides. The original common homothety makes the normalized state constant along its orbits, so its rank is at most two. On an open set where $h_1 h_2 h_3 \neq 0$, Theorem 4.3 leaves rank two, which the additional common rotation excludes by Theorem 6.1. Hence $h_1 h_2 h_3 = 0$ on that neighbourhood of the solution. Analytic continuation takes this identity back to the original germ.

Consequently, if the original germ had $h_1 h_2 h_3 \neq 0$, the cylindrical alternative of Lemma 7.5 would be impossible at every noncentral point of either half-axis.

7.4 Removal of the axis at a homogeneous limit

At a homogeneous limit the weighted connection tends to zero. Recovering a regular deformation at the axis requires control of the physical connection: its weighted value contains a factor r . We use the development from an axial initial state to express the radial flow in physical coordinates.

Under the contrary hypothesis $h_1 h_2 h_3 \neq 0$, the preceding argument leaves only the homogeneous alternative at every noncentral axis point. The normalized connection and its derivative blocks tend to zero. Weighted homogeneity identifies these orbits with solutions having finite physical initial data on the axis.

Lemma 7.8 (Coordinates near a regular point of the axis). *A radial orbit in the invariant state set whose full limit set is contained in the set of homogeneous states extends analytically through its noncentral physical axis point.*

Proof. Fix a physical transverse ray $x = x_{\text{ax}} + r e_r$, with e_r, e_z fixed in Cartesian coordinates. Include a principal frame $\mathbf{E} \in SO(3)$. The physical equations along this ray are the autonomous analytic system

$$\frac{dS}{dr} = \sum_i v_i \mathcal{Q}_i(S), \quad \frac{d\mathbf{E}}{dr} = \mathbf{E} \Gamma(v, S), \quad v = \mathbf{E}^\top e_r, \quad a_0 = \mathbf{E}^\top e_z. \quad (7.15)$$

Starting from arbitrary sufficiently small finite data $(S_0, \mathbf{E}_0)$ at $r = 0$, solve (7.15) up to $r = 1$. Denote its normalized state at $r = 1$ by $\Psi(S_0, \mathbf{E}_0)$. Here the orthonormal pair (v, a_0) is identified with its principal frame $\mathbf{E}$ along the fixed physical ray. At $S_0 = 0$, $\Psi(0, \mathbf{E}_0) = (0, \mathbf{E}_0)$. Order the state blocks by $\mathbf{u}, \boldsymbol{\tau}, \mathbf{p}, \mathbf{E}$. Since a weight- d derivative has weight $d + 1$, the linearized physical system at zero is triangular in these blocks with zero diagonal. Its time-one map has identity diagonal; the frame-initial-value block is also the identity. Therefore $D\Psi(0, \mathbf{E}_0)$ is invertible and Ψ is an analytic coordinate map near the manifold of homogeneous states.

Weighted homogeneity gives

$$\Phi_{\mathcal{R}}^t \Psi(S_0, \mathbf{E}_0) = \Psi(e^{dt} S_0, \mathbf{E}_0), \quad e^{dt} S_0 = (e^t p_0, e^{2t} \tau_0, e^{3t} u_0). \quad (7.16)$$

Rescale the solution of (7.15) from $[0, e^t]$ to $[0, 1]$; each component has the asserted weight.

The frame part of the manifold of homogeneous states is compact. A finite cover by the maps Ψ gives one neighbourhood in which the negative radial flow is governed by (7.16). Since the entire limit set consists of homogeneous states, the state orbit eventually enters this neighbourhood. It then has the unique finite physical initial data determined by Ψ^{-1} , and is exactly the regular solution of (7.15) issued from the axis.

All physical constraints and their differential consequences hold for $r > 0$. The finite physical state just obtained allows their continuous passage to $r = 0$. Proposition 2.5 reconstructs an analytic deformation on a neighbourhood of the axis point. Fix its target frame and position to agree with the continued germ at one point of the ray. Uniqueness along the ray then identifies the two germs.

Choose a ball about the axis point on which this realization is defined. Uniqueness of the state and both frame equations identifies their continuations along paths in the ball minus the source axis. A meridian circle closes in the analytic realization, so the branches whose lifted angles differ by a full turn coincide. $\square$

If the original germ satisfies $h_1 h_2 h_3 \neq 0$, every axial limit is homogeneous. Lemma 7.8 therefore extends the continued strain through every noncentral point of both half-axes. The identification of angular lifts makes this strain single-valued and analytic on $\mathbb{R}^3 \setminus \{0\}$. Its constant principal values, closed projector pressure forms and common source similarity are preserved under continuation.

7.5 The global contradiction on punctured space

We finish with a geometric consequence of regularity on punctured space. The spherical flux of a principal projector bounds its pressure constant. For one principal label the bounds confine a curve on which κ_i is nonzero and constant, whereas $\vartheta_i(Z)$ increases strictly.

Lemma 7.9 (Rigidity on spheres in punctured space). *Suppose the strain tensor of a universal deformation has constant, pairwise distinct principal values and is single-valued and smooth on $\mathbb{R}^3 \setminus \{0\}$ and admits the common source similarity $Z = x + \Omega x$. Then h_i vanishes identically for one fixed principal label i .*

Proof. The punctured space is simply connected. The three real principal line bundles are orientable, so choose a global orthonormal principal frame and its dual coframe α_i . The closed projector pressure forms have global potentials

$$\operatorname{div} P_i = \nabla \phi_i, \quad \sum_i \phi_i = 0, \quad Z \phi_i = K_i, \quad \sum_i K_i = 0. \quad (7.17)$$

The last identities follow from similarity invariance of the closed forms and connectedness; their constants have no prescribed signs.

We first prove the strict pressure bounds

$$-1 < K_i < 2. \quad (7.18)$$

Write $\varrho = |x|$, $n = x/\varrho$, and $L = \Omega x$. Similarity invariance and compactness of a regular unit sphere give $|\nabla P_i| = O(\varrho^{-1})$; hence the pressure has at most logarithmic growth. The field L is divergence-free and tangent to every sphere. On a shell $\delta < |x| < R$,

$$\int \nabla \phi_i \cdot L = 0, \quad \operatorname{div}(P_i L) = \nabla \phi_i \cdot L,$$

since $P_i : \Omega = 0$. The inner flux is $O(\delta^3)$, so

$$\int_{|x|=R} n \cdot P_i L = 0.$$

Also $\operatorname{div}(P_i Z) = K_i + 1$. Integration and the preceding cancellation give the exact identity

$$1 + K_i = \frac{3}{4\pi} \int_{\mathbb{S}^2} n \cdot P_i(Rn) n \, dS(n). \quad (7.19)$$

Projection positivity first gives the non-strict bounds. If $K_i = -1$, the continuous nonnegative integrand is everywhere zero and the corresponding principal line is tangent to the whole sphere. Its orientation gives a nowhere-zero tangent vector field. Such a field $e(n)$ would give a homotopy $\cos(t)n + \sin(t)e(n)$, $0 \leq t \leq \pi$, between the identity and the antipodal map, whose degrees are 1 and -1 . This is impossible. If $K_i = 2$, both other principal lines are everywhere tangent and give the same contradiction. Thus (7.18) holds. Choose one fixed index with

$$-1 < K_i \leq 0, \quad \delta_i = 1 + K_i > 0. \quad (7.20)$$

Use the forms ϑ_i and coefficients κ_i in (2.16), and set

$$\ell_i = \vartheta_i(Z). \quad (7.21)$$

The structure equation (2.16) reads

$$d\vartheta_i = \kappa_i \vartheta_j \wedge \vartheta_k. \quad (7.22)$$

Applying d once more and noting $d(\vartheta_j \wedge \vartheta_k) = 0$ yields

$$D_i \kappa_i = 0. \quad (7.23)$$

Furthermore $\mathcal{L}_Z \vartheta_i = \delta_i \vartheta_i$, and e_i lies in the kernel of $d\vartheta_i$. Cartan's formula, evaluated on e_i , gives

$$D_i \ell_i = \delta_i e^{\phi_i} > 0. \quad (7.24)$$

Along the similarity flow $x \mapsto e^t \exp(t\Omega)x$,

$$\phi_i(x) = K_i \log \varrho + \Phi_i(\exp(-\log \varrho \Omega)n), \quad h_i(x) = \varrho^{-1} H_i(\exp(-\log \varrho \Omega)n).$$

The functions Φ_i, H_i are smooth on the compact unit sphere. Consequently fixed positive constants c, C satisfy

$$\begin{aligned} c\varrho^{K_i} &\leq e^{\phi_i} \leq C\varrho^{K_i}, & |h_i| &\leq C/\varrho, \\ |\kappa_i| &\leq C\varrho^{2K_i-1}, & |\ell_i| &\leq C\varrho^{1+K_i}. \end{aligned} \quad (7.25)$$

Suppose κ_i is nonzero at one point and follow its maximal unit-speed e_i curve $\gamma : (a, b) \rightarrow \mathbb{R}^3 \setminus \{0\}$. By (7.23), κ_i is a fixed nonzero number along the curve. Since $2K_i - 1 < 0$, equation (7.25) confines the entire curve to some ball $\varrho \leq R$. On that ball ℓ_i is bounded, but

$$\frac{d}{dt} \ell_i(\gamma(t)) \geq \delta_i c R^{K_i} > 0$$

by (7.20). Both endpoints a, b must be finite. Unit speed makes γ Cauchy at each finite endpoint; each has a finite Euclidean position limit. A nonzero limit would allow ordinary ODE continuation through a smooth point, contradicting maximality. Thus both limits are the origin. Since $1 + K_i > 0$, (7.25) makes ℓ_i tend to zero at both ends. That contradicts its strict increase with a positive lower derivative bound. Hence κ_i , and therefore this fixed h_i , vanishes identically. $\square$

Proof of Theorem 7.1. A hypothetical open set with $h_1 h_2 h_3 \neq 0$ is excluded directly by Lemma 7.2 when both angular velocities are zero. Otherwise inversion makes $\omega_s \neq 0$. Continuation and Lemma 7.5 give the two exhaustive axis alternatives. The cylindrical alternative contradicts the two-rotation theorem by Lemma 7.7. The homogeneous alternative removes every noncentral axis point by Lemma 7.8. The remaining single-valued strain on punctured space contradicts Lemma 7.9. $\square$

Theorems 6.1 and 7.1 exclude every common similarity supplied by a state map of rank two. Together with Section 4, this leaves the case of rank three.

8 The algebraic dimension bound

The rank two argument used a similarity along a fibre of the normalized state map. At rank three that map is locally injective, so there is no such fibre to which the argument can be applied. We instead consider the algebraic relations satisfied by the state and bound the dimension of its Zariski closure. A closure of dimension at most two cannot contain a local state image of rank three.

The projection onto the two abnormality ratios separates the argument into two cases. Over the ratio plane we prove that the generic state fibre is finite. Over a ratio curve we prove that it has dimension at most one. The required estimates are therefore

$\dim \mathcal{B}$	$\dim Y_{\text{gen}}$	$\dim Y$
2	0	2
1	≤ 1	≤ 2 .

Here Y and $\mathcal{B}$ will denote the state closure and its ratio image, respectively. The distinction also determines what must be proved at infinity: the plane case requires an empty generic projective boundary, whereas the curve case requires only a finite one. The algebraic equations determine possible boundary states; the differential equations determine which of them can belong to the closure of a solution.

Theorem 8.1 (The rank bound). *For every triple of positive, pairwise distinct, constant principal values, the retained state of a solution on $h_1 h_2 h_3 \neq 0$ satisfies $\text{rank } d\mathbf{X} \leq 2$.*

8.1 The state image and the ratio map

Fix a connected solution chart in the localization (3.3). Put

$$\kappa_0 = \frac{a-b}{a(b-1)}, \quad q = 1 + \kappa_0 = \frac{b(a-1)}{a(b-1)}, \quad \eta = \kappa_0 \frac{h_2}{h_1}, \quad \zeta = \frac{\kappa_0 h_3}{q h_1}. \quad (8.1)$$

The constants κ_0, q are nonzero. The localization becomes $\eta \zeta (\eta - \zeta) (\eta + 1) (\zeta - 1) \neq 0$. The six off-diagonal entries of $\hat{\mathbf{p}}$ and three affine coordinates R_i , together with η, ζ , form rational coordinates for $\mathbf{X}$. The change of coordinates is given by (C.1) in Appendix C.1; its coefficients are invertible in this localization. It preserves differential rank.

We may therefore compute the state fibre dimension using the nine coordinates consisting of the off-diagonal entries and the R_i , with the ratios fixed.

Let $\mathcal{R}_{\text{norm}}$ be the localized complex coordinate ring and write $E_i \mathbf{X} = \mathcal{V}_i(\mathbf{X})$ for the state equations. The state is analytic by Section 2. We take the Zariski closure of the image of this solution. Its ideal records the algebraic relations along the solution, and the chain rule makes that ideal invariant under the state derivations. This invariance supplies additional equations on its projective boundary.

The finite compatibility ideal is contained in this vanishing ideal. Its zero set may have components to which the state derivations are not tangent. Taking the vanishing ideal of the

solution image selects an irreducible variety preserved by those derivations, on which every polynomial relation may be differentiated. The finite local ideals below consist of relations in this invariant ideal.

Lemma 8.2 (The invariant prime ideal). *The vanishing ideal*

$$\mathfrak{p} = \{P \in \mathcal{R}_{\text{norm}} : P(\mathbf{X}) = 0 \text{ on the solution chart}\}$$

is prime and is preserved by each $\mathcal{V}_i$. Its zero set Y is the irreducible Zariski closure of the state image.

Proof. Pullback takes $\mathcal{R}_{\text{norm}}$ to the ring of complex valued real analytic functions on the connected chart. That ring has no zero divisors, by the identity theorem, so the kernel is prime. If $P \in \mathfrak{p}$, the chain rule gives $(\mathcal{V}_i P)(\mathbf{X}) = E_i(P(\mathbf{X})) = 0$. $\square$

In particular, the normalized relations induced by $\mathcal{J}_{\text{raw}}$, and all their state derivatives, vanish on Y . Let $\mathcal{B}$ be the Zariski closure of the ratio image (η, ζ) , and let Y_{gen} denote the generic fibre of $Y \rightarrow \mathcal{B}$.

Lemma 8.3 (Algebraic dimension and differential rank). *One has*

$$\text{rank } d\mathbf{X} \leq \dim Y = \dim \mathcal{B} + \dim Y_{\text{gen}}. \quad (8.2)$$

Proof. At a point of rank ρ , choose ρ state coordinates with independent differentials. Their local image contains an open subset of $\mathbb{R}^\rho$, on which no nonzero complex polynomial vanishes. They are algebraically independent on Y , so $\dim Y \geq \rho$. The equality follows by adding transcendence degrees in $\mathbb{C} \subset \mathbb{C}(\mathcal{B}) \subset \mathbb{C}(Y)$. $\square$

Lemma 3.1 gives $\dim \mathcal{B} \in \{1, 2\}$. Thus (8.2) reduces the rank bound to the two fibre estimates above. Passing to a projective closure turns these into boundary questions: an affine fibre of positive dimension has a nonempty boundary, and a fibre of dimension at least two has a boundary of positive dimension.

These assertions can be checked after extending the generic base field to its algebraic closure. If an affine irreducible variety of positive dimension had no boundary in its projective closure, it would be both affine and proper and hence zero dimensional. More precisely, the homogenizing coordinate defines a nonzero section on the irreducible projective closure. When its zero set is nonempty, that zero set has codimension one. Thus a finite boundary bounds the affine dimension by one, while an empty boundary forces the affine variety to be finite.

8.2 Finite fibres over the ratio plane

Suppose that $\dim \mathcal{B} = 2$. We work over the function field of the ratio plane. A positive dimensional affine generic fibre would have a point at infinity in its projective closure. The closure of such a point gives a boundary component dominant over the ratio plane. It is this type of component that must be excluded.

8.2.1 The relative projective boundary

Give the six off-diagonal coordinates weight one and the three R_i weight two. Write $\bar{p} = \tau \hat{p}$ and $(\mathbf{v})_j = \tau^2 R_j$ in the weighted projective closure. The construction in Appendix C.2 gives a proper modification equal to Y on the affine open set $\tau \neq 0$. Choose an open neighbourhood $\mathcal{U}$ of $(\eta, \zeta) = (1, -1)$ in the ratio plane as in that construction. A boundary component dominating the plane is proper over $\mathcal{U}$. Its image is closed and contains the generic point, hence is all of $\mathcal{U}$. It therefore meets the fibre over $(1, -1)$. Throughout this argument the entries of $\bar{p}$ are ordered as $(12, 13, 21, 23, 31, 32)$.

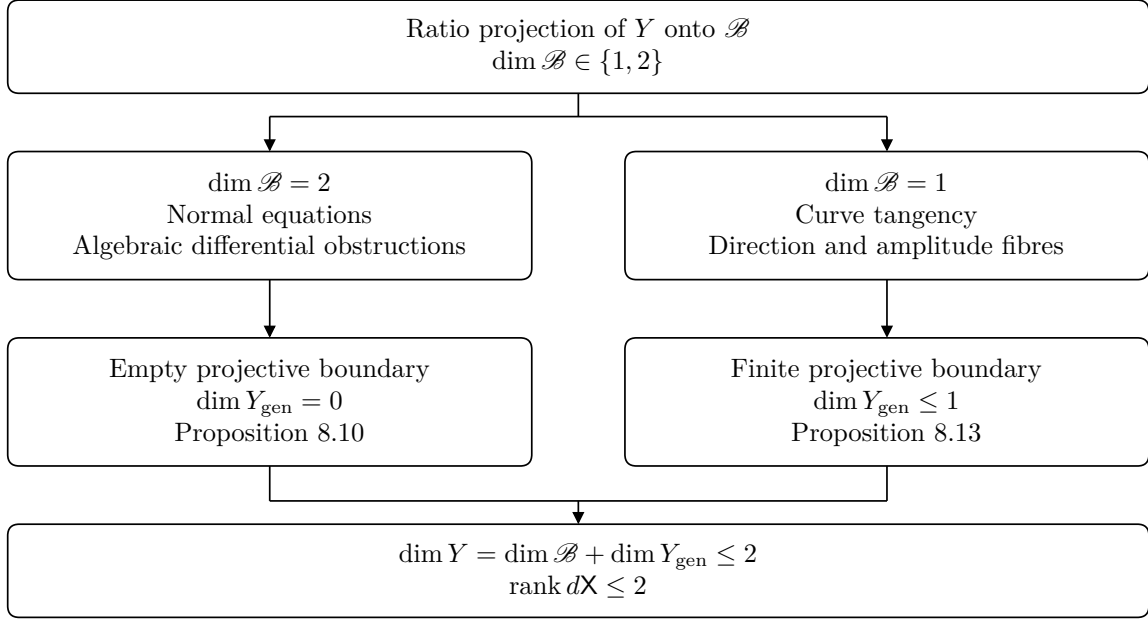


Figure 3: The fibre dimension estimates for the ratio projection.

The weights reflect the degrees of the state equations: frame coefficients have degree one, and their first derivative coordinates have degree two. A single ordinary projective chart would conflate these two growth scales. The modification separates their relative orders by blowing up the loci where the weighted derivative coordinates vanish. Properness brings a dominant boundary component to the chosen ratio fibre; the blowup charts give regular local equations at its points.

Lemma 8.4 (Boundary coverage). *Fix positive, pairwise distinct principal values and suppose that $\dim \mathcal{B} = 2$. Consider a formal arc with generic point in Y and special point on the boundary over $(\eta, \zeta) = (1, -1)$. After a finite extension it lifts to the modification and the finite covers of Appendix C.2, with the following alternatives.*

- (i) *Put $n = \text{ord } \tau > 0$ and $s_0 = \min_j \text{ord}(\mathbf{v})_j$, with $s_0 = \infty$ if $\mathbf{v} = 0$ identically. The valuation regions are*

$$s_0 = 0, \quad 0 < s_0 < n/2, \quad s_0 = n/2, \quad n/2 < s_0 < n, \quad n \leq s_0 \leq \infty.$$

They lie, respectively, in the original weighted charts, the supercritical charts, the common critical chart, the subcritical charts and the ordinary charts.

- (ii) *In the intermediate regions $0 < s_0 < n$, the leading representative $\overline{\mathbf{R}}$ has a nonempty coordinate support. A support of size one leaves four possible $\bar{p}$ coordinates and fifteen nonempty supports. A support of size two leaves, up to cyclic relabeling, a single paired support, two coordinate axes, a mixed support, or a nonzero paired group together with at least one remaining coordinate. A support of size three has the three paired groups $(12, 32)$, $(13, 23)$ and $(21, 31)$.*
- (iii) *On each support stratum, the simultaneous ranks of any finite collection of regular coefficient matrices, together with the zero and nonzero loci of finitely many regular coefficient factors, define a finite locally closed refinement.*

Every closed point of a boundary component dominating the ratio plane over $(1, -1)$ admits an arc of this form.

Proof. The two blowups are proper and have their centres in $\tau = 0$. Hence an arc from the affine locus lifts by the valuative criterion. The auxiliary covers are finite and surjective, so the lift extends to them after a finite extension of its complete discrete valuation ring. A ramification index multiplies all orders by the same integer.

If $s_0 = 0$, a nonzero $(\mathbf{v})_j$ selects an original weighted chart, including points with $\bar{p} = 0$. If $s_0 > 0$, some $\bar{p}$ coordinate is nonzero. For finite s_0 , choose an entry $(\mathbf{v})_j$ of least order. The chart equations are

chart	coordinates	orders
W_O	$\mathbf{v} = \tau \mathbf{R}$	$\text{ord } R_j \geq 0 \quad (s_0 \geq n)$
W_S	$\mathbf{v} = \kappa \delta \bar{\mathbf{R}}, \quad \tau = \kappa^2 \delta$	$(\text{ord } \kappa, \text{ord } \delta) = (n - s_0, 2s_0 - n)$
W_T	$\mathbf{v} = \sigma \bar{\mathbf{R}}, \quad \tau = \sigma^2 \beta_T$	$(\text{ord } \sigma, \text{ord } \beta_T) = (s_0, n - 2s_0).$

Nonnegativity gives the regions in (i). At $s_0 = n/2$, δ and β_T are units and $\delta\beta_T = 1$ on the overlap. If $s_0 = \infty$, the ordinary chart has $\mathbf{R} = 0$.

In an intermediate chart, choose $(\mathbf{v})_j$ of least order and normalize $\bar{\mathbf{R}} = \mathbf{v}/(\mathbf{v})_j$. The finite auxiliary coordinates e^{alg} of Appendix C.2 are regular on the lifted arc and satisfy

$$L(\bar{p})\bar{\mathbf{R}} = \frac{\tau}{(\mathbf{v})_j} e^{\text{alg}}, \quad \text{ord } \frac{\tau}{(\mathbf{v})_j} = n - s_0 > 0.$$

Thus $L(\bar{p})\bar{\mathbf{R}} = 0$ at its special point. Writing out the matrix L of Appendix C.1 gives

$$\begin{aligned} \bar{p}_{12}\bar{R}_1 &= \frac{\zeta - \eta}{\zeta(\eta + 1)} \bar{p}_{32}\bar{R}_3, \\ \bar{p}_{13}\bar{R}_1 &= \frac{\zeta - \eta}{\eta(\zeta - 1)} \bar{p}_{23}\bar{R}_2, \\ \bar{p}_{21}\bar{R}_2 &= \frac{\eta(\zeta - 1)}{\zeta(\eta + 1)} \bar{p}_{31}\bar{R}_3. \end{aligned}$$

Their coefficients are units in the ratio localization. With only $\bar{R}_3 \neq 0$, they force $\bar{p}_{31} = \bar{p}_{32} = 0$, leaving the other four entries. With $\bar{R}_2\bar{R}_3 \neq 0$ and $\bar{R}_1 = 0$, they force $\bar{p}_{23} = \bar{p}_{32} = 0$ and pair $\bar{p}_{21}$ with $\bar{p}_{31}$. If this pair is the only nonzero group, the support is paired. If it is zero, the remaining coordinates give the two axes or their mixed support. If the pair and at least one remaining coordinate survive, they give the last alternative in (ii). Cyclic permutations give three paired, six axis and three mixed configurations. When all three $\bar{R}_i$ are nonzero, the displayed relations give the three groups in (ii). These incidence relations therefore allow $3 \cdot 15 + 3 \cdot 7 + 7 = 73$ possible coordinate support patterns.

On a fixed support, a matrix has rank $r > 0$ precisely where all its $(r + 1)$ -minors vanish and at least one r -minor is nonzero; rank zero means that every entry vanishes. These are finitely many locally closed conditions. Each additional coefficient factor is then partitioned into its zero and nonzero loci. Intersecting these finite partitions proves (iii). On a paired chart put $\bar{p}_{12} = 1$, $\ell = \bar{p}_{32}$ and use $k = a\kappa_0$, $m = 1 + k$ as in Appendix C.1. The two conditions $m\ell^2 = \pm k$ are disjoint, since $2k \neq 0$. Each is considered both on and off the material resonance $k + q = 0$. The local ideals are formed in the full local ring, with the coefficient factors that vanish at the special point retained as parameters.

Finally, the strict transform $\tilde{Y}$ is the irreducible closure of Y in the modification. It is Noetherian, being of finite type over $\mathbb{C}$. Its generic point lies in Y and specializes to every closed point of $\tilde{Y}$. A discrete valuation ring realizes this specialization [27, Lemma 28.5.10, Tag 054F]; completing it gives the required formal arc. A boundary component dominating the ratio plane meets the selected fibre by properness. $\square$

Let J be the ideal of the closure of Y in a boundary chart. It is saturated:

$$J : \tau^\infty = J.$$

The homogenized compatibility equations and their derivatives, after division by powers invertible on the affine open set, generate an ideal $I \subset J$. If $\tau^N \in I$, saturation gives $1 \in J$, so the chart contains no point of the closure of Y . The same conclusion follows by excluding formal arcs whose generic points lie in $\tau \neq 0$: a point of the closure of an affine open set lies on such an arc after a field extension. Normal equations are useful here because they compare the first nonzero terms of an approaching arc. A nilpotent normal coordinate vanishes on the arc, and a system of weighted initial equations can force several normal coordinates to vanish together.

Saturation also justifies division by an exceptional power when the quotient is regular: if $\tau^d F \in J$, then $F \in J$. This is the ideal-theoretic reason the divided equations remain necessary equations for the solution closure. We retain τ in the local equations. A boundary point belonging to the closure of the affine solution must admit an arc with $\tau \neq 0$ at its generic point. If the equations in I force $\tau = 0$ identically on every such arc, that point is excluded from the closure.

Lemma 8.5 (Exclusion by normal equations). *Work in the completed local ring of a boundary chart. Let $g_1, \dots, g_r$ be normal coordinates vanishing at the point, and suppose that $\tau^N \in I + (g_1, \dots, g_r)$ for some integer $N > 0$. Assign positive integer weights to the g_j . Suppose that finitely many equations in I , expanded in the g_j with the parameters along the family retained, have no terms below their respective weighted degrees. After reducing the coefficients modulo the maximal ideal, suppose that their initial forms have the origin as their only common zero over the algebraic closure of the residue field. Then no formal arc through the point, annihilating I , has a generic point in $\tau \neq 0$.*

Proof. If some g_j is nonzero along such an arc, minimize the quotient of its order by its weight. After a finite ramified extension, the first coefficients of the g_j at this weighted order form a nonzero vector. The first terms of the equations make every initial form vanish on that vector, a contradiction. Hence all g_j vanish on the arc, and the inclusion $\tau^N \in I + (g_1, \dots, g_r)$ gives $\tau = 0$. $\square$

There are two ways in which the local equations will be used. First, suppose that all the equations vanish on a specified family, and that their linear coefficient matrix in its normal coordinates has full column rank. An invertible minor permits implicit elimination of these coordinates in the completed local ring. The zero normal coordinates already solve the full equations, so uniqueness of the implicit solution makes the eliminated coordinates zero. If τ is in this normal ideal, the family has no approaching affine arc.

Second, a coefficient minor may vanish on a subfamily. On that locus we eliminate only the coordinates belonging to an invertible submatrix, using the full equations rather than their linear parts. The equations in the remaining coordinates then have well-defined weighted initial forms. If these forms have only the zero solution and $\tau^N \in I + (g_1, \dots, g_r)$, the preceding lemma again excludes an affine arc. The weighted initial forms include the quadratic terms created by implicit elimination. Lemmas C.6 and C.7 give the corresponding local reductions.

Some boundary families require a normal ideal containing products or squares of coordinates. If the compatibility relations belong to such an ideal and their classes span its quotient by the maximal ideal, Nakayama's lemma makes them generate the entire ideal. Membership in the ideal is checked for the full polynomials, and spanning is checked in this quotient. Thus the conclusion is an inclusion of ideals in the completed local ring, even when the family has a nonreduced normal structure. The powers and products are retained until this inclusion has been obtained. On the ordinary chart of the modification, $\mathbf{v} = \tau \mathbf{R}$, with $\mathbf{R}$ a regular coordinate vector on that chart.

Lemma 8.6 (Reduction of the relative boundary). *Near the fibre $(\eta, \zeta) = (1, -1)$, every irreducible component of the boundary of Y dominating the ratio plane is contained, up to a*

signed cyclic relabeling, in the family

$$\begin{aligned} \mathcal{F}_{\text{in}} : \quad \bar{p} &= (1, 0, 0, 0, 0, \ell), \quad \mathbf{R} = (\lambda_{\text{in}}(v)\ell, u, v), \quad \tau = 0, \quad \ell \neq 0, \\ \lambda_{\text{in}}(v) &= \frac{(\zeta - \eta)v - 2\eta(\zeta + 1)}{\zeta(\eta + 1)}. \end{aligned} \quad (8.3)$$

Here $\bar{p} = \tau\hat{p}$ and $\mathbf{R} = \tau\mathbf{R}$ are ordinary homogeneous coordinates. The homogeneous regularizations of the state derivations on the ordinary projective closure preserve the surviving boundary component.

Proof. Let a boundary component dominate the ratio base. By properness it meets the fibre $(1, -1)$. We examine the approaches from the affine locus to this fibre, using the necessary ideal $I \subset J$ and Lemma 8.5.

For an approaching arc, use the orders n, s_0 and the support alternatives of Lemma 8.4. On each support we retain the coefficient factors at a rank drop and their intersections with the special principal value loci in the local coefficient ring.

The local equations on these supports have two exclusion mechanisms. Where a normal coefficient matrix has full column rank, its selected relations generate the required normal ideal and force a power of τ into I . On its rank strata, one first eliminates the coordinates belonging to an invertible submatrix using the full local equations. The remaining normal equations have initial homogeneous forms with no nonzero common zero, or generate an ideal containing the required exceptional coordinate. Lemma 8.5 forces $\tau = 0$ on such an arc. At the four-point family, implicit elimination gives two normal ideals describing the local families through those points. The remaining equations generate their intersection. The eliminated expression for τ belongs to both ideals, hence to their intersection. Pulling this membership back gives τ in the original local ideal, including at the intersection of the two families. Appendix C.8 gives the generators.

The conic endpoint with $\bar{p}_{23} = 0$ and one nonzero leading derivative coordinate requires an exact relative ideal. On its intermediate chart, $\mathbf{v} = \kappa\delta\bar{\mathbf{R}}$ and $\tau = \kappa^2\delta$. Six complete equations with an invertible Jacobian define an implicit graph. The reference conic ideal on this graph is generated by one coordinate and the square of another. A first residual is divisible by $\bar{p}_{23}$, and its regular quotient, together with a second residual, has an invertible coefficient matrix in these two generators. Thus $\bar{p}_{23}$ times the reference ideal belongs to the necessary ideal. The coordinate κ belongs to the reference ideal, while δ is divisible by $\bar{p}_{23}$, giving

$$\kappa\delta \in I, \quad \tau = \kappa^2\delta \in I.$$

Appendix C.9 gives this local proof with η, ζ retained as parameters. Since $I \subset J$ and $J : \tau^\infty = J$, this endpoint is excluded from the relative closure of the affine state image.

For each coefficient used in elimination, its nonvanishing set permits division and its zero locus is treated with the original equations. This gives a further partition of each support by coefficient rank. On the paired supports, the coefficient rank can drop on the two amplitude hypersurfaces

$$(1 + a\kappa_0)\ell^2 = a\kappa_0, \quad (1 + a\kappa_0)\ell^2 = -a\kappa_0.$$

Their difference is $2a\kappa_0 \neq 0$, so they are disjoint. The material resonance in these strata is

$$a\kappa_0^2 = (1 + a\kappa_0)q \iff a\kappa_0 + q = 0.$$

It is kept as a parameter condition when either amplitude hypersurface is considered. In the critical missing-row stratum, the original equations would require

$$1 + \frac{b(a-1)^2}{a(b-1)^2} = 0,$$

which is impossible because $a, b > 0$. On the canonical paired stratum, simultaneous vanishing of the elimination minors at the material resonance would require

$$-\kappa_0(1 + \kappa_0) = \frac{3}{4}.$$

But $-\kappa_0(1 + \kappa_0) = \frac{1}{4} - (\kappa_0 + \frac{1}{2})^2 \leq \frac{1}{4}$. On the inactive paired stratum, elimination at the resonance leaves one normal coordinate whose quadratic coefficient is

$$-\frac{16q}{\kappa_0} \left((\kappa_0 + \frac{1}{2})^2 + 3 \right) \neq 0.$$

The remaining equation is a unit times the square of that coordinate in the completed local ring. That coordinate must therefore vanish on every reduced formal arc annihilating I , and the implicit equations force the exceptional coordinate to vanish. The other coefficient zero sets give either an ideal containing τ or the initial forms of Lemma 8.5, as specified in Appendix C.8.

On the other intermediate supports we use differential invariance to pass from the selected fibre to a neighbourhood in the base. The state derivations extend after homogeneous regularization on the weighted Rees model with integer weights $(n, 2n - s_0)$. Put $\gamma = 2 - s_0/n$. For each monomial of a regularized derivation, the exponent of the exceptional parameter is affine in γ . The endpoint degree bounds in Appendix C.6 make this exponent nonnegative at $\gamma = 1, 2$, and therefore throughout $1 < \gamma < 2$. On the supports with two available horizontal fields, their base minor is nonzero and the resulting boundary component dominates the ratio base. On full derivative support with at least two surviving amplitude pairs, the induced derivations fix the amplitude and direction parameters and act independently on η, ζ . The ratios are algebraically independent over the common constant field, so the compatibility equations can be compared coefficientwise (Appendix C.3). On a transverse local chart, Lemma C.4 transfers the radical inclusion from the selected fibre to the full local ring: the formal flows preserve J and multiply τ by a unit. The supports lacking these fields were treated by the relative local identities above.

Coefficient comparison here takes place in a function field where the direction parameters are common constants and the ratios are algebraically independent over those constants. Formal transfer uses two regular lifted fields with an invertible base minor. Their formal flows preserve the saturated ideal and send the boundary equation to a unit multiple of itself, so they transport a radical inclusion forcing $\tau = 0$ from the selected fibre to the relative local ring. At a point where the base minor vanishes, the relative local identities above supply the exclusion.

In the ordinary chart, the remaining local ideals give the radical inclusion

$$(\bar{p}_{13}, \bar{p}_{21}, \bar{p}_{23}, \bar{p}_{31}, (\mathbf{R})_1 - \lambda_{\text{in}}(v)\bar{p}_{32}) \subset \sqrt{I + (\tau)}.$$

Here the projective normalization is $\bar{p}_{12} = 1$, and $u = (\mathbf{R})_2$, $v = (\mathbf{R})_3$. The normal ideals giving this inclusion are written in Appendix C.8. Their squares and cubes force the residual normal coordinates to vanish on the reduced boundary; the quadratic change of coordinates on the resonant stratum has the same radical. Hence a nonempty open subset of the component satisfies (8.3). Since this family is relatively closed, irreducibility puts the generic point of the component in it. On the ordinary projective closure, Lemma C.3 applies to the invariant prime $\mathfrak{p}$ and its saturated homogeneous closure. Its regularized derivations preserve the minimal primes of the boundary ideal, which gives the last assertion. $\square$

The surviving family still has the three boundary coordinates ℓ, u, v . Reduction to this family therefore does not yet give a fibre dimension bound. The next argument uses its induced differential equations: it first makes a boundary coordinate algebraic over the ratios and the common constants, and then excludes the resulting algebraic differential equation.

8.2.2 Differential equations on the ordinary boundary family

The necessary homogenized equations vanish on $\mathcal{F}_{\text{in}}$. To exclude a dominant component in this family, the normal coefficient matrices must be singular: an invertible normal matrix would put τ in the compatibility ideal, contradicting saturation. The induced derivations also preserve the component. The singular normal matrices make certain boundary coordinates algebraic, while differential invariance gives equations incompatible with that algebraicity.

On $\mathcal{F}_{\text{in}}$, set

$$\Theta = (\zeta - \eta - 1)v - 2\eta\zeta, \quad \beta = \frac{\eta(\zeta - 1)}{\zeta(\eta + 1)}. \quad (8.4)$$

Away from $(v + 2)\Theta = 0$, two boundary derivations have independent base components and can be normalized to induce $\partial_\eta, \partial_\zeta$. Their equations and normal terms are given in Appendix C.10.

The normalized fields send (η, ζ) to $(1, 0)$ and $(0, 1)$, respectively. We use their common constant field to compare coefficients. After fixing η , the second field induces $d/d\zeta$ on the ratio field. The loci $v = -2$ and $\Theta = 0$, where the normalization fails, are treated separately in Proposition 8.9.

Lemma 8.7 (Algebraicity on the ordinary boundary family). *Let E be the function field of an invariant irreducible component of the boundary contained in $\mathcal{F}_{\text{in}}$ and dominant over the ratio plane. Suppose that the component is contained in neither $v = -2$ nor $\Theta = 0$. Let C be the common constant field of the two normalized horizontal derivations. Then η, ζ are algebraically independent over C . Away from $v = 2\eta$, the coordinate v is algebraic over $C(\eta, \zeta)$, including on $u = 0$. On $v = 2\eta$, the coordinate u/β is algebraic over that field.*

Proof. The two derivations annihilate ℓ and, where it is defined, the quantity

$$\frac{u^2}{\beta^2(v + 2)(v - 2\eta)}.$$

A polynomial relation of least total degree between η, ζ over C would have both partial derivatives zero. In characteristic zero this is impossible.

Thus the two ratio coordinates remain independent even after adjoining the first integrals to the constant field. This is the field over which the coefficient comparison must be made. A polynomial equation with coefficients depending on ℓ and the displayed first integral is an equation over $C(\eta, \zeta)$, because both of those quantities lie in C . To prove algebraicity of v , it remains to show that at least one of the necessary polynomial equations is nonzero in this field; the vanishing of all its coefficients would supply no such conclusion.

Consider the matrix of linear normal terms of the compatibility relations along the family. A unit maximal minor would make the full relations generate the normal ideal, which contains τ . Lemma 8.5 would exclude that point from the closure of Y . The normal minors consequently vanish on the component. Substitution of the displayed constant quantity gives polynomial equations for v over $C(\eta, \zeta)$. Their coefficient factorization in Appendix C.10 separates the two amplitude hypersurfaces

$$(1 + a\kappa_0)\ell^2 = \pm a\kappa_0.$$

Off these hypersurfaces, two coefficient vectors have a minor equal to 63. Their simultaneous vanishing forces $\kappa_0\ell^2 + q = 0$ and the displayed first integral to equal $-q$. On the positive amplitude hypersurface, a linear combination of two coefficient conditions is a nonzero multiple of $a\kappa_0^2$. On the negative hypersurface, the constant coefficient in v is a unit times $(a\kappa_0)^4$ times the first integral, which is nonzero here. On the remaining locus where the first integral equals $-q$, the coefficient of v^4 has two parameter coefficients with a linear combination $-24a\kappa_0^2$. Its additional factor $a\kappa_0 + q$ can vanish only on the negative amplitude hypersurface already treated. Hence the normal minors give a nonzero equation for v .

On $u = 0$, one retains the mixed normal generator $un_{\perp}$ and the quadratic generator $n_{\perp}^2$, with $n_{\perp}$ as in (C.9). Two leading coefficients of the resulting equations in v contain, up to nonzero factors, $5\kappa_0\ell^2 + 4q$ and $4\kappa_0\ell^2 + 5q$. They cannot both vanish, since

$$5(4\kappa_0\ell^2 + 5q) - 4(5\kappa_0\ell^2 + 4q) = 9q \neq 0.$$

The factorization of the coupled normal columns in the same appendix then gives a nonzero polynomial in v .

The mixed generator $un_{\perp}$ is kept even though the value of u is zero on the subfamily. Dividing it by u would change the local ideal and would remove precisely this case. The quadratic generator $n_{\perp}^2$ records the remaining normal equation when the linear column degenerates. The two coefficient factors above show that the resulting polynomial conditions cannot both become identically zero; they give algebraicity of v on this subfamily without inverting u .

On $v = 2\eta$, the normal equations instead give a polynomial in u . After unit factors are removed, its three-column normal factor has highest ζ coefficient

$$-16(1 + a\kappa_0)q(\eta + 1)^5\ell^2 \neq 0.$$

Thus this polynomial is nonzero, and u/β is algebraic because β belongs to $C(\eta, \zeta)$ and is nonzero. At $u = 0$ the corresponding normal generators contain τ , excluding the component directly. $\square$

We now hold η fixed and regard ζ as the independent variable. An algebraic function of ζ defines a finite covering of the projective line. Residues of differentials and pole orders on this covering give obstructions to the induced boundary equations.

Lemma 8.8 (Algebraic differential obstructions). *Let K be a field of characteristic zero and let $E/K(\zeta)$ be finite. Fix $\eta \in K$ with $\eta(\eta + 1) \neq 0$. If $v \in E$ satisfies*

$$\frac{dv}{d\zeta} = \frac{(v - 2\eta)(v + 2)}{(\zeta - \eta - 1)v - 2\eta\zeta}$$

with nonzero denominator, then $v = -2$ or $v = 2\eta$. The logarithmic derivative of a nonzero algebraic function in E cannot be a nonzero constant in K .

Proof. Extend the constant field algebraically and pass to the smooth projective curve of the resulting function field. If v is nonconstant, its equation gives

$$d\left(\frac{\zeta}{v + 2}\right) = -\frac{(\eta + 1)v}{(v - 2\eta)(v + 2)^2} dv. \quad (8.5)$$

A nonconstant rational function on a smooth projective curve maps onto the projective line, so there is a point above $v = 2\eta$. The differential on the right of (8.5), considered first on the v -line, has residue $-\eta/[2(\eta + 1)]$ there. If $v - 2\eta$ has order $e > 0$ at a point of the covering curve, pullback multiplies this residue by e . It remains nonzero. On the other hand, in a local parameter the derivative of a Laurent series has no coefficient of order -1 . The exact differential on the left therefore has residue zero at every point. This contradiction makes v constant. Its differential equation then reduces to $(v - 2\eta)(v + 2) = 0$.

For the logarithmic assertion, a nonzero algebraic function has the form t^e times a unit at any point, so its logarithmic differential has at most a simple pole there. At a point above $\zeta = \infty$, write ζ as t^{-r} times a unit with $r > 0$. Then $d\zeta$ has pole order $r + 1 \geq 2$. Equality of a logarithmic differential with a nonzero constant multiple of $d\zeta$ is impossible. $\square$

Proposition 8.9 (Exclusion of the ordinary boundary family). *No invariant irreducible component of the boundary contained in $\mathcal{F}_{\text{in}}$ dominates the ratio plane.*

Proof. The hypersurface $\Theta = 0$ is transverse to an induced derivation: on it the derivative of Θ is $4(\zeta - 1)\eta(\eta + 1)/(\zeta - \eta - 1)$. Thus it cannot contain an invariant dominant component. Away from this hypersurface and $v = -2$, the normalized derivation D_ζ fixes η and satisfies

$$D_\zeta v = \frac{(v - 2\eta)(v + 2)}{\Theta}, \quad D_\zeta \ell = 0, \quad D_\zeta(u/\beta) = \frac{v + 1 - \eta}{\Theta}(u/\beta). \quad (8.6)$$

By Lemma 8.7, the field $C(\eta)(\zeta, v)$ is finite over $C(\eta)(\zeta)$. It is preserved by D_ζ , whose extension is unique in characteristic zero. Lemma 8.8 gives $v = 2\eta$. On the locus $v = 2\eta$, the normal relations exclude $u = 0$, while

$$D_\zeta \log(u/\beta) = -\frac{1}{2\eta}. \quad (8.7)$$

The algebraicity of u/β contradicts the second assertion of Lemma 8.8.

On $v = -2$, use the relative normal generators in (C.11)–(C.13). The ideals J_9, J_{10}, J_{11} correspond to the ordinary stratum and to the two amplitude hypersurfaces above. Each has τ among its linear generators. The compatibility polynomials belong to the relevant ideal, and their coefficient classes span its quotient by the maximal ideal times that ideal. Nakayama's lemma therefore gives

$$\tau \in J_j \subset I, \quad j = 9, 10, 11.$$

On the negative amplitude hypersurface a quadratic change of normal coordinates produces J_{11} and leaves τ unchanged. The coefficient matrices and this change are given in Appendix C.11. If $a\kappa_0 + q = 0$, the unit coefficient matrix is obtained at the base point $(1, -2)$ in place of $(1, -1)$. Thus $\tau \in I$ on a nonempty base open set, contrary to saturation for a component dominant over the ratio plane. $\square$

Dominance is used here only to pass to a nonempty open set of the base. An irreducible component dominating the ratio plane has its generic point over every such open set. Thus a unit coefficient matrix near $(1, -2)$ excludes the same putative component even if that matrix is singular at $(1, -1)$. Here the coefficient matrix is computed for the relative normal ideal on $v = -2$ at the replacement base point.

Proposition 8.10 (Finite fibres over the ratio plane). *If the state image is dominant over the ratio plane, its generic state fibre is finite.*

Proof. An affine generic fibre of positive dimension would give a boundary component dominant over the ratio plane. Properness forces it to meet $(\eta, \zeta) = (1, -1)$. Lemma 8.6 puts its generic point in $\mathcal{F}_{\text{in}}$, contradicting Proposition 8.9. The generic boundary is empty. Consequently the affine generic fibre is also proper. A proper affine variety over a field is finite; see Lemma C.1. $\square$

In particular, the estimate concerns the generic fibre over $\mathbb{C}(\eta, \zeta)$. Special fibres over lower dimensional subsets of the ratio plane may have larger algebraic zero sets; they do not alter the transcendence degree of $\mathbb{C}(Y)$ over the ratio field. They will enter the proof only if the entire ratio image is a curve, which is the next case.

8.3 Fibres over an algebraic ratio curve

Suppose that $\dim \mathcal{B} = 1$. If η is constant, the cyclic relabeling

$$(a', b') = (b/a, 1/a), \quad \eta' = -\zeta/\eta, \quad \zeta' = -1/\eta$$

makes η' nonconstant. We may therefore work in $\mathbb{C}(\mathcal{B})$ with

$$\varsigma = \frac{d\zeta}{d\eta}, \quad L = \eta\varsigma - \zeta.$$

The derivative with respect to η extends uniquely from $\mathbb{C}(\eta)$ to the function field of the curve, since the extension is finite and separable. In particular, every derivative of ς occurring below belongs to this field.

Tangency gives $E_i\zeta = \varsigma E_i\eta$. On the solution open set where $d\eta \neq 0$, differentiating this equation gives the curve compatibility relations. After clearing denominators, they belong to $\mathfrak{p}$ by analyticity and constrain the generic fibre of Y .

The slope ς is a rational function on the ratio curve. Pointwise elimination treats its value as a parameter. Along the curve, the equations $\zeta' = \varsigma$ and the differentiated compatibility relations further constrain the exceptional direction families.

The smaller base permits a larger state fibre. An affine irreducible component of dimension $d \geq 1$ has boundary of dimension $d - 1$ in its projective closure. We therefore need only show that its boundary is finite. Tangency to the ratio curve adds the equations that will give this bound.

We make the parameter distinctions at the generic point of $\mathcal{B}$. Since η is nonconstant, every nonzero polynomial in η is a unit in $\mathbb{C}(\mathcal{B})$. A polynomial in the curve coordinates either vanishes identically or is a unit there. Thus isolated exceptional ratio values do not form a further generic branch; a coefficient zero set matters precisely when it contains the ratio curve. The original localization is

$$\eta\zeta(\eta - \zeta)(\eta + 1)(\zeta - 1) \neq 0.$$

The additional characteristic conditions are

$$\varsigma - 1 = 0, \quad L + 1 = 0, \quad L + \varsigma = 0, \quad \varsigma = 0, \quad L = 0. \quad (8.8)$$

Write

$$\begin{aligned} g_* &= \eta\zeta(\eta - \zeta)(\zeta - 1)(\eta + 1)(1 + \eta - \zeta)(\varsigma - 1)(L + 1)(L + \varsigma)\varsigma L. \\ \mathcal{P}_0 &= \eta\zeta(\eta^2 - 1)\varsigma^2 - 2(\eta^2 - \zeta^2)\varsigma + \eta\zeta(1 - \zeta^2). \end{aligned} \quad (8.9)$$

The factors of g_* arise from the reconstruction and direction equations, and $\mathcal{P}_0$ is the additional quadratic factor of the direction determinant. At the generic point, first separate $\zeta = \eta + 1$, then the five conditions in (8.8). On their complement, g_* is a unit, and $\mathcal{P}_0$ either vanishes or is a unit. This is the partition in Table 2.

Each row of the table is a condition in the function field of one irreducible curve, rather than a condition at an isolated point. A product vanishes in this field only if one of its factors vanishes identically. The rows are taken in order, so intersections are assigned to the first applicable row. After these rows have been removed, the factors in $g_*\mathcal{P}_0$ are units and may be used in rational reconstruction. The equations before those divisions are used in the earlier rows.

8.3.1 Characteristic ratio curves

The first three conditions in (8.8) integrate, respectively, to

$$\zeta = \eta + c, \quad \zeta = 1 + c\eta, \quad \zeta = c(\eta + 1).$$

Their common branch $\zeta = \eta + 1$ is treated separately below. The undivided zero pair equations treat a persistent vanishing opposite pair. On the remaining chart, two compatibility equations form an affine system for two pair coordinates over three other pair coordinates. Its three coefficient rank strata give the following dimension counts.

At rank two, the first two coordinates are reconstructed. The remaining equations give two coprime polynomials in the three base coordinates, so their common zero set has dimension at most one. At rank one, consistency imposes a Cramer numerator as well as the determinant. The determinant is irreducible and does not divide that numerator, again

Table 2: Branches at the generic point of the ratio curve $\mathcal{B}$. The rows are considered in the displayed order, each outside the preceding rows.

Condition	Algebraic argument	Fibre conclusion	Proof location
$\zeta = \eta + 1$	Rational reconstruction and coprime quintic initial forms	$\dim Y_{\text{gen}} \leq 1$	Lemma 8.11; Appendix C.15
$\varsigma = 1, L = -1$, or $L = -\varsigma$	Affine pencils, Cramer numerators and compatibility	$\dim Y_{\text{gen}} \leq 1$	Appendix C.14
$\varsigma = 0$ or $L = 0$	The first nonzero homogeneous terms of the original equations	$\dim Y_{\text{gen}} \leq 1$	Appendix C.16
$g_* \neq 0, \mathcal{P}_0 = 0$	Direction equations and tangency to the projected factors	Finite projective boundary	Appendices C.16 and C.17
$g_* \mathcal{P}_0 \neq 0$	Direction and amplitude bounds; slope elimination	Finite projective boundary	Proposition 8.12; Lemma C.13

giving base dimension at most one. The third compatibility equation cuts the affine kernel line in finitely many points and adds no fibre dimension. At rank zero, and on the zero-column loci where this last equation can degenerate, the compatible base is finite and the residual equations bound the remaining fibre dimension by one. Appendix C.14 gives the determinants, Cramer numerators and residual equations on these strata.

The last two conditions in (8.8) give $\zeta = c$ and $\zeta = c\eta$. On these branches the highest homogeneous terms can cancel; the next nonzero terms of the original equations give the required bound, as shown in Appendix C.16.

Lemma 8.11 (The ratio line $\zeta = \eta + 1$). *On the persistent branch $\zeta = \eta + 1$, the generic state fibre has dimension at most one.*

Proof. Tangency makes a fixed diagonal multiple of the normalized curl matrix symmetric. The original relations exclude a persistent zero opposite pair. On the remaining open set, three pair coordinates reconstruct the complete state rationally. Two necessary polynomials in those coordinates have coprime leading forms of degree five for every admissible triple of principal values and ratio. A common factor of the two polynomials would have a leading homogeneous factor dividing both quintic forms, contrary to their coprimality. A common zero set of dimension at least two in these three coordinates would have a hypersurface component and hence a common irreducible factor. Its dimension is therefore at most one, and rational reconstruction transfers the bound to the state fibre. Appendix C.15 gives the reconstruction, the leading forms and the zero pair argument. $\square$

8.3.2 Directions and amplitudes at infinity

On $g_* \neq 0$, the first tangencies give the rational coordinate change (C.28). It expresses the three R_i rationally in the six off-diagonal frame coefficients. Consequently a dimension bound for these six coordinates is also a bound for the complete generic state fibre.

Write the three opposite pairs as (w_i, m_i) in these coordinates. At infinity each nonzero pair has direction $(w_i : m_i) \in \mathbb{P}^1$. Once direction representatives are chosen, their amplitudes are defined up to one common multiplier. The mixed equations first constrain these three projective directions. For each permitted choice, the cubic and quartic equations constrain the relative amplitudes. A finite set of directions with finite projective amplitude fibres gives a finite boundary. Directions with a zero coordinate are treated by retaining the homogeneous equations on that coordinate support.

A direction specifies the ratio within a pair, whereas an amplitude specifies its size relative to the other pairs. If all three pairs are nonzero, the amplitudes form an open subset of $\mathbb{P}^2$; with two pairs they form an open subset of $\mathbb{P}^1$, and with one pair there is no relative amplitude parameter. This separation avoids counting the common homogeneous multiplier as a boundary coordinate. It also shows why a direction bound alone is insufficient: even one fixed direction can carry an amplitude curve.

The mixed equations are bilinear in pairs of directions. An invertible coefficient block determines a projective map between its two directions. With three invertible blocks, the direction fibre is the fixed point set of their composition. A nonidentity projective map of $\mathbb{P}^1$ has at most two fixed points. Its fixed point equation vanishes identically precisely when the representing matrix is scalar. With singular blocks, the determinant and trace relations give the exceptional cases in Appendix C.16.

Projection of these exceptional equations gives finitely many plane polynomials. A factor depending only on η is a unit. For a nonvertical irreducible factor f , characteristic zero gives separability in ζ , so f_ζ is nonzero at the generic point of $f = 0$. Tangency therefore gives $\varsigma = -f_\eta/f_\zeta$. Substituting this slope into the original equations and clearing denominators gives polynomials whose joint greatest common divisor with f is 1 in $\mathbb{Q}(\eta)[\zeta]$, as proved in Appendix C.16. Their Bézout identity remains valid in $\mathbb{C}(\mathcal{B})$, where the nonzero denominators in η are units. It contradicts simultaneous vanishing on a persistent curve. The coefficient zero sets use the original bilinear equations and their tangency relations before elimination. The mixed direction fibre consequently has at most two points at the generic ratio point.

For proper supports and the seven full support patterns with some $m_i = 0$, the cubic and quartic relations also give finite projective amplitude fibres. Passing to squared amplitudes is a finite map, and the cubics are linear in those squares. For full support, rank zero is excluded by the pure cubic coefficients and tangency to their vanishing loci. At rank at least two, the projective kernel is at most a point, whose inverse image contains only finitely many amplitude points. At rank one, the projective kernel is a line; a positive dimensional fibre would require a common line component of the remaining quartic equations after the substitution. For two nonzero pairs with amplitudes x, y , quartic combinations give $\pm 4x^2y^2$, so there is no amplitude curve on that support. For three nonzero pairs, if the defining linear form of the kernel line has a zero coefficient, the pure quartic terms $xy(x+y)^2$ and $xy(x^2+y^2)$ force $xy = 0$. The six resulting configurations have coefficient ideals, augmented by their first two curve derivatives, containing a unit on $g_* \neq 0$ (Appendix C.16). They therefore cannot persist on the ratio curve.

On $m_1m_2m_3 \neq 0$, put $t_i = w_i/m_i$. One direction requires separate treatment:

$$t_1 = \frac{\varsigma}{2\zeta}, \quad t_2 = t_3 = -\frac{L}{2\eta\zeta}.$$

For this distinguished direction, the amplitude equations are conics in the squared amplitudes. Since $g_* \neq 0$, $\eta\varsigma/\zeta \neq 0, 1$. The determinants in Appendix C.16 give a nondegenerate conic, and the coefficient minor

$$-2(\eta\varsigma/\zeta - 1)^2 \neq 0$$

shows that the first two conics are not proportional. If $\eta\varsigma/\zeta = 1/2$, both degenerate, but the third conic is nondegenerate and not proportional to either. At the other permitted values at least one of the first two is nondegenerate. A nondegenerate conic is irreducible, so its intersection with a conic not proportional to it has no common component. Bézout's theorem makes their projective intersection finite. The finite squaring map transfers this conclusion to the original pair amplitudes.

For full support directions other than the distinguished one, use the affine coordinates (U, V, w_*) of (C.30). The distinguished direction is their origin. The original equations are

linear in the auxiliary vector

$$\xi = (UV, U, V, Q_*(w_*))^T,$$

where Q_* is the polynomial in (C.31). Its coordinates also satisfy

$$Q_*(w_*)\xi_2\xi_3 - \xi_1\xi_4 = 0.$$

The original relations imply that $\xi = 0$ forces the distinguished direction, so $\xi \neq 0$ on the present branch. The five linear equations give the matrix M_* ; four of the equations give the square matrix M , as defined in Appendix C.19.

The linear equations determine possible auxiliary vectors. The displayed quadratic relation then imposes the product condition satisfied by the original coordinates. Eliminating the direction parameter gives two necessary conditions: $\mathcal{P}_1 = 0$ for the linear equations and $\mathcal{P}_2 = 0$ for their compatibility with the quadratic relation. These polynomials are defined in Appendix C.20.

8.3.3 The quadratic condition and the remaining directions

On $\mathcal{P}_0 = 0$, retain the original direction equations. Appendix C.17 treats both determinant roots, the leading coefficient, their collision locus and the reconstruction minors. For a projected plane polynomial f , its curve tangency and $\mathcal{P}_0 = 0$ imply

$$\eta\zeta(\eta^2 - 1)f_\eta^2 - 2(\zeta^2 - \eta^2)f_\eta f_\zeta - \eta\zeta(\zeta^2 - 1)f_\zeta^2 = 0.$$

After removal of the localization factors, the straight branches and $(\eta + 1)\zeta = \eta$, each norm or minor polynomial f is coprime to its displayed tangency polynomial in $\mathbb{Q}(\eta)[\zeta]$ (Appendix C.17). The corresponding Bézout identities exclude a persistent curve on $f = 0$.

Substituting the tangent slope $\varsigma = -f_\eta/f_\zeta$ in $\mathcal{P}_0$ and multiplying by f_ζ^2 gives the displayed polynomial. A nonvertical irreducible curve contained in $f = 0$ and satisfying $\mathcal{P}_0 = 0$ would therefore make both polynomials vanish in its function field, contrary to their Bézout identity. On the remaining curve $(\eta + 1)\zeta = \eta$, the original full support direction ideal is the unit ideal over $\mathbb{Q}(\eta)$.

For the distinguished direction, $\varsigma' = \varsigma L/(\eta\zeta)$. Differentiation of $\mathcal{P}_0 = 0$ therefore gives

$$\eta\zeta((\mathcal{P}_0)_\eta + \varsigma(\mathcal{P}_0)_\zeta) + \varsigma L(\mathcal{P}_0)_\varsigma = 0.$$

Its resultant with $\mathcal{P}_0$ has, outside the localization factors and finite η -levels, only the factors $\zeta + 1$, $\zeta + \eta$ and $\eta^2 + \zeta^2 + 1$. The first two are the straight branches. The last is strictly positive on the real ratio image, so it is nonzero in $\mathbb{C}(\mathcal{B})$. Together with the proper support and amplitude bounds, these exclusions give a finite generic projective boundary on $\mathcal{P}_0 = 0$.

On $g_*\mathcal{P}_0 \neq 0$, the determinant defines a finite degree two divisor in the projective direction parameter. The linear equations define a kernel line on this divisor. Taking the norm of the quadratic relation along the divisor gives a polynomial condition on (η, ζ, ς) , which must hold together with the condition for existence of the kernel.

Vanishing of the affine leading coefficient places a root at infinity. Vanishing of the discriminant makes two roots coincide. The homogeneous determinant defines a divisor of length two in either case, with multiplicity. Forming the norm on this divisor includes both configurations.

Proposition 8.12 (Conditions for the remaining directions). *Let $\mathcal{B}$ be the algebraic ratio curve of a solution, with η nonconstant, and suppose that $g_*\mathcal{P}_0 \neq 0$ at its generic point. A compatible full support direction with $m_1m_2m_3 \neq 0$, distinct from the distinguished direction, satisfies $\mathcal{P}_1 = \mathcal{P}_2 = 0$ in $\mathbb{C}(\mathcal{B})$.*

Proof. The nonzero auxiliary vector is annihilated by M_* , so its five maximal minors have a common root in the projective direction parameter. Write each of these quadratic minors in the basis of the three homogeneous monomials of degree two. Evaluation at that root gives a nonzero vector annihilated by their coefficient matrix. The matrix consequently has rank at most two, and all its three by three minors vanish.

The identities in Appendix C.20 give the ideal of these coefficient minors as $(\mathcal{P}_1)$. After $\mathcal{P}_1$ is factored out, the residual minors have a linear combination equal to $\varsigma^2(\varsigma - 1)^2$ times a nonzero polynomial in η . This is a unit at the generic point of the present curve, so the vanishing minors imply $\mathcal{P}_1 = 0$. The same unit identities make the two constant columns of the transformed four-row matrix independent. Their six minor charts cover the parameter base.

On each chart, Schur reduction leaves a two by two linear pencil. The coefficient ideals for an identically zero determinant and for rank at most two at a root both contain a nonzero polynomial in η times $\varsigma(\varsigma - 1)\mathcal{P}_0$ (Appendix C.21). This element is a unit in the current function field. The determinant is therefore a nonzero homogeneous quadratic, and the matrix has rank three on its zero divisor. This divisor is finite flat of degree two, and its kernel is an invertible line on which the quadratic relation restricts.

On the determinant divisor the reduced pencil has rank one, because its rank zero locus has been excluded by the coefficient ideal. A nonzero entry gives a local generator of its kernel; on overlaps these generators differ by a unit. Thus the kernel is a line also at a multiple determinant root. The quadratic relation restricts to a section on this line. The local kernel generators and their unit transition factors define it on the entire determinant divisor.

Construct the norm on the ambient regular parameter open set. Locally trivialize the kernel line and write the restricted quadratic relation as an element of the finite locally free algebra of the determinant divisor. Its norm is the determinant of multiplication by this element. Vanishing at a compatible root makes multiplication noninvertible in that root's local algebra, hence makes the norm zero. This statement also holds when that local algebra is nonreduced at a double root.

The two adjugate columns give local kernel representatives. Their resultants are nonzero, so the norm is a nonzero section and defines a Cartier divisor. Every irreducible component of this divisor occurs in both resultants. Their common factor therefore bounds its support by $\mathcal{P}_0\mathcal{P}_2 = 0$. On the present open set $\mathcal{P}_0$ is a unit, leaving $\mathcal{P}_2 = 0$. The homogeneous determinant includes roots at infinity. Lemmas C.10 and C.12 give the Schur reduction and the polynomial identity for this norm. $\square$

The conditions $\mathcal{P}_1 = \mathcal{P}_2 = 0$ constrain a direction at a fixed ratio point. For them to hold along the ratio curve, their derivatives with respect to its nonconstant coordinate η must also vanish. Since $\varsigma' = \varsigma$, differentiation gives

$$(\mathcal{P}_j)_\eta + \varsigma(\mathcal{P}_j)_\varsigma + (\mathcal{P}_j)_\varsigma\varsigma' = 0, \quad j = 1, 2. \quad (8.10)$$

Cross multiplication eliminates ς' without division and gives the polynomial C_* of Appendix C.23. Put $R = \text{Res}_\varsigma(\mathcal{P}_1, \mathcal{P}_2)$ and $T = \text{Res}_\varsigma(\mathcal{P}_1, C_*)$.

Multiplying the first differentiated equation by $(\mathcal{P}_2)_\varsigma$ and the second by $(\mathcal{P}_1)_\varsigma$, then subtracting, cancels the ς' terms. The identity also holds where either partial derivative is zero. The actual curve thus supplies a common root in ς for each resultant pair and satisfies $R = T = 0$. These differentiated conditions express persistence along the curve.

Lemma C.13 gives a common factor D_0 and

$$\gcd(R/D_0, T/D_0) = 1 \quad \text{in } \mathbb{Q}(\eta)[\varsigma].$$

The Bézout identity for these quotients has coefficients in $\mathbb{Q}(\eta)[\varsigma]$. Clearing its coefficients gives an identity with a nonzero polynomial in η on the right. That polynomial is a unit in

$\mathbb{C}(\mathcal{B})$, because η is nonconstant. If D_0 were nonzero in this field, $R = T = 0$ would make both quotients vanish, contradicting the identity. Hence $D_0 = 0$. Since the curve function field has no zero divisors, one of its seven irreducible factors vanishes identically:

$$\zeta = 0, \quad \zeta = -1, \quad \zeta = -\eta, \quad \zeta = \eta, \quad \zeta = 1, \quad \zeta = \eta + 1, \quad (\eta + 1)\zeta = \eta. \quad (8.11)$$

The factors ζ , $\zeta - \eta$, $\zeta - 1$ and $\zeta - \eta - 1$ are units on $g_*\mathcal{P}_0 \neq 0$. On $\zeta = -1$, differentiation gives $\varsigma = 0$; on $\zeta = -\eta$, it gives $L = 0$. On $(\eta + 1)\zeta = \eta$, $\varsigma = (\eta + 1)^{-2}$ and direct substitution gives $\mathcal{P}_0 = 0$. These three cases are the earlier rows of Table 2.

Each alternative therefore contradicts a factor being a unit in $g_*\mathcal{P}_0$. No full support direction distinct from the distinguished one can persist on the current ratio curve.

Proposition 8.13 (Fibres over an algebraic ratio curve). *If the ratio image is an algebraic curve, the generic state fibre has dimension at most one.*

Proof. After cyclic relabeling take η nonconstant. The characteristic branches, $\zeta = \eta + 1$, and $\mathcal{P}_0 = 0$ have the asserted bound. Outside them, the proper supports, the directions with some $m_i = 0$, and the distinguished direction have finite projective amplitude fibres over a finite direction set. Every other direction would satisfy $\mathcal{P}_1 = \mathcal{P}_2 = 0$ and would therefore force one of the excluded conditions (8.11). The generic projective boundary is finite. Its affine fibre has dimension at most one, and the rational state reconstruction transfers this bound to all nine fibre coordinates. $\square$

8.4 Proof of the rank bound

Proof of Theorem 8.1. On a connected localized chart, $\dim \mathcal{B} = 0$ is excluded by Lemma 3.1. If $\dim \mathcal{B} = 2$, Proposition 8.10 gives $\dim Y_{\text{gen}} = 0$. If $\dim \mathcal{B} = 1$, Proposition 8.13 gives $\dim Y_{\text{gen}} \leq 1$. Lemma 8.3 yields $\text{rank } dX \leq 2$ in both cases.

By Lemma 3.2, the localized locus is dense on the branch $h_1h_2h_3 \neq 0$. Every minor of order three of the smooth state map (3.2) vanishes there. Continuity gives its vanishing at every remaining point of that branch. $\square$

9 Classical branches and proof of the main theorem

The rank exclusions establish the integrability hypothesis in Kafadar's classification. We combine this result with the classical classification on regular invariant charts, continue the resulting germs analytically, and prove the converse by constructing the pressure. The families are those in (2.3)–(2.8).

9.1 Integrability of a principal plane field

Proof of Theorem 2.1. Suppose first that $h_1h_2h_3$ were nonzero at a point. After shrinking, it is nonzero on an open ball U . Lemma 3.2 says that each zero set of a normalization factor has empty interior in U . These sets are closed, so the complement of their finite union contains a nonempty open chart satisfying (3.3). On a connected such chart, the retained state map X has constant rank by Theorem 3.3. Since its domain is three dimensional, its rank is one of 0, 1, 2, 3.

Ranks zero and one contradict Theorem 4.3. Rank three contradicts Theorem 8.1. At rank two, Proposition 5.1 gives one nonzero common similarity generator with equal source and target scales. The normal forms in Proposition 5.3 leave exactly the six pairings with zero dilation factor or a pair with a common nonzero dilation factor. The former are excluded by Theorem 6.1, and the latter by Theorem 7.1. Thus every possible rank is excluded, proving

$$h_1h_2h_3 \equiv 0. \quad (9.1)$$

By Lemma 2.3, B is analytic on each inverse-function chart. Its distinct constant eigenvalues are λ_i^{-1} , and its labelled spectral projectors are the analytic tensors

$$P_i = \prod_{j \neq i} \frac{B - \lambda_j^{-1} I}{\lambda_i^{-1} - \lambda_j^{-1}}.$$

Projecting and normalizing fixed vectors gives analytic unit eigenvectors on a small ball; their signs may be chosen to orient the frame. Each $h_i = e_i \cdot \text{curl } e_i$ is analytic and is unchanged when e_i changes sign. A finite product of analytic functions on a connected domain can vanish identically only if one factor vanishes identically. Applied to (9.1), this gives a fixed label with $h_i \equiv 0$. Finally, the nowhere-zero one-form $\alpha_i = e_i^\flat$ satisfies $\alpha_i \wedge d\alpha_i = h_i \text{vol}_g$, so Frobenius gives the asserted local surfaces. $\square$

9.2 Constant principal values

Throughout this subsection the eigenvalues of $c = B^{-1}$ are constant. Thus c is a positive definite flat metric with $\det c = 1$, and $\text{div } c$ and $\text{div } c^{-1}$ are local gradients. By Lemma 2.3, c is analytic.

Proposition 9.1 (Kafadar’s classification). *Under these hypotheses on constant principal values the following statements hold.*

1. *If c has a fixed Euclidean eigenvector on a neighbourhood, the deformation is locally in $\mathcal{F}_0$ or $\mathcal{F}_5$.*
2. *If at least two eigenvalues coincide, c has a fixed Euclidean eigenvector locally. Consequently this case is also contained in $\mathcal{F}_0 \cup \mathcal{F}_5$.*
3. *If the eigenvalues are distinct and one principal direction is normal to a local foliation by surfaces, the deformation is locally in $\mathcal{F}_0$ or $\mathcal{F}_5$.*

Proof. These are the applications of Theorems 1, 2, and 3 of [11, p. 20], respectively. Theorem 1 assumes that the Cartesian components of c are represented by their local Taylor series. Lemma 2.3 supplies this hypothesis. By [11, §5, pp. 25–27, equation (5.1)], besides constant Cartesian tensors, the deformation has the form

$$r = AR, \quad \theta = C\Theta + D \log R, \quad z = EZ, \quad A^2 CE = 1.$$

After translations and fixed Euclidean coordinate changes, this is $\mathcal{F}_5$ with $(\alpha, k, \beta, \gamma) = (A, C, D, E)$. A constant Cartesian metric c has an affine local Euclidean realization, giving $\mathcal{F}_0$. Two local realizations of the same flat metric differ by a fixed Euclidean motion.

Theorem 2, proved in [11, §3, pp. 20–23], gives the fixed eigenvector in the repeated-value case, after which Theorem 1 applies. If all three values coincide, positivity and $\det c = 1$ give $c = I$, whose realizations are rigid motions.

For the third assertion, use Theorem 3 and its proof in [11, §4, pp. 23–25], with the Taylor hypothesis in its final fixed-direction step supplied as above. Its geometric assumption is exactly the existence of a principal direction normal to coordinate surfaces. In the notation of the present paper this means $\alpha_i \wedge d\alpha_i = 0$, or $h_i = 0$, on a neighbourhood. Frobenius’ theorem gives the local foliation. $\square$

Corollary 9.2 (The constant-invariant classification). *If both invariants are constant on a neighbourhood, the universal deformation is locally in $\mathcal{F}_0$ or $\mathcal{F}_5$.*

Proof. The characteristic polynomial fixes the eigenvalues of c . For repeated values apply the second part of Proposition 9.1. For distinct values, Theorem 2.1 supplies a fixed principal label with h_i identically zero on a neighbourhood. Apply Proposition 9.1(3). $\square$

9.3 Reduction of the invariant map

For constant invariants, two choices of energy supplied the equilibrium equations used above. When the invariants vary, derivatives of the arbitrary energy give additional restrictions. They reduce the invariant map to one local parameter and place its level surfaces in the setting of the classical regular patch classifications.

Work on a spatial inverse-function chart and identify vectors and one-forms using the Euclidean metric. Put

$$u = dI_1, \quad v = dI_2, \quad \beta = (\operatorname{div} B)^b, \quad \delta = (\operatorname{div} c)^b.$$

For a symmetric tensor A and a one-form ω , write $A\omega = (A\omega^\sharp)^b$. Expanding the divergence in the original stress equation gives

$$\frac{1}{2} dp_W = W_1\beta - W_2\delta + W_{11}Bu + W_{12}(Bv - cu) - W_{22}cv. \quad (9.2)$$

Universality requires the right-hand side to be closed for every W . Conversely, on a sufficiently small simply connected chart, this closedness supplies the pressure for each W by the Poincaré lemma. This is the coordinate-free form of the pressure calculation in [3, pp. 468–469, equations (2.1)–(2.2)].

Lemma 9.3 (Rank of the invariant map). *Every universal deformation satisfies*

$$dI_1 \wedge dI_2 = 0 \quad (9.3)$$

at every point. On a chart where one invariant has nonzero differential, that invariant may be used as a local parameter U , and

$$I_1 = i_1(U), \quad I_2 = i_2(U), \quad B dU = \lambda(U) dU, \quad c dU = \lambda(U)^{-1} dU.$$

Here $\lambda > 0$, and at least one of i'_1, i'_2 is nonzero.

Proof. At a specified point the third-order jet of the function $W(I_1, I_2)$ can be prescribed arbitrarily, for example by choosing a cubic polynomial. In the exterior derivative of (9.2), the coefficients of $W_{111}, W_{222}, W_{112}, W_{122}$ give, respectively,

$$u \wedge Bu = 0, \quad v \wedge cv = 0. \quad (9.4)$$

$$\begin{aligned} v \wedge Bu + u \wedge Bv - u \wedge cu &= 0, \\ v \wedge Bv - v \wedge cu - u \wedge cv &= 0. \end{aligned} \quad (9.5)$$

Suppose $u \wedge v \neq 0$ at a point. The same condition holds on a smaller neighbourhood. The pure identities give $Bu = \alpha u$ and $cv = \gamma v$, where α, γ are smooth positive functions. The first mixed identity then reduces to

$$(\gamma^{-1} - \alpha)u \wedge v = 0.$$

Thus u and v belong to a common two dimensional eigenspace of B with eigenvalue α . Symmetry and $\det B = 1$ imply that its three eigenvalues are $\alpha, \alpha, \alpha^{-2}$ throughout this neighbourhood. Consequently

$$I_1 = 2\alpha + \alpha^{-2}, \quad I_2 = \alpha^2 + 2\alpha^{-1},$$

which makes $u \wedge v = 0$, a contradiction. This proves (9.3).

On a chart with $dI_1 \neq 0$, take $U = I_1$; the constant-rank theorem gives $I_2 = i_2(U)$. If instead $dI_2 \neq 0$, use $U = I_2$. The appropriate identity in (9.4) gives the asserted principal direction, even if the other invariant is constant. On each connected level patch of U , the characteristic polynomial of B has constant coefficients, while the continuous function λ takes values in its finite set of positive roots. Thus λ is constant on that level patch. Evaluating it on a local transverse section shows that the resulting function of U is smooth, and analytic under Lemma 2.3. $\square$

Lemma 9.4 (Pressure on a regular invariant chart). *On a chart as in Lemma 9.3, universality implies*

$$\beta = b(U)dU, \quad \delta = d(U)dU \quad (9.6)$$

for scalar functions b, d . Conversely, the principal-direction relations of that lemma together with (9.6) make (9.2) locally exact for every strain energy W .

Proof. The last three terms in (9.2) are $k_W(U)dU$, where

$$k_W = \lambda i'_1 W_{11} + (\lambda i'_2 - \lambda^{-1} i'_1) W_{12} - \lambda^{-1} i'_2 W_{22}.$$

They are already closed. The coefficients of the remaining first and second derivatives of the strain energy in its exterior derivative are

$$\begin{aligned} d\beta &= d\delta = 0, \\ i'_1 dU \wedge \beta &= 0, \\ i'_2 dU \wedge \beta - i'_1 dU \wedge \delta &= 0, \\ i'_2 dU \wedge \delta &= 0. \end{aligned}$$

If $i'_1 \neq 0$, the second identity gives $dU \wedge \beta = 0$, and the middle mixed identity gives $dU \wedge \delta = 0$. If $i'_2 \neq 0$, use the last and mixed identities in the reverse order. Hence both forms are multiples of dU . Their closedness makes their coefficients constant on connected level patches of U , proving (9.6). Substitution in (9.2) proves the converse: its right-hand side is a scalar function of U times dU . $\square$

The preceding lemmas place each regular invariant chart in the setting of the classical classification.

9.4 Variable invariants on regular patches

Proposition 9.5 (Classical classification on a regular patch). *Let V be a regular invariant chart, and restrict to a subchart on which the multiplicities of the eigenvalues of c are constant. Then the deformation is locally one of $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$. Moreover the eigenvalue corresponding to dU is simple on this subchart. Every nonempty open set on which the invariant map has rank one contains a nonempty subchart to which this statement applies.*

Proof from the classical classification. Denote the eigenvalue of c along dU by c_1 and the remaining two by c_2, c_3 . These labels are local, with c_1 determined by the gradient. There are three cases.

If the three values are distinct, Ericksen's classification [3, §4, pp. 476–485] applies. The level sets of the invariant parameter are parallel planes or concentric circular cylinders; the spherical possibility is excluded in this case. The resulting models are the block, sector, and cylindrical families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$.

If $c_1 \neq c_2 = c_3$, the gradient eigenvalue is simple and [3, §5, pp. 485–486] completes the classification. The planar and cylindrical possibilities give already listed degenerations, and the spherical case gives $\mathcal{F}_4$, including eversion with its compensating angular orientation.

If c_1 is repeated, Ericksen's §6, pp. 486–489, covers only an additional integrability assumption on the remaining principal direction. The unrestricted case uses [16, Theorem 1.1, p. 138, proof pp. 154–158], with the corrections in [17, p. 406]. For nonconstant invariants their result reduces the geometry to concentric circular cylinders; the curves tangent to the distinguished simple principal direction are circular helices. Martin and Carlson then test the two cylindrical models under the condition that the eigenvalue along the invariant gradient is repeated and show that only constant-invariant degenerations remain [18, §3, pp. 107–108]. This case therefore cannot occur on V . Their footnote on p. 106 records the needed scope of

Ericksen's cylindrical calculation: although it occurs in the section headed by distinct eigenvalues, the cylindrical classification itself requires universality and cylindrical level surfaces, so it applies to the preceding reduction.

These are the classes B_1, B_2, B_3 of [18, p. 106]. See also [16, Theorem 1.2, pp. 138 and 159].

Finally, a nonempty regular invariant open set has a smaller open set of constant eigenvalue multiplicity: either the discriminant of its characteristic polynomial is nonzero somewhere, or it vanishes throughout and one restricts to a repeated-value patch. A triple eigenvalue would give $B = I$. At such a point both invariants attain their lower bound 3, so both differentials vanish; hence no triple-value point lies in the regular invariant set. In the repeated-value alternative the two distinct eigenvalues have a positive gap locally, which fixes whether the gradient belongs to the simple or repeated eigenspace. The three cases above apply on this smaller patch. $\square$

The classical result supplies a deformation on a regular invariant patch, after restricting to constant eigenvalue multiplicity. Ericksen removes the common critical set $\{dI_1 = dI_2 = 0\}$ at the start of the variable invariant argument [3, p. 469]. To include that set and points where principal values coincide, consider the analytic invariants pulled back to a connected reference neighbourhood. Either both are constant, giving the classification already proved, or their common critical set has empty interior. In the latter case the proposition supplies a classical germ on a nonempty open subset. The remaining step is to carry this fixed germ to the chosen point.

9.5 Continuation of a classical germ

The cylindrical and spherical charts used in the classical formulas may omit an axis or a centre at which the Cartesian deformation is regular. For example, the removable-axis part of $\mathcal{F}_3$ is

$$\begin{aligned}\chi_1(X, Y, Z) &= \sqrt{A} \operatorname{Rot}(\delta Z + \theta_0) \operatorname{diag}(1, k)(X, Y), \\ \chi_3(X, Y, Z) &= Z/(Ak) + z_0, \\ A &> 0, \quad k \in \{1, -1\}.\end{aligned}\tag{9.7}$$

Here δ is arbitrary. For $\mathcal{F}_5$, a regular interior axis requires $\beta = 0$ and $k = \pm 1$, giving an affine map.

A classical germ fixes both its parameters and its source and target Euclidean coordinates. On the regular domain of its formula, the analytic identity theorem propagates equality with the deformation. At a boundary of that domain, boundedness of the deformation gradient either excludes the boundary or forces a removable coordinate singularity. The following proposition describes these continuations.

Proposition 9.6. *Let Ω be connected and let $\chi : \Omega \rightarrow \mathbb{R}^3$ be a real analytic local diffeomorphism with $\det D\chi = 1$. Suppose that on a nonempty open subset it equals one of (2.3)–(2.8), in fixed source and target Euclidean coordinates and with fixed parameters. Then every point of Ω has a neighbourhood on which χ is a continuation of that same model. The only removable coordinate boundaries are (9.7) for $\mathcal{F}_3$, a rigid motion for $\mathcal{F}_4$, and an affine map for $\mathcal{F}_5$.*

Proof. Shrink the initial open subset to a connected patch. Keep the parameters and the two Euclidean coordinate changes selected on the initial patch. Denote by $\mathcal{D}$ the full regular source domain of its displayed formula: for $\mathcal{F}_1$ it is $\{2\alpha X + d > 0\}$; for $\mathcal{F}_2$ and $\mathcal{F}_5$ it is $\{R > 0\}$; for $\mathcal{F}_3$ it is $\{R > 0, AR^2 + B_0 > 0\}$; and for $\mathcal{F}_4$ it is $\{R > 0, \varepsilon R^3 + B_0 > 0\}$. For $\mathcal{F}_0$ take $\mathcal{D} = \mathbb{R}^3$. Here R is distance from the fixed axis or centre. Positive roots are used throughout.

Let V be the connected component of $\Omega \cap \mathcal{D}$ containing the initial patch. In the cylindrical models, use the angular covering of $\mathcal{D}$, with the lift of Θ fixed on that patch. The formula defines a single analytic Cartesian map G_{mod} on this covering. If $\pi : \tilde{V} \rightarrow V$ is the connected covering component containing the lifted patch, the analytic identity theorem gives

$$G_{\text{mod}} = \chi \circ \pi \quad \text{on } \tilde{V}. \quad (9.8)$$

The projection is onto V , by path lifting. Thus the same model gives the germ of χ at every point of V . The models without a source angle use the identical argument without a covering.

By (9.8), continuation around a loop in V returns the same germ. For a loop of winding number n , single-valuedness requires $\beta n = 0$ in $\mathcal{F}_2$, $\kappa n = 0$ and $kn \in \mathbb{Z}$ in $\mathcal{F}_3$, and $kn \in \mathbb{Z}$ in $\mathcal{F}_5$.

Suppose that $q \in \Omega \cap \partial V$. Then $q \notin \mathcal{D}$: otherwise a small ball about q contained in $\Omega \cap \mathcal{D}$ and meeting V would lie in the same component. Consequently q is a finite axis, centre, or root boundary of the model. Choose a closed ball about q contained in Ω ; $D\chi$ is bounded there. Since equality on V includes equality of derivatives, we examine the coordinate boundaries.

The block and straightening models. For $\mathcal{F}_1$ the only boundary is $r = 0$, where

$$|\partial_X \chi| = |\alpha|/r \rightarrow \infty.$$

For $\mathcal{F}_2$ the boundary is $R = 0$, and its derivative along the unit angular direction satisfies

$$|D\chi(e_\Theta)| = \frac{\sqrt{\beta^2 + \gamma^2}}{R} \geq \frac{|\beta|}{R} \rightarrow \infty.$$

Both contradict boundedness, since $\alpha\beta \neq 0$.

The cylindrical model. In $\mathcal{F}_3$, $A(k\gamma - \delta\kappa) = 1$ in particular implies $A \neq 0$. If $r \rightarrow 0$ and $R \rightarrow R_* > 0$, then

$$|\partial_R \chi| = |A|R/r \rightarrow \infty.$$

If $R \rightarrow 0$ and $r \rightarrow r_* > 0$, then

$$|D\chi(e_\Theta)| = \frac{\sqrt{k^2 r^2 + \kappa^2}}{R} \rightarrow \infty, \quad (9.9)$$

because k and κ cannot both vanish. The only remaining possibility is $R \rightarrow 0$ and $r \rightarrow 0$. It forces $B_0 = 0$, $A > 0$, and $r = \sqrt{A}R$. Equation (9.9) first gives $\kappa = 0$.

The punctured ball obtained by removing the source axis from a sufficiently small ball about q is connected and lies in $\Omega \cap \mathcal{D}$. It meets V and hence lies entirely in V . On any angular wedge in this punctured ball, (9.8) supplies one fixed angle branch. At the axial level $Z = Z_*$ of q , continuity gives $\chi_\perp(q) = 0$, and differentiability gives a real linear map $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$L(\cos \Theta, \sin \Theta) = \sqrt{A}(\cos(k\Theta + \delta Z_* + \theta_0), \sin(k\Theta + \delta Z_* + \theta_0)) \quad (9.10)$$

on an open interval of angles. Its norm is constant on that interval. The quadratic identity $v^\top(L^\top L - AI)v = 0$ on an open circular arc implies $L^\top L = AI$. Hence $L/\sqrt{A}$ is orthogonal; its angular rate on that arc is $+1$ or -1 . Differentiating (9.10) therefore yields $k = \pm 1$.

The volume condition now gives $\gamma = 1/(Ak)$, and the original model has the Cartesian expression

$$\begin{aligned} \chi_\perp(X, Y, Z) &= \sqrt{A} \operatorname{Rot}(\delta Z + \theta_0) \operatorname{diag}(1, k)(X, Y), \\ \chi_3(X, Y, Z) &= Z/(Ak) + z_0. \end{aligned} \quad (9.11)$$

This is analytic on all of $\mathbb{R}^3$, has determinant one, and agrees with the initial model. The parameter δ remains arbitrary.

The spherical model. For $\mathcal{F}_4$, a boundary with $r \rightarrow 0$, $R \rightarrow R_* > 0$ is excluded by the radial stretch $R^2/r^2 \rightarrow \infty$. A boundary with $R \rightarrow 0$, $r \rightarrow r_* > 0$ is excluded by the tangential stretch $r/R \rightarrow \infty$. If both radii vanish, then $B_0 = 0$ and positivity of both radii gives $\varepsilon = 1$. Thus $r = R$, $Q \in \text{SO}(3)$, and $\chi(X) = Q_0 + Q(X - P)$ is the rigid continuation of the same model.

The logarithmic cylindrical model. The only boundary of $\mathcal{F}_5$ is $R = 0$. As above, a punctured ball about an interior axis point belongs to V , so fixed radial rays in an angular wedge may be used. Identifying the transverse target plane with $\mathbb{C}$, their transverse difference quotients are

$$\frac{\chi_\perp(R \cos \Theta, R \sin \Theta, Z_*) - \chi_\perp(q)}{R} = \alpha \exp(i(\beta \log R + k\Theta + \theta_0)).$$

They cannot converge as $R \downarrow 0$ unless $\beta = 0$. The same open-arc linearity argument as in (9.10) then gives $k = \pm 1$. The continuation is the affine map

$$\chi_\perp = \alpha \text{Rot}(\theta_0) \text{diag}(1, k)(X, Y), \quad \chi_3 = Z/(\alpha^2 k) + z_0. \quad (9.12)$$

There is no coordinate boundary for $\mathcal{F}_0$. In each removable case just found, the fixed model has an analytic Cartesian expression on all of $\mathbb{R}^3$; equality on the original patch and the identity theorem give equality with χ throughout Ω . In every other case no point of $\Omega \cap \partial V$ is possible. Then V is nonempty, open and relatively closed in connected Ω , so $V = \Omega$. This proves the proposition. $\square$

9.6 Sufficiency of the classical models

Proposition 9.7 (Sufficiency of the classical models). *Every regular local model in $\mathcal{F}_0, \dots, \mathcal{F}_5$, with the parameter ranges in (2.3)–(2.8), is an orientation preserving incompressible universal deformation. The assertion includes regular Cartesian continuations across coordinate axes or centres.*

Proof. Fix an arbitrary sufficiently smooth energy $W(I_1, I_2)$, and put

$$\Sigma = 2W_1 B - 2W_2 B^{-1}. \quad (9.13)$$

It suffices to construct a local scalar p with $\nabla p = \text{div } \Sigma$. All tensor components in cylindrical or spherical coordinates below refer to orthonormal bases.

For a symmetric tensor whose cylindrical components depend only on r , the Euclidean divergence is

$$\text{div } \Sigma = \left(\Sigma'_{rr} + \frac{\Sigma_{rr} - \Sigma_{\theta\theta}}{r} \right) e_r + \left(\Sigma'_{r\theta} + \frac{2\Sigma_{r\theta}}{r} \right) e_\theta + \left(\Sigma'_{rz} + \frac{\Sigma_{rz}}{r} \right) e_z. \quad (9.14)$$

In particular, if $B_{r\theta} = B_{rz} = 0$, then B^{-1} has the same radial block structure. Both invariants, hence both W_i , depend only on r , and $\Sigma_{r\theta} = \Sigma_{rz} = 0$. For any fixed $r_* > 0$ in the chart,

$$p(r) = \Sigma_{rr}(r) + \int_{r_*}^r \frac{\Sigma_{rr}(s) - \Sigma_{\theta\theta}(s)}{s} ds \quad (9.15)$$

therefore gives equilibrium. The possibly nonzero component $\Sigma_{\theta z}(r)$ contributes nothing to (9.14).

The homogeneous family. For $\mathcal{F}_0$, the matrix $B = LL^\top$ and both invariants are constant. Thus Σ is constant in Cartesian coordinates, and any constant pressure suffices. The signed Jacobian is $\det L = 1$.

Bending and cylindrical deformation. For $\mathcal{F}_1$ and $\mathcal{F}_3$, respectively, the deformation gradients are

$$F_1 = \begin{pmatrix} \alpha/r & 0 & 0 \\ 0 & \beta r & 0 \\ 0 & \gamma & (\alpha\beta)^{-1} \end{pmatrix}, \quad F_3 = \begin{pmatrix} AR/r & 0 & 0 \\ 0 & kr/R & \delta r \\ 0 & \kappa/R & \gamma \end{pmatrix}. \quad (9.16)$$

Their signed determinants are

$$\det F_1 = 1, \quad \det F_3 = A(k\gamma - \delta\kappa) = 1. \quad (9.17)$$

The first matrix uses a Cartesian reference basis; the second uses a cylindrical reference basis. Their left Cauchy–Green tensors have the radial block structure required above. For $\mathcal{F}_3$ one has

$$\begin{aligned} B_{rr} &= \frac{A^2 R^2}{r^2}, & B_{\theta\theta} &= r^2 \left(\frac{k^2}{R^2} + \delta^2 \right), \\ B_{\theta z} &= r \left(\frac{k\kappa}{R^2} + \delta\gamma \right), & B_{zz} &= \frac{\kappa^2}{R^2} + \gamma^2. \end{aligned} \quad (9.18)$$

Since $R^2 = (r^2 - B_0)/A$, all these entries are functions of r . Consequently (9.15) proves universality for both families.

Straightening. For $\mathcal{F}_2$, with a Cartesian current basis and a cylindrical reference basis,

$$F_2 = \begin{pmatrix} \alpha R & 0 & 0 \\ 0 & \beta/R & 0 \\ 0 & \gamma/R & (\alpha\beta)^{-1} \end{pmatrix}, \quad \det F_2 = 1. \quad (9.19)$$

Here $R^2 = 2(x - x_0)/\alpha$. Thus the Cartesian components of B depend only on x , and $B_{xy} = B_{xz} = 0$. The same is true of B^{-1} and Σ , so $\operatorname{div} \Sigma = (\partial_x \Sigma_{xx})e_x$. A pressure is

$$p(x) = \Sigma_{xx}(x) = 2\alpha^2 R^2 W_1 - \frac{2W_2}{\alpha^2 R^2}, \quad (9.20)$$

with W_i evaluated at (I_1, I_2) .

Spherical deformation. For $\mathcal{F}_4$, differentiation in Cartesian coordinates gives

$$F_4 = Q \left(r'n \otimes n + \frac{r}{R}(I - n \otimes n) \right), \quad r' = \varepsilon \frac{R^2}{r^2}. \quad (9.21)$$

In particular,

$$\det F_4 = (\det Q) r' \frac{r^2}{R^2} = \varepsilon^2 = 1. \quad (9.22)$$

Writing $e_r = Qn$, the two eigenvalues of B are

$$\begin{aligned} b_r &= \frac{R^4}{r^4}, & b_t &= \frac{r^2}{R^2}, \\ \Sigma &= \Sigma_r e_r \otimes e_r + \Sigma_t (I - e_r \otimes e_r), \\ \Sigma_r &= 2W_1 b_r - 2W_2 b_r^{-1}, & \Sigma_t &= 2W_1 b_t - 2W_2 b_t^{-1}. \end{aligned} \quad (9.23)$$

All coefficients depend only on r , since $R^3 = \varepsilon(r^3 - B_0)$. The spherical divergence formula yields

$$\begin{aligned} \operatorname{div} \Sigma &= \left(\Sigma'_r + \frac{2(\Sigma_r - \Sigma_t)}{r} \right) e_r, \\ p(r) &= \Sigma_r(r) + 2 \int_{r_*}^r \frac{\Sigma_r(s) - \Sigma_t(s)}{s} ds. \end{aligned} \quad (9.24)$$

Logarithmic cylindrical deformation. For $\mathcal{F}_5$ the orthonormal gradient and strain tensor are

$$F_5 = \begin{pmatrix} \alpha & 0 & 0 \\ \alpha\beta & \alpha k & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad B = \begin{pmatrix} \alpha^2 & \alpha^2\beta & 0 \\ \alpha^2\beta & \alpha^2(\beta^2 + k^2) & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix}. \quad (9.25)$$

Their determinant and invariants are

$$\det F_5 = \alpha^2 k \gamma = 1, \quad (9.26)$$

$$I_1 = \alpha^2(1 + \beta^2 + k^2) + \gamma^2, \quad I_2 = \frac{1 + \beta^2 + k^2}{\alpha^2 k^2} + \gamma^{-2}.$$

In particular, W_1, W_2 are constants for the fixed arbitrary energy. Set

$$M_W = \alpha^2 W_1 + \frac{W_2}{\alpha^2 k^2}. \quad (9.27)$$

Direct inversion of the planar block gives $\Sigma_{rr} - \Sigma_{\theta\theta} = 2M_W(1 - \beta^2 - k^2)$ and $\Sigma_{r\theta} = 2\beta M_W$. Hence

$$\begin{aligned} \operatorname{div} \Sigma &= \frac{2M_W(1 - \beta^2 - k^2)}{r} e_r + \frac{4\beta M_W}{r} e_\theta, \\ p(r, \theta) &= 2M_W \left[(1 - \beta^2 - k^2) \log \frac{r}{r_*} + 2\beta(\theta - \theta_*) \right]. \end{aligned} \quad (9.28)$$

The angle is a local coordinate as in Section 2.2.

On an inverse-function neighbourhood of a regular Cartesian continuation, Σ is C^2 . The identity $\operatorname{curl} \operatorname{div} \Sigma = 0$, established off the coordinate singularity, extends there by continuity. The local Poincaré lemma then supplies a pressure on a ball containing the point. For example, the twisted-axis model (9.7) has $B_{rr} = A$, $B_{\theta\theta} = A + \delta^2 r^2$ and $(B^{-1})_{rr} = (B^{-1})_{\theta\theta} = A^{-1}$. Its two invariants are smooth functions of r^2 , and (9.15) is itself a smooth function of r^2 up to an additive constant. $\square$

9.7 Proof of the main theorem

Proof of Theorem 2.2. Fix a reference point and a connected inverse-function ball U about it. Lemma 2.3 makes χ and the pulled-back invariants analytic. We first obtain one classical germ on a nonempty open subset of U ; Proposition 9.6 will then carry that fixed model to the chosen point.

If the invariants are not both constant, one of their differentials is nonzero somewhere. Lemma 9.3 and Proposition 9.5 yield a classical germ in $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_4$ after shrinking.

If the invariants are constant, their characteristic polynomial $t^3 - I_1 t^2 + I_2 t - 1$ fixes the positive principal values. Corollary 9.2 supplies a germ in $\mathcal{F}_0 \cup \mathcal{F}_5$.

In every case a single classical model agrees with χ on an open patch, with fixed parameters and fixed Euclidean coordinate changes. Proposition 9.6 therefore supplies its regular Cartesian germ at the original point. This proves necessity. Proposition 9.7 proves sufficiency. $\square$

10 Conclusion and discussion

For constant, pairwise distinct principal stretches, the universal equilibrium and compatibility equations force one principal direction to be normal to a local family of surfaces (Theorem 2.1). This supplies the geometric hypothesis in Kafadar's classification [11]. Together with the earlier classifications, it gives the six classical families in Theorem 2.2. Interior analyticity permits continuation through critical points of the principal invariant map and removable coordinate singularities, so the local classification holds throughout the regular deformation domain.

The proof uses the scale invariance of the principal frame equations. After normalization, fibres of positive dimension determine common Euclidean similarities of the reference and current configurations. The equations for these similarities exclude states of rank less than three on the locus where all principal abnormalities are nonzero. At rank three, projection onto the two abnormality ratios gives an algebraic dimension bound: the generic state fibre is finite over the ratio plane and has dimension at most one over a ratio curve. Thus the Zariski closure of the state image has dimension at most two, which excludes rank three.

Global realizations of the classical families on prescribed domains require further compatibility conditions. The local formulas must define a single valued deformation, and global injectivity imposes restrictions on the domain and the family parameters. Equilibrium also has a global condition: on a multiply connected current domain, each closed pressure form must have vanishing periods in order to admit a single valued pressure. For cylindrical models on a full annulus, angular periodicity of the deformation and the periods of the pressure forms impose restrictions on the family parameters.

For incompressible anisotropic constitutive classes that admit every isotropic strain energy, universality implies the isotropic equilibrium equations. Theorem 2.2 therefore restricts any such universal deformation to a classical family. The remaining problem is to classify the material preferred direction fields that satisfy the additional universality constraints on these deformations. Yavari and Goriely [32] derived these constraints for transversely isotropic, orthotropic and monoclinic solids and described direction fields with the symmetries of the classical deformations. A specific question is whether the additional equations force these symmetries, or allow other direction fields on the same deformation families.

A The geometric differential system

The state polynomials are defined over

$$\mathbb{K} = \mathbb{Q}(a, b), \quad \mathcal{R}_{\text{raw}} = \mathbb{K}[p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23}, p_{31}, p_{32}, p_{33}, \tau_0, \dots, \tau_5, u_0, u_1].$$

A.1 Pressure, compatibility, and signs in the principal frame

The coefficients in (2.14) satisfy

$$d\alpha_k(e_i, e_j) = \sum_{\ell} \epsilon_{ij\ell} p_{k\ell}, \quad [e_i, e_j] = - \sum_{k, \ell} \epsilon_{ij\ell} p_{k\ell} e_k.$$

For constant w , the Koszul coefficients in (2.17) satisfy

$$\Gamma_{ij}^k - \Gamma_{ji}^k = C_{ij}^k, \quad w_k \Gamma_{ij}^k + w_j \Gamma_{ik}^j = 0. \quad (\text{A.1})$$

These are the torsion-free and metric compatibility identities. For $T = \sum w_i e_i \otimes e_i$, differentiation gives

$$(\text{div } T)_i = (w_i - w_j) p_{jk} - (w_i - w_k) p_{kj}.$$

The exterior derivative of its dual one-form is (2.19), with frame term $\sum_{\ell} A_{\ell} p_{\ell i}$.

The projector reduction follows from

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & a^{-1} & b^{-1} \end{pmatrix} = - \frac{(a-1)(a-b)(b-1)}{ab}. \quad (\text{A.2})$$

It is nonzero on the entire admissible principal value domain. Consequently closedness of the divergences of B and c , together with $\sum_i \text{div } P_i = 0$, gives closedness separately for all three projector divergences.

A.2 Construction of the finite differential system

We construct the vector fields of Proposition 2.4 over the coefficient ring $\mathbb{Q}[a, b, P_{\text{mat}}^{-1}]$. Rows and columns of every matrix in this paper are numbered from one.

Order 27 formal variables $d_{i,jk}$ lexicographically, starting with $d_{1,11}$, and let $\partial_i^0 p_{jk} = d_{i,jk}$. Construct a column E of 24 expressions as follows: first the nine quantities $w_\ell R_{kij}^\ell(w)$ for (ℓ, k) and (i, j) in the ordered list $(2, 3), (3, 1), (1, 2)$, for $w = (1, 1, 1)$; next the same nine for $w = (1, 1/a, 1/b)$; finally the three expressions (2.19) for $w = (1, a, b)$ and for $w = (1, 1/a, 1/b)$. Derivatives in this definition are ∂_i^0 . Write

$$E = M^{(0)}d - r^{(0)}.$$

Set

$$\begin{aligned} I_0 &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 20, 21, 22, 23, 24), \\ J_0 &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 22, 27), \\ K_0 &= (15, 21, 23, 24, 25, 26). \end{aligned}$$

Write $Q = (\mathcal{Q}_i^{p_{jk}})_{i,j,k}$ for the twenty-seven component column in the derivative order just specified. Using the derivative coordinate column τ of (2.21), set $Q_{K_0} = \tau$, and use

$$Q_{J_0} = (M_{I_0, J_0}^{(0)})^{-1} \{r_{I_0}^{(0)} - M_{I_0, K_0}^{(0)} \tau\}. \quad (\text{A.3})$$

The determinant and substitution identities are

$$\det M_{I_0, J_0}^{(0)} = -\frac{(a-1)^4(a-b)^5(b-1)^6}{32a^6b^8}, \quad M^{(0)}Q - r^{(0)} = 0. \quad (\text{A.4})$$

The free derivative coordinates are those in (2.20).

For the first prolongation introduce 18 variables $T_{i,r}$, $1 \leq i \leq 3$, $0 \leq r \leq 5$, in that order, and set

$$\partial_i^1 = \sum_{j,k} \mathcal{Q}_i^{p_{jk}} \partial_{p_{jk}} + \sum_r T_{i,r} \partial_{\tau_r}.$$

Form the 27 commutator expressions

$$\partial_i^1 \mathcal{Q}_j^{p_{k\ell}} - \partial_j^1 \mathcal{Q}_i^{p_{k\ell}} - \sum_m C_{ij}^m \mathcal{Q}_m^{p_{k\ell}}, \quad (\text{A.5})$$

in pair order $(i, j) = (2, 3), (3, 1), (1, 2)$ and lexicographic (k, ℓ) . Write their column as $M^{(1)}T - r^{(1)}$. Set

$$\begin{aligned} I_1 &= (1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 15, 16, 17, 21, 24), \\ J_1 &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 18), \quad K_1 = (13, 15). \end{aligned}$$

Set $T_{K_1} = (u_0, u_1)^\top$ and solve the selected rows as in (A.3). Denote the resulting 18 functions by $\mathcal{Q}_i^{\tau_r}$. Thus $u_0 = D_3\tau_0$ and $u_1 = D_3\tau_2$. The determinant is

$$\det M_{I_1, J_1}^{(1)} = -\frac{2(a-b)^2}{a^2(b-1)^2}. \quad (\text{A.6})$$

The unselected residuals generate three cubic conditions C_1, C_2, C_3 given in Appendix B.1.

Next introduce six variables $U_{i,r}$, $1 \leq i \leq 3$, $0 \leq r \leq 1$, and put

$$\partial_i^2 = \sum_{j,k} \mathcal{Q}_i^{p_{jk}} \partial_{p_{jk}} + \sum_{r=0}^5 \mathcal{Q}_i^{\tau_r} \partial_{\tau_r} + \sum_{r=0}^1 U_{i,r} \partial_{u_r}.$$

Apply the commutator formula to the six coordinates τ_r , using

$$\partial_i^2 \mathcal{Q}_j^{\tau_r} - \partial_j^2 \mathcal{Q}_i^{\tau_r} - \sum_m C_{ij}^m \mathcal{Q}_m^{\tau_r}.$$

The resulting 18 expressions form the column $M^{(2)}U - r^{(2)}$. The six rows

$$I_2 = (1, 3, 4, 6, 7, 9)$$

have determinant

$$\det M_{I_2, \{1, \dots, 6\}}^{(2)} = \frac{b(a-1)}{2a(b-1)}. \quad (\text{A.7})$$

Invert this submatrix to define the six functions $\mathcal{Q}_i^{u_r}$. Substituting these functions for $U_{i,r}$ in ∂_i^2 gives the three fields $\mathcal{Q}_i$.

Every solution satisfies $D_i \mathbf{S} = \mathcal{Q}_i(\mathbf{S})$. Its coordinate weights are the d_ℓ defined in Section 2, and $\mathcal{Q}_i^\ell$ has weight $d_\ell + 1$. All coefficient denominators divide powers of P_{mat} in (2.22).

The residual equations and their differential ideal are specified below.

A.3 The residual equations

Use the derivative order and the selections just specified. For any one of these selections write

$$M = (M_J \mid M_K), \quad y_J = (M_{I,J})^{-1}(r_I - M_{I,K}y_K).$$

The entries of M are rational functions of the principal value ratios; the entries of r are state polynomials. Matrix inversion therefore takes place over $\mathbb{K}$, while multiplication by r takes place in $\mathcal{R}_{\text{raw}}$. The determinants are (A.4)–(A.7). The identity $M^{(0)}Q - r^{(0)} = 0$ gives

$$D_i h_i = -2h_i(p_{jk} - p_{kj}), \quad (i, j, k) \text{ cyclic}. \quad (\text{A.8})$$

After substitution of the functions $\mathcal{Q}_i^{\tau_r}$, the unselected rows of the first prolongation have the three monic representatives (B.1)–(B.3) in the lexicographic order of $\mathcal{R}_{\text{raw}}$.

The last solve uses all eighteen commutators of the six τ coordinates. Keep their twelve nonzero residuals in row order, followed by

$$D_1 C_1, D_2 C_1, D_3 C_1, D_1 C_2, D_2 C_2, D_3 C_2, D_1 C_3, D_2 C_3, D_3 C_3.$$

Here $D_i C_j$ denotes the polynomial $\mathcal{Q}_i(C_j)$. This defines the ordered quartic polynomials $\mathcal{E}_0, \dots, \mathcal{E}_{20}$.

A.4 Weights and the differential ideal

Weighted homogeneity of the three linear systems gives

$$\sum_{\ell=1}^{17} d_\ell \mathbf{S}_\ell \frac{\partial \mathcal{Q}_i^j}{\partial \mathbf{S}_\ell} = (d_j + 1) \mathcal{Q}_i^j. \quad (\text{A.9})$$

For the quintic polynomials, select quartic rows

$$(1, 2, 4, 6, 7, 11, 12, 13, 14, 16, 19, 20).$$

Differentiate each in directions 1, 2, 3, in that order, obtaining thirty-six rows. The selected derivative positions are

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, \\ 21, 22, 23, 26, 28, 31, 32, 35).$$

The six u_0, u_1 components of the bracket defects (2.24) correspond to the three cyclic pairs.

Every solution annihilates $\mathcal{J}_{\text{raw}}$, by Proposition 2.4.

B Normalization and the equations at ranks zero and one

B.1 Algebraic reconstruction of the normalized state

Use the scalar coefficients $\Delta_a, \Delta_b, \Delta_{ab}$ introduced in Section 3.

The cubic residuals of (A.5) are

$$\begin{aligned}
 C_1 = & p_{11}^2 p_{23} - p_{11}^2 p_{32} + \frac{p_{21} \tau_3 (a-b)}{ab-a} + \frac{p_{22} \tau_1 (a-b)}{ab-a} \\
 & + \left(\frac{p_{21}^2 p_{23} (-a+b)}{ab-a} + \frac{p_{22} \tau_4 (-a+b)}{ab-a} \right) \\
 & + \left(\frac{p_{21}^2 p_{32} (-a+b)}{ab-a} + \frac{p_{31} \tau_2 (-a+b)}{2ab-2a} \right) \\
 & + \left(\frac{p_{11} p_{32} p_{33} (a-b)}{ab-b} + \frac{p_{33} \tau_4 (a-b)}{ab-b} \right) \\
 & + \left(\frac{p_{21} p_{22} p_{31} (2a-2b)}{ab-a} + \frac{p_{21} p_{31} p_{33} (-a+b)}{ab-b} \right)
 \end{aligned} \tag{B.1}$$

$$\begin{aligned}
 C_2 = & p_{11} p_{12} p_{32} - p_{11} \tau_3 + p_{11} \tau_5 - p_{12}^2 p_{13} - p_{12}^2 p_{31} - p_{12} \tau_1 \\
 & + \left(\frac{p_{13} p_{22}^2 (a-b)}{ab-a} + \frac{p_{22}^2 p_{31} (-a+b)}{ab-a} \right) \\
 & + \left(\frac{p_{32} \tau_2 (-a+b)}{2ab-2a} + \frac{p_{33} \tau_5 (-a+b)}{ab-b} \right) \\
 & + \left(\frac{p_{12} p_{32} p_{33} (-a+b)}{ab-b} + \frac{p_{21} p_{22} p_{32} (a-b)}{ab-a} \right) \\
 & + \left(\frac{p_{22} p_{31} p_{33} (a-b)}{ab-b} \right)
 \end{aligned} \tag{B.2}$$

$$\begin{aligned}
 C_3 = & p_{11} p_{12} p_{33} - p_{11} p_{13} p_{32} - p_{11} p_{23} p_{31} - p_{11} p_{31} p_{32} - p_{11} \tau_0 - p_{12} p_{13}^2 - p_{13} \tau_1 \\
 & + (2p_{11} p_{13} p_{23} + \frac{p_{11} \tau_2 (ab-b)}{2ab-2a} - p_{12} p_{13} p_{31}) \\
 & + \left(\frac{p_{12} p_{33}^2 (-a+b)}{ab-b} + \frac{p_{21} p_{23}^2 (a-b)}{ab-a} \right) \\
 & + \left(\frac{p_{21} p_{33}^2 (a-b)}{ab-b} + \frac{p_{22} \tau_0 (-a+b)}{ab-a} \right) \\
 & + \left(\frac{p_{11} p_{21} p_{22} (-ab+b)}{ab-a} + \frac{p_{23} \tau_3 (-a+b)}{ab-a} \right) \\
 & + \left(\frac{p_{21} p_{22} p_{33} (-a+b)}{ab-a} + \frac{p_{21} p_{23} p_{32} (a-b)}{ab-a} \right)
 \end{aligned} \tag{B.3}$$

Write the ordered quartic equations of Appendix A.2 as $M\mathbf{u} - \mathbf{r}$, with the residuals of the second prolongation followed by $\mathcal{Q}_i C_j$ in cubic-major order. Let M_0 select rows (1, 4) and M_1 select rows (13, 17); S_0, S_1 denote these row-selection matrices. For

$$X_0 = \Delta_b h_1 h_2, \quad Y_0 = \Delta_a h_1 h_3, \quad Z_0 = \Delta_{ab} h_2 h_3$$

their determinants are

$$\det M_0 = \frac{2(6X_0 + Y_0 + Z_0)}{\Delta_a}, \quad \det M_1 = \frac{\Delta_{ab}(X_0 - Y_0 - Z_0)}{2\Delta_a^2}.$$

The matrix

$$N = \frac{\Delta_a}{2} \text{adj}(M_0) S_0 + \frac{2\Delta_a^2}{\Delta_{ab}} \text{adj}(M_1) S_1 \tag{B.4}$$

satisfies $NM = 7\Delta_b h_1 h_2 \text{Id}_2$. Thus $\mathbf{u} = N\mathbf{r}/(7\Delta_b h_1 h_2)$.

B.2 The square system for constant ratios

Use the normalized abnormalities $\hat{h}_2 = h_2/h_1$, $\hat{h}_3 = h_3/h_1$ and R_1, R_2, R_3 in Lemma 3.1. Differentiate the relations

$$p_{13} - R_1 p_{31} = 0, \quad p_{21} - R_2 p_{12} = 0, \quad p_{32} - R_3 p_{23} = 0$$

in directions 2, 3, 1, respectively. The ratios $\hat{h}_2, \hat{h}_3$ and hence the R_i are constant. Insert (A.3) and substitute

$$\hat{\mathbf{p}} = \begin{pmatrix} 1 & \hat{p}_{12} & R_1 \hat{p}_{31} \\ R_2 \hat{p}_{12} & \hat{h}_2 & \hat{p}_{23} \\ \hat{p}_{31} & R_3 \hat{p}_{23} & \hat{h}_3 \end{pmatrix}. \quad (\text{B.5})$$

All six free first jets cancel. The three coefficients of $(\hat{p}_{31}^2, \hat{p}_{12}^2, \hat{p}_{23}^2)$ give the matrix displayed in that lemma, whose determinant is 4.

B.3 Exclusion of an open normalization divisor

Proof of Lemma 3.2. Suppose $\mu_1 = 0$ on an open set. Put

$$q = \frac{\Delta_b}{\Delta_a}, \quad d = 1 - q, \quad g = \frac{a-1}{b-1} = 1 - ad, \\ b = \frac{a(1-d)}{1-ad}, \quad z = h_1/h_2 = \hat{h}_2^{-1}.$$

Here $a > 0$ and $qdg h_2 z \neq 0$. The values $z = \pm d$ throughout an open piece already contradict Lemma 3.1. We may therefore shrink to $z^2 \neq d^2$.

Differentiating $h_3 - qh_2 = 0$ in the frame equations gives $p_{32} = -p_{23}$ and

$$\tau_0 = \tau_3 = \tau_5 = 0, \quad \tau_2 = 2h_2(p_{21} - p_{12}), \quad \tau_1 = -p_{12}p_{31} + h_2p_{23}z(1 + 2z/d).$$

In this subsection use the h_2 -normalized coefficients

$$\frac{\mathbf{p}}{h_2} = \begin{pmatrix} z & X & Y \\ U & 1 & W \\ V & -W & q \end{pmatrix}.$$

The two remaining cubic conditions and one differentiated equation relating opposite entries give

$$(X^2 + d)Y + 2z(1 + z/d)XW = 0, \quad (\text{B.6})$$

$$(dq - Y^2)X + 2z(1 - z/d)YW = 0, \quad (\text{B.7})$$

$$2X^2Y^2 + (d - z)Y^2 - q(d + z)X^2 = 0, \quad (\text{B.8})$$

$$-(1 + aq)(z^2 + d^2) + 2(1 + qg)z - 4XU - 4gYV = 0. \quad (\text{B.9})$$

Equations (B.6)–(B.7) show that $X = 0$ if and only if $Y = 0$. If this occurs on an open piece, (B.9) confines z to the zeros of

$$Q(z) = -(1 + aq)(z^2 + d^2) + 2(1 + qg)z.$$

This is nonzero: if $aq \neq -1$ its quadratic coefficient is nonzero, whereas $aq = -1$ gives $Q = 4z$. Continuity forces z constant, contradicting Lemma 3.1. Thus a solution has a smaller open set with $XY \neq 0$.

The unnormalized function $s = p_{23}$ satisfies $D_2 s = D_3 s = 0$. Since

$$[D_2, D_3] = -h_1 D_1 - p_{21} D_2 - p_{31} D_3,$$

$D_1s = 0$, so s is constant. The D_1p_{23} and D_1p_{32} equations give

$$UX = -\frac{gW^2}{ad} - \frac{(z-d)^2}{4d}, \quad VY = \frac{W^2}{ad} + \frac{q(z+d)^2}{4d}. \quad (\text{B.10})$$

Differentiate $\tau_0 = \tau_3 = \tau_5 = 0$ using the original fields. The equation $D_3\tau_0 = 0$ sets $u_0 = 0$; $D_3\tau_3 = 0$ determines τ_4/h_2^2 with coefficient 2; and $D_2\tau_5 = 0$ determines u_1/h_2^3 with coefficient $q/2$. The remaining equations are $D_2\tau_0 = 0$ and $D_3\tau_5 = 0$. Substitution of (B.10) and

$$W = -\frac{d(X^2 + d)Y}{2z(z+d)X}, \quad R = X^2, \quad Y^2 = \frac{q(z+d)R}{2R+d-z}$$

gives two cubics $F_1(R), F_2(R)$. The last denominator satisfies $(2R+d-z)Y^2 = q(z+d)X^2 \neq 0$.

Write $F_i = \sum_{k=0}^3 f_{i,k}R^k$, with coefficients and differential identities given in Appendices B.7 and B.4.

Put $e = ad - a - 1$. A useful combination of the two necessary cubics is

$$F_1 + F_2 = 4adz(z+d)H, \quad H = 4R(R+d)^2 + eK, \quad (\text{B.11})$$

where

$$K = 2(d^2 + 3dz - z^2)R^2 + (2d^3 + 4d^2z - 9dz^2 + z^3)R + 3dz^2(z-d).$$

The multiplier is nonzero on the current chart, so $F_1 = H = 0$. On the exceptional principal value relation $e = 0$, one has $d = 1 + a^{-1} > 0$ and $R = X^2 > 0$. Thus $H = 4R(R+d)^2 > 0$, a contradiction.

Suppose now $e \neq 0$, keeping the principal values fixed. The cubic H has three distinct Laurent roots at $z = \infty$, with the expansions

$$\begin{aligned} R_0 &= -3d + O(z^{-1}), \\ R_1 &= \frac{1}{2}z + \frac{1}{2e} + O(z^{-1}), \\ R_2 &= \frac{1}{2}ez^2 + O(z). \end{aligned} \quad (\text{B.12})$$

For $t = z^{-1}$, substitute respectively $R = r$, $R = (2t)^{-1} + c$, and $R = t^{-2}y$ into H , and multiply by t^3, t^3, t^6 . The limiting equations are $e(r + 3d) = 0$, $1/2 - ec = 0$, and $2y^2(2y - e) = 0$. The derivatives at these roots are $e, -e, e^2$, so the analytic implicit function theorem gives the displayed Laurent expansions.

Substitution in the first cubic gives

$$\begin{aligned} F_1(R_0) &= 12ade z^6 + O(z^5), \\ F_1(R_1) &= a(d-1)(ad-1)z^6 + O(z^5), \\ F_1(R_2) &= ae^3 z^9 + O(z^8). \end{aligned} \quad (\text{B.13})$$

Each leading coefficient is nonzero because $a, d, d-1, ad-1, e$ are nonzero. Hence F_1 and H have no common root over the Laurent-series field, and their resultant is a nonzero polynomial in z . Equivalently, since the leading R coefficient of H is 4, the root-product formula gives

$$\text{Res}_R(F_1, H) = -768a^3d(d-1)(ad-1)e^4z^{21} + O(z^{20}). \quad (\text{B.14})$$

A common root therefore confines z to a finite set. Continuity makes z locally constant, contrary to Lemma 3.1.

Finally cyclic relabeling transforms (a, b) to $(b/a, 1/a)$ and $(1/b, a/b)$. Their $h'_3 - q'h'_2$ are respectively μ_2 and $-(\Delta_a/\Delta_{ab})\mu_3$. These nonzero multiples reduce $\mu_2 = 0$ and $\mu_3 = 0$ to the case just proved. Each zero set is closed with empty interior, and their finite union has dense open complement. $\square$

B.4 The cubic equations on an open normalization divisor

On a solution patch with $\mu_1 = 0$, use the spectral parameters

$$q = 1 - d, \quad g = 1 - ad, \quad b = \frac{a(1-d)}{1-ad}.$$

The chart has $a > 0$, $b > 0$ and $a(a-1)d(1-d)(1-ad) \neq 0$. On the subdomain of Appendix B.3 satisfying $h_2XYz(z^2 - d^2) \neq 0$, the weighted substitution is

$$\begin{aligned} \mathbf{p}/h_2 &= \begin{pmatrix} z & X & Y \\ U & 1 & W \\ V & -W & q \end{pmatrix}, \\ (\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)/h_2^2 &= (0, -XV + Wz(1 + 2z/d), 2(U - X), 0, T, 0), \\ (u_0, u_1)/h_2^3 &= (J_{\text{jet}}, K_{\text{jet}}). \end{aligned}$$

Substitution into the fields $\mathcal{Q}_i$ gives three zero derivatives of $h_3 - qh_2$, two zero transverse derivatives of $p_{23} + p_{32}$, and

$$\frac{D_1(p_{23} + p_{32})}{h_2^2} = \frac{ad}{4g} \left\{ -(1 + aq)(z^2 + d^2) + 2(1 + qg)z - 4XU - 4gYV \right\}.$$

The three cubic relations restrict, in their fixed order, to

$$0, \quad -\{(X^2 + d)Y + 2z(1 + z/d)XW\}, \quad (dq - Y^2)X + 2z(1 - z/d)YW.$$

Their indicated linear combination is precisely the undivided biquadratic (B.8).

Define

$$e_{3i+j} = (D_{i+1}\tau_{k_j})/h_2^3 \Big|_{\text{substitution}}, \quad (k_0, k_1, k_2) = (0, 3, 5), \quad 0 \leq i, j \leq 2.$$

They satisfy

$$e_6 = J_{\text{jet}}, \quad \partial_T e_7 = 2, \quad \partial_{K_{\text{jet}}} e_5 = q/2.$$

Consequently $J_{\text{jet}} = 0$, $e_7 = 0$, $e_5 = 0$ determine the three auxiliary coordinates. Put

$$r_1 = e_3 - \frac{1}{2}(\partial_T e_3)e_7, \quad r_2 = e_8 - \frac{1}{2}(\partial_T e_8)e_7.$$

Both expressions are independent of $T, J_{\text{jet}}, K_{\text{jet}}$. Now substitute (B.10) and set

$$W = -\frac{d(X^2 + d)Y}{2z(z + d)X}, \quad L = 2X^2 + d - z, \quad Y^2 = \frac{q(z + d)X^2}{L}.$$

If F_1, F_2 are the two cubics defined in Appendix B.7, the differential identities are

$$r_1 = \frac{(d-1)X}{4a(ad-1)z^2Y(z+d)L^2} F_1(X^2), \tag{B.15}$$

$$r_2 = \frac{(d-1)X}{4az^2Y(z+d)L^2} F_2(X^2). \tag{B.16}$$

Since $LY^2 = q(z+d)X^2 \neq 0$, the multipliers in (B.15)–(B.16) are nonzero on this subdomain.

B.5 Orbit equations for two Killing generators

B.5.1 Planar orbit metrics

If $q_i = 0$ and $q_j q_k > 0$ in the translation case, the orbit metric has matrix $R_\theta \text{diag}(\lambda_i, m) R_\theta^\top$, where R_θ is planar rotation through θ and $m = \lambda_j \sin^2 \alpha + \lambda_k \cos^2 \alpha$. A translation depending on z removes the mixed normal-orbit terms of c and leaves a positive normal lapse. The flat Gauss equation gives

$$\det \frac{d}{dz} (R_\theta \text{diag}(\lambda_i, m) R_\theta^\top) = -(\lambda_i - m)^2 (\theta')^2 = 0.$$

If $\theta' \neq 0$, then $m = \lambda_i$ fixes the mixing angle because $\lambda_j \neq \lambda_k$. Consequently $|v_j|$ is constant. The affine vector v_j must be constant, which contradicts its rotating nonzero direction. Thus $\theta' = 0$.

B.5.2 Cylindrical orbit metrics

The nonradial projector components satisfy

$$[r(U'_i + 2U_i/r)]' = 0, \quad (V'_i + V_i/r)' = 0,$$

whose integrations give (4.3). Their projector identities are $U_i^2 + V_i^2 = q_i(1 - q_i)$ and $-(U_i U_j + V_i V_j) = q_i q_j$.

Choose an interval with $q_i = 0$. Two zero q 's give the radial principal direction. Otherwise $q_j q_k > 0$ and e_i is tangent to the cylindrical orbits. Their metric Γ in coordinates (ϕ, z) has the fixed relative eigenvalue λ_i :

$$\det\{\Gamma - \lambda_i \text{diag}(r^2, 1)\} = 0. \quad (\text{B.17})$$

Develop c into Euclidean space. The two commuting independent Killing fields become either two translations or a rotation and an axial translation. In the first case Γ is constant. Equation (B.17) gives $\Gamma_{zz} = \lambda_i$, $\Gamma_{\phi z} = 0$, and the principal direction is axial away from isolated coincidences, hence everywhere by continuity.

In the second case

$$\Gamma = R(r)^2 u u^\top + v v^\top$$

for constant independent two-vectors u, v . The developing map may contain arbitrary radial angular and axial shifts, which do not change this orbit matrix or its volume determinant. Constant volume gives $R^2(R')^2 \det[u, v]^2 = \text{constant } r^2$, so

$$R^2 = K r^2 + L, \quad K \neq 0.$$

The r^4 coefficient of (B.17) is $-\lambda_i K u_z^2$, so $u_z = 0$. Write

$$\Gamma = \begin{pmatrix} a_0 r^2 + n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix}.$$

The two coefficients of (B.17) are

$$(a_0 - \lambda_i)(n_{22} - \lambda_i) = 0, \quad n_{11}(n_{22} - \lambda_i) - n_{12}^2 = 0. \quad (\text{B.18})$$

If $n_{22} = \lambda_i$, then $n_{12} = 0$ and the eigenvector is axial unless $\Gamma = \lambda_i \text{diag}(r^2, 1)$ identically. In that exceptional isotropic orbit block, the two-plane projection identity below says $U_j^2 + V_j^2$ is constant. The Laurent sum-of-squares argument makes U_j, V_j constant, and then $V_j = 0$. Since $q_j q_k > 0$, the other two eigenvectors lie in the radial-azimuthal plane and e_i is again axial.

If $n_{22} \neq \lambda_i$, equations (B.18) give $a_0 = \lambda_i$ and a fixed coordinate kernel (u_0, v_0) for $\Gamma - \lambda_i \text{diag}(r^2, 1)$. If either component vanishes, e_i is axial or azimuthal, both satisfying $h_i = 0$. Otherwise its physical direction is (ru_0, v_0) and the tangent direction perpendicular to it gives

$$V_j = \kappa r U_j, \quad U_j = \beta_{j,0} + \beta_{j,-2}/r^2, \quad \kappa \neq 0.$$

The other relative orbit eigenvalue is $m = n_{22} + n_{11}/r^2$. In the plane spanned by e_r and that tangent direction, the two eigenvalues of c are λ_j, λ_k , whence

$$(\lambda_j - \lambda_k)^2 (U_j^2 + V_j^2) = (\lambda_j - m)(m - \lambda_k). \quad (\text{B.19})$$

Its r^2 coefficient forces $\beta_{j,0} = 0$, and its r^{-4} coefficient then gives $(\lambda_j - \lambda_k)^2 \beta_{j,-2}^2 = -n_{11}^2$. Thus $\beta_{j,-2} = n_{11} = 0$, contradicting $U_j^2 + V_j^2 = q_j q_k > 0$. This excludes the remaining cylindrical branch.

B.6 The nonabelian similarity equations

For the equivariant map (4.4), the fixed trace equations give the following planar identities. Let $F_\perp = Df_\perp$, $J_\perp = \det F_\perp \neq 0$ be constant, $\tilde{\varphi} = \varphi \circ f_\perp^{-1}$, $s = |\nabla \varphi|^2$, and $t = |\nabla \tilde{\varphi}|^2$. Constancy of the two spectral traces gives

$$\begin{aligned} |F_\perp|^2 + s &= E, & |F_\perp|^2/J_\perp^2 + t/\kappa^2 + 1/\kappa^2 &= E', \\ t &= c_0 s + d_0, & c_0 &= \kappa^2/J_\perp^2 > 0. \end{aligned} \quad (\text{B.20})$$

Constant determinant gives the inverse identity

$$\text{div}_x(c^{-1}) = (\Delta_Y f^{-1}) \circ f. \quad (\text{B.21})$$

Its axial component is $-(\Delta \tilde{\varphi}) \circ f_\perp / \kappa$. Closedness and the same target homogeneity argument imply $\Delta \tilde{\varphi} = 0$. On a chart with $\nabla \varphi \neq 0$, both height functions have nonzero gradient. Let φ^* be a local harmonic conjugate of φ and put $\mathcal{W} = \varphi + i\varphi^*$. Adjusting its imaginary additive constant gives $(x\partial_x + y\partial_y + \omega R_z)\mathcal{W} = \mathcal{W}$. In the complex source coordinate $z = x + iy$,

$$(1 + i\omega)z\mathcal{W}' = \mathcal{W}, \quad \mathcal{W} = c_h z^{1/(1+i\omega)}, \quad c_h \neq 0.$$

Choose a local branch and set $u = \text{Re } \mathcal{W}$, $v = \text{Im } \mathcal{W}$, $\theta = \arg \mathcal{W}$. Then

$$s = K e^{2\omega\theta}, \quad K > 0.$$

Pull a target harmonic conjugate of $\tilde{\varphi}$ back to a function $w(u, v)$. Homogeneity gives $uw_u + vw_v = w$, so $a_0 = w_u$, $b_0 = w_v$ depend only on θ . The two planar metrics are $(du^2 + dv^2)/s$ and $(du^2 + dw^2)/t$. Their determinant and trace give

$$b_0 = J_\perp c_0 + J_\perp d_0/s, \quad (\text{B.22})$$

$$a_0^2 = C_0 - c_0 s + C_1/s - J_\perp^2 d_0^2/s^2, \quad (\text{B.23})$$

where $C_0 = c_0 E - d_0 - 1 - J_\perp^2 c_0^2$ and $C_1 = d_0 E - 2J_\perp^2 c_0 d_0$. Equality of mixed derivatives gives $a_0' \cos \theta + b_0' \sin \theta = 0$.

If $\omega \neq 0$, use s as a local coordinate. Differentiation of (B.23) and the mixed-derivative identity give

$$s^4 \left(\frac{d(a_0^2)}{ds} \right)^2 \cos^2 \theta - 4J_\perp^2 d_0^2 a_0^2 \sin^2 \theta = 0.$$

Substituting $s = K e^{2\omega\theta}$ gives a finite sum of distinct complex exponentials with exponents $2n\omega$ and $2n\omega \pm 2i$. They are linearly independent on every open interval, by the Vandermonde determinant obtained by successive differentiation. Yet the coefficient of $e^{(8\omega+2i)\theta}$ is $c_0^2 K^4/4 \neq 0$. This is impossible.

The case $\omega = 0$ is treated directly in Section 4.

B.7 The cubics on the normalization zero sets

Write the two cubics in Lemma 3.2 as $F_i = \sum_{k=0}^3 f_{i,k} R^k$. Their coefficients are

$$\begin{aligned}
f_{1,0} = & a^2 d^8 + a^2 d^7 z - 2a^2 d^7 - 2a^2 d^6 z \\
& + a^2 d^6 - 18a^2 d^5 z^3 + a^2 d^5 z - 6a^2 d^4 z^4 \\
& + 18a^2 d^4 z^3 + 18a^2 d^3 z^5 + 6a^2 d^3 z^4 \\
& + 6a^2 d^2 z^6 - 18a^2 d^2 z^5 - 6a^2 d z^6 - 2ad^7 \\
& - 2ad^6 z + 4ad^6 + 2ad^5 z - 2ad^5 + 18ad^4 z^3 \\
& + 6ad^3 z^4 - 24ad^3 z^3 - 18ad^2 z^5 - 6adz^6 \\
& + 24adz^5 + d^6 + d^5 z - 2d^5
\end{aligned} \tag{B.24}$$

$$\begin{aligned}
f_{1,1} = & 2a^2 d^7 + 12a^2 d^6 z - 4a^2 d^6 \\
& + 36a^2 d^5 z^2 - 14a^2 d^5 z + 2a^2 d^5 \\
& - 20a^2 d^4 z^3 - 36a^2 d^4 z^2 + 2a^2 d^4 z \\
& - 62a^2 d^3 z^4 + 20a^2 d^3 z^3 - 18a^2 d^2 z^5 \\
& + 62a^2 d^2 z^4 - 2a^2 d z^6 + 18a^2 d z^5 \\
& + 2a^2 z^6 - 4ad^6 - 14ad^5 z + 10ad^5 - 36ad^4 z^2 \\
& + 12ad^4 z - 6ad^4 + 20ad^3 z^3 + 44ad^3 z^2 \\
& + 62ad^2 z^4 - 36ad^2 z^3 + 18adz^5 - 72adz^4 \\
& + 2az^6 + 2d^5 + 2d^4 z - 6d^4
\end{aligned} \tag{B.25}$$

$$\begin{aligned}
f_{1,2} = & a^2 d^6 + 11a^2 d^5 z - 2a^2 d^5 \\
& + 52a^2 d^4 z^2 - 12a^2 d^4 z + a^2 d^4 \\
& + 54a^2 d^3 z^3 - 52a^2 d^3 z^2 + a^2 d^3 z \\
& + 16a^2 d^2 z^4 - 54a^2 d^2 z^3 + 4a^2 d z^5 \\
& - 16a^2 d z^4 - 4a^2 z^5 - 2ad^5 - 12ad^4 z + 8ad^4 \\
& - 52ad^3 z^2 + 6ad^3 z - 6ad^3 - 54ad^2 z^3 \\
& + 64ad^2 z^2 - 16adz^4 + 60adz^3 - 4az^5 + d^4 \\
& + d^3 z - 6d^3
\end{aligned} \tag{B.26}$$

$$f_{1,3} = 2ad^3 - 4ad^2 z - 2ad^2 - 4adz^2 - 2d^2 \tag{B.27}$$

$$\begin{aligned}
f_{2,0} = & -a^2 d^8 - a^2 d^7 z + 2a^2 d^7 + 2a^2 d^6 z \\
& - a^2 d^6 + 6a^2 d^5 z^3 - a^2 d^5 z + 6a^2 d^4 z^4 \\
& - 6a^2 d^4 z^3 - 6a^2 d^3 z^5 - 6a^2 d^3 z^4 \\
& - 6a^2 d^2 z^6 + 6a^2 d^2 z^5 + 6a^2 d z^6 + 2ad^7 \\
& + 2ad^6 z - 4ad^6 - 2ad^5 z + 2ad^5 - 6ad^4 z^3 \\
& - 6ad^3 z^4 + 24ad^3 z^3 + 6ad^2 z^5 + 6adz^6 \\
& - 24adz^5 - d^6 - d^5 z + 2d^5
\end{aligned} \tag{B.28}$$

$$\begin{aligned}
f_{2,1} = & -2a^2d^7 - 4a^2d^6z + 4a^2d^6 \\
& - 12a^2d^5z^2 + 6a^2d^5z - 2a^2d^5 \\
& + 12a^2d^4z^2 - 2a^2d^4z + 30a^2d^3z^4 \\
& + 22a^2d^2z^5 - 30a^2d^2z^4 + 2a^2dz^6 \\
& - 22a^2dz^5 - 2a^2z^6 + 4ad^6 + 6ad^5z - 10ad^5 \\
& + 12ad^4z^2 + 4ad^4z + 6ad^4 - 28ad^3z^2 \\
& - 30ad^2z^4 + 36ad^2z^3 - 22adz^5 + 72ad^4z^4 \\
& - 2az^6 - 2d^5 - 2d^4z + 6d^4
\end{aligned} \tag{B.29}$$

$$\begin{aligned}
f_{2,2} = & -a^2d^6 - 3a^2d^5z + 2a^2d^5 \\
& - 20a^2d^4z^2 + 4a^2d^4z - a^2d^4 \\
& - 38a^2d^3z^3 + 20a^2d^3z^2 - a^2d^3z \\
& - 24a^2d^2z^4 + 38a^2d^2z^3 - 4a^2dz^5 \\
& + 24a^2dz^4 + 4a^2z^5 + 2ad^5 + 4ad^4z - 8ad^4 \\
& + 20ad^3z^2 + 26ad^3z + 6ad^3 + 38ad^2z^3 \\
& - 32ad^2z^2 + 24ad^2z - 60ad^2z^3 + 4az^5 - d^4 \\
& - d^3z + 6d^3
\end{aligned} \tag{B.30}$$

$$f_{2,3} = -2ad^3 + 20ad^2z + 2ad^2 + 20ad^2z^2 + 2d^2 \tag{B.31}$$

The resultant of the divisor cubics. The 6×6 Sylvester determinant of F_1, F_2 satisfies

$$\text{Res}_R(F_1, F_2) = -1024(d-1)(ad-1)a^3d^4z^6(z+d)^9P_{12}(a, d, z), \tag{B.32}$$

which defines $P_{12} \in \mathbb{Q}[a, d, z]$. Its leading coefficient and specialization at $ad - a - 1 = 0$ are

$$\begin{aligned}
[z^{12}]P_{12} &= 48a^3(ad - a - 1)^4, \\
P_{12}|_{d=1+1/a} &= \frac{55296z^4(a+1)^2(az - a - 1)(az + a + 1)}{a}.
\end{aligned} \tag{B.33}$$

If $aq \neq -1$, the leading coefficient is nonzero. If $aq = -1$, the specialization is a nonzero polynomial of degree six for $a > 0$. A common root R therefore gives $P_{12} = 0$. Continuity makes z locally constant, contradicting Lemma 3.1.

B.8 Equations for a dilation and a commuting rotation

For the map (4.5), the derivative in spherical source and cylindrical target orthonormal bases has columns

$$F_f = [(U, \omega U, V)^\top, (U', U\psi', V')^\top, (0, kU/\sin \theta, 0)^\top].$$

The matrix C_f has fixed eigenvalues, and the Jacobian is given by (4.6).

For a symmetric tensor T whose spherical physical components depend only on θ , define

$$\mathcal{A}_T = T'_{r\theta} + \cot \theta T_{r\theta} + 3T_{rr} - \text{tr } T, \quad \mathcal{E}_T = T'_{\theta\phi} + 2 \cot \theta T_{\theta\phi} + 3T_{r\phi}.$$

Its radial and azimuthal divergences are $\mathcal{A}_T/r$ and $\mathcal{E}_T/r$. Curl differentiation gives the pressure equations

$$\mathcal{A}_{C_f} = \alpha_{C_f}, \quad \sin \theta \mathcal{E}_{C_f} = \beta_{C_f}, \quad \mathcal{A}_{C_f^{-1}} = \alpha_B, \quad \sin \theta \mathcal{E}_{C_f^{-1}} = \beta_B, \tag{B.34}$$

with real constants $\alpha_{C_f}, \beta_{C_f}, \alpha_B, \beta_B$.

Contracting $\operatorname{div} C_f = F_f^\top \Delta f$ with D, R and using $|\Delta f|^2 = 2 \operatorname{tr} C_f + 2f \cdot \Delta f$ gives the first equation in (4.7).

Here $A_0 = 2(\operatorname{tr} C_f + \alpha_{C_f} - (\omega/k)\beta_{C_f})$ and $B_0 = 2(\operatorname{tr} C_f^{-1} + \alpha_B)$. The inverse norm equation follows from (B.21) and $|\Delta f^{-1}|^2 = 2 \operatorname{tr} C_f^{-1} + 2f^{-1} \cdot \Delta f^{-1}$. The middle equation is the volume identity (4.6).

B.9 The two Legendre equations

Lemma B.1. *Suppose (4.7) holds on an open interval with $|x|, |y| < 1$ and $\mathcal{N} > 0$. Then either*

$$\mathcal{N} = s_0 > 0, \quad y = (J/k)s_0^{-3/2}x + n,$$

or

$$\mathcal{N} = u + b_n x^2 > 0, \quad ub_n \neq 0, \quad y = \frac{Jx}{ku\sqrt{\mathcal{N}}}, \quad \widetilde{\mathcal{N}} = \frac{1}{u} - \frac{ub_n k^2}{J^2} y^2. \quad (\text{B.35})$$

Proof. The first equation has the general solution

$$\mathcal{N} = A_0/6 + b_0 P_2(x) + c_1 Q_2(x), \quad P_2 = \frac{3x^2 - 1}{2}, \quad Q_2 = \frac{P_2}{2} \log \frac{1+x}{1-x} - \frac{3x}{2}.$$

Suppose $c_1 \neq 0$. Since Q_2 tends to opposite infinities at the two endpoints, the maximal component I of $\{\mathcal{N} > 0\}$ containing the initial interval has an interior endpoint $\xi \in (-1, 1)$ where $\mathcal{N} = 0$. Analyticity makes this a finite-order zero, say of order $m \geq 1$. The integral of $\mathcal{N}^{-3/2}$ therefore diverges there. Integrate $y' = (J/k)\mathcal{N}^{-3/2}$ along I . Starting at $|y| < 1$, it attains $y = \varepsilon = \pm 1$ before reaching ξ . At this attained point $\mathcal{N} > 0$ and $y' \neq 0$, so $x(y)$ and $\widetilde{\mathcal{N}} = 1/\mathcal{N}(x(y))$ are analytic.

The inverse Legendre identity extends along I by the analytic identity theorem. Subtract $B_0/6$ from $\widetilde{\mathcal{N}}$ and expand at $y = \varepsilon$. Its Taylor coefficients obey

$$a_{n+1} = \frac{\varepsilon[6 - n(n+1)]}{2(n+1)^2} a_n.$$

The coefficient vanishes at $n = 2$, so the analytic solution is $\widetilde{\mathcal{N}} = \alpha + \beta y^2$. If $\alpha \neq 0$, the volume and reciprocal equations imply

$$\frac{d}{dx} \left(\frac{y}{\sqrt{\mathcal{N}}} \right) = (J/k)\alpha, \quad \alpha\mathcal{N} = 1 - \beta((J/k)\alpha x + q_0)^2.$$

This makes $\mathcal{N}$ polynomial, contradicting its nonzero logarithmic mode. If $\alpha = 0$, positivity gives $\beta > 0$ and fixed nonzero sign of y , and $\mathcal{N}' = -2(J/k)\sqrt{\beta} \operatorname{sgn} y \neq 0$. A nonconstant affine $\mathcal{N}$ has nonconstant forward Legendre expression, again impossible. Thus $c_1 = 0$ and $\mathcal{N} = u + b_n x^2$.

If $b_n = 0$, integrate directly. If $b_n \neq 0$ and $u = 0$, the volume primitive makes $\widetilde{\mathcal{N}}$ a nonconstant affine function of y , contradicting the inverse equation. If $ub_n \neq 0$, integration gives

$$y = \frac{Jx}{ku\sqrt{\mathcal{N}}} + n, \quad \widetilde{\mathcal{N}} = \frac{1}{u} - \frac{ub_n k^2}{J^2} (y - n)^2.$$

The inverse Legendre expression has linear coefficient $8ub_n k^2/J^2$, so $n = 0$. $\square$

B.10 The phase and norm parameters

In the nonconstant case (B.35), the axial component in (4.5) is $V = \nu x$, $\nu = J/(ku) \neq 0$. In cylindrical source coordinates (r, ϕ, z) , the spherical radius is $\sqrt{r^2 + z^2}$ and $x = z/\sqrt{r^2 + z^2}$. Put $t = z/r$. Since

$$R^2 = (r^2 + z^2)U^2 = (r^2 + z^2)(\mathcal{N} - V^2) = ur^2 + (u + b_n - \nu^2)z^2,$$

put $a_n = u$ and $m = u + b_n - \nu^2$. The coefficient a_n is a norm parameter; the fixed principal value ratios remain a, b . The map takes the form

$$R = r\sqrt{Q}, \quad Z = \nu z, \quad \Phi = k\phi + \omega \log r + \psi_c(t), \quad Q = a_n + mt^2 > 0.$$

Here the cylindrical phase is

$$\psi_c(t) = \psi\left(\arccos \frac{t}{\sqrt{1+t^2}}\right) + \frac{\omega}{2} \log(1+t^2).$$

Its physical derivative is

$$F_f = \begin{pmatrix} a_n/\sqrt{Q} & 0 & mt/\sqrt{Q} \\ \sqrt{Q}(\omega - t\psi'_c) & k\sqrt{Q} & \sqrt{Q}\psi'_c \\ 0 & 0 & \nu \end{pmatrix}, \quad \det F_f = a_n k \nu \neq 0.$$

The azimuthal pressure equation is

$$(1+t^2)(Q\psi'_c)' - tQ\psi'_c = K_0, \\ Q\psi'_c = c_1\sqrt{1+t^2} + K_0 t.$$

Indeed $r(\operatorname{div} C_f)_\phi/k = 2a_n\omega - tQ\psi'_c + (1+t^2)(Q\psi'_c)'$ must be constant. Multiplying $\operatorname{tr} F_f^\top F_f - E = 0$ by Q gives

$$P(t) + 2c_1 t \{K_0(1+t^2) - \omega Q\} \sqrt{1+t^2} = 0, \quad (\text{B.36})$$

where

$$P(t) = a_n^2 + m^2 t^2 + (k^2 + \omega^2)Q^2 + (1+t^2)\{c_1^2(1+t^2) + K_0^2 t^2\} \\ - 2\omega K_0 t^2 Q + (\nu^2 - E)Q.$$

The polynomial and square-root coefficients vanish separately: otherwise $1+t^2$ would be a square in $\mathbb{R}(t)$, contrary to its odd irreducible-factor valuation. The fourth-degree coefficient of P is

$$k^2 m^2 + (K_0 - \omega m)^2 + c_1^2.$$

Reality and $k \neq 0$ force $m = K_0 = c_1 = 0$. Thus $\psi'_c = 0$ and now $Q = a_n > 0$ follows from the target radius condition.

For the constant norm case $\mathcal{N} = a_n > 0$, write $y = d + \ell x$, $\ell \neq 0$, $X_0 = 1 - x^2$, $W_0 = 1 - (d + \ell x)^2$. Let $h = X_0 W_0 \psi_x$. With an integration constant c_{phase} , the angular pressure equation gives

$$h = c_{\text{phase}} + \{\beta_{C_f}/(ka_n) - 3\omega(1-d^2)\}x + 3\omega d \ell x^2 + \omega \ell^2 x^3.$$

The constant trace equation, multiplied by $X_0 W_0$, is the polynomial identity

$$(E/a_n - 1)X_0 W_0 = \omega^2 X_0 W_0^2 + \ell^2 X_0^2 + k^2 W_0^2 + h^2. \quad (\text{B.37})$$

As an identity in $\mathbb{R}[x]$, (B.37) can be evaluated at $x = \pm 1$, giving $W_0(\pm 1) = h(\pm 1) = 0$. They yield $d = 0$, $\ell^2 = 1$, $c_{\text{phase}} = 0$, $\beta_{C_f}/(ka_n) = 2\omega$, and then $\psi_x = -\omega x/(1-x^2)$. This gives the cylindrical map described above.

Consequently the commuting dilation-rotation family is

$$f(r, \phi, z) = (\sqrt{a_n} r \cos \Theta, \sqrt{a_n} r \sin \Theta, \nu z + z_0), \quad \Theta = k\phi + \omega \log r + \theta_0, \quad (\text{B.38})$$

with $a_n > 0$, $k\nu \neq 0$, arbitrary real ω , and local angles. Its metric is

$$c = \begin{pmatrix} a_n(1+\omega^2) & a_n\omega k & 0 \\ a_n\omega k & a_n k^2 & 0 \\ 0 & 0 & \nu^2 \end{pmatrix}_{(e_r, e_\phi, e_z)}.$$

The axial vector is principal, and the other two principal vectors have constant angle with e_r . Their curls are axial, so all three coefficients h_i vanish. Conversely, for $e = \cos \alpha e_r + \sin \alpha e_\phi$ the projector divergence is the gradient of $\cos(2\alpha) \log r + \sin(2\alpha) \phi$; the axial projector has zero divergence. Hence the family is locally universal.

For prescribed positive, pairwise distinct eigenvalues λ_i of the displayed metric, choose $\nu^2 = \lambda_i$ and order the other two as $\lambda_j < \lambda_k$. The parameter restrictions are

$$a_n \in [\lambda_j, \lambda_k], \quad k^2 = \lambda_j \lambda_k / a_n^2, \quad \omega^2 = (\lambda_k - a_n)(a_n - \lambda_j) / a_n^2.$$

Determinant normalization imposes $a_n k \nu = 1$.

C Algebraic details for the case of maximal rank

We give the differential relations and local ideal identities used in Section 8, followed by the elimination on ratio curves. The coefficient fields are rational function fields or their finite algebraic extensions; the final coprimality argument uses a reduction to a finite field.

C.1 Coefficient rings and differential relations

Use the constants and rescaled ratios defined in Section 8.1.

Set

$$k = a\kappa_0, \quad m = 1 + k = \frac{a-1}{b-1}.$$

The constants κ_0, q, k, m are nonzero and $mq = (b/a)m^2 > 0$. Define the three affine fibre coordinates R_i by

$$\begin{aligned} \hat{\tau}_1 &= -\hat{p}_{12}(\hat{p}_{13} + \hat{p}_{31}) + \left(1 - \frac{\eta\zeta}{\eta - \zeta}\right) \hat{p}_{23} \\ &\quad - \frac{\eta\zeta}{\eta - \zeta} \hat{p}_{32} - \frac{\eta\zeta}{2(\eta - \zeta)} R_1, \\ \hat{\tau}_2 &= -\frac{2\eta}{\kappa_0(1 + \eta)} \hat{p}_{12} + \frac{2\eta^2}{\kappa_0(1 + \eta)} \hat{p}_{21} - \frac{\eta}{\kappa_0(1 + \eta)} R_3, \\ \hat{\tau}_3 &= \hat{p}_{21}(\hat{p}_{23} + \hat{p}_{32}) + \frac{\zeta - \eta + \eta\zeta}{\kappa_0(1 - \zeta)} \hat{p}_{13} \\ &\quad + \frac{\zeta}{\kappa_0(1 - \zeta)} \hat{p}_{31} + \frac{\zeta}{2\kappa_0(1 - \zeta)} R_2. \end{aligned} \tag{C.1}$$

The coefficient of each R_i is invertible in the stated localization. Thus the six $\hat{p}_{ij}$ with $i \neq j$ and three R_i , together with η, ζ , are obtained from the retained state X in (3.2) by an invertible rational coordinate change. Its inverse recovers $\hat{h}_2 = \eta/\kappa_0$ and $\hat{h}_3 = q\zeta/\kappa_0$, while (C.1) recovers the three retained derivatives. Explicitly, the forward change is

$$\begin{aligned} R_1 &= \frac{2(\eta - \zeta)}{\eta\zeta} \{-\hat{p}_{12}(\hat{p}_{13} + \hat{p}_{31}) + \hat{p}_{23} - \hat{\tau}_1\} - 2(\hat{p}_{23} + \hat{p}_{32}), \\ R_2 &= \frac{2\kappa_0(1 - \zeta)}{\zeta} \{\hat{\tau}_3 - \hat{p}_{21}(\hat{p}_{23} + \hat{p}_{32})\} - \frac{2(\zeta - \eta + \eta\zeta)}{\zeta} \hat{p}_{13} - 2\hat{p}_{31}, \\ R_3 &= -\frac{\kappa_0(1 + \eta)}{\eta} \hat{\tau}_2 - 2\hat{p}_{12} + 2\eta\hat{p}_{21}. \end{aligned}$$

It therefore preserves differential rank. Write $R = (R_1, R_2, R_3)$.

The affine frame coefficients are $\hat{p}_{ij} = p_{ij}/h_1$; ordinary homogeneous coefficients are written $\bar{p}_{ij}$, and their accompanying homogeneous vector is $\mathbf{R} = \tau R$. The original weighted boundary instead uses $\mathbf{v} = \tau^2 R$. The intermediate charts use the representative $\bar{\mathbf{R}}$ defined in

Appendix C.2. Support statements refer to the corresponding homogeneous representative of the affine vector. The localization in the principal value ratios requires

$$ab(a-1)(b-1)(a-b) \neq 0;$$

Sign arguments use the positivity of the principal values. The ratio localization contains

$$\eta\zeta(\eta-\zeta)(\eta+1)(\zeta-1) \neq 0.$$

Further localizations will be specified in the corresponding charts.

Let Φ be the rational map from $(x, \hat{h}_2, \hat{h}_3)$ to the full normalized state $\hat{\mathbf{S}}$, defined by (C.1), the cubic reconstruction and (B.4). If $\mathcal{E}_j$, $0 \leq j \leq 20$, denote the ordered weight-four compatibility polynomials of Appendix A, put $Q_j = \mathcal{E}_j \circ \Phi$. The pushforwards of the normalized state fields under the coordinate change are

$$\begin{aligned} \mathcal{V}_i f &= \sum_{j=1}^9 V_{ij} \partial_{x_j} f + B_{i1} \partial_{\hat{h}_2} f + B_{i2} \partial_{\hat{h}_3} f, \\ x &= (\hat{p}_{12}, \hat{p}_{13}, \hat{p}_{21}, \hat{p}_{23}, \hat{p}_{31}, \hat{p}_{32}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3). \end{aligned} \quad (\text{C.2})$$

Thus $V_{ij} = \mathcal{V}_i x_j$ and $B_{i1} = E_i \hat{h}_2$, $B_{i2} = E_i \hat{h}_3$, with the latter expressions pulled back by Φ .

Write F_{rs} for the part of bidegree (r, s) in $(\hat{p}, \mathbf{R})$ and $F^{[d]}$ for the ordinary homogeneous part of degree d . Let $(j_0, \dots, j_5) = (10, 11, 13, 15, 18, 19)$. For each ν , choose the scalar c_ν and the off-diagonal coordinate $\hat{p}_{\mu\nu}$ so that $c_\nu \hat{p}_{\mu\nu} Q_0$ has the same highest weighted part as Q_{j_ν} , and define

$$R_\nu = Q_{j_\nu} - c_\nu \hat{p}_{\mu\nu} Q_0, \quad 0 \leq \nu \leq 5.$$

The highest weighted parts determine this choice uniquely, and $\deg R_\nu \leq 4$. Set

$$\alpha = \frac{\zeta - \eta}{\zeta(\eta + 1)}, \quad \beta = \frac{\eta(\zeta - 1)}{\zeta(\eta + 1)}, \quad L = \begin{pmatrix} \hat{p}_{12} & 0 & -\alpha \hat{p}_{32} \\ \hat{p}_{13} & -\alpha \hat{p}_{23}/\beta & 0 \\ 0 & \hat{p}_{21} & -\beta \hat{p}_{31} \end{pmatrix}.$$

Then

$$\det L = \alpha(\hat{p}_{12}\hat{p}_{23}\hat{p}_{31} - \hat{p}_{13}\hat{p}_{21}\hat{p}_{32}).$$

The factorization of the degree-four relations is determined by

$$D = \begin{pmatrix} \hat{p}_{31} & 0 & 0 \\ -\hat{p}_{32} & 0 & 0 \\ 0 & \hat{p}_{21} & 0 \\ 0 & 0 & \hat{p}_{12} \\ 0 & 0 & \hat{p}_{13} \\ 0 & -\hat{p}_{23} & 0 \end{pmatrix}, \quad \mathbf{B}_{\nu+1,i} = \partial_{\mathbf{R}_i}(R_\nu)_{31}, \quad \mathbf{B}_L = \frac{\mathbf{B} \operatorname{adj} L}{\det L}.$$

All entries of $\mathbf{B}_L$ are polynomials. Define

$$C = \begin{pmatrix} (\mathbf{B}_L)_{1,*}/\hat{p}_{31} \\ (\mathbf{B}_L)_{3,*}/\hat{p}_{21} \\ (\mathbf{B}_L)_{4,*}/\hat{p}_{12} \end{pmatrix}, \quad o = \begin{pmatrix} (R_0)_{40}/\hat{p}_{31} \\ (R_2)_{40}/\hat{p}_{21} \\ (R_3)_{40}/\hat{p}_{12} \end{pmatrix}.$$

These quotients are polynomials and satisfy

$$\mathbf{B}_L = DC, \quad ((R_\nu)_{40})_{\nu=0}^5 = Do, \quad (R_\nu^{[4]})_{\nu=0}^5 = D(CLR + o).$$

There is a unique coefficient g_i for which $(\mathcal{V}_i \mathbf{R}_i)_{12} = g_i(Q_0)_{12}$. Define

$$\hat{E}_i = \mathcal{V}_i - g_i Q_0 \partial_{\mathbf{R}_i}, \quad (c_1^{\operatorname{diag}}, c_2^{\operatorname{diag}}, c_3^{\operatorname{diag}}) = \left(\frac{2(\eta - \zeta)}{\eta\zeta}, -\frac{2(\zeta - 1)}{\zeta}, \frac{\kappa_0(\eta + 1)}{\eta q} \right).$$

If $\sigma = (3, 2, 1)$, the diagonal cubic parts are

$$(\widehat{E}_i \mathbf{R}_i)^{[3]} = c_i^{\text{diag}} (CLR + o)_{\sigma(i)}.$$

For each row j of C , put

$$\begin{aligned} P_{1j,*} &= (0, 0, 0, C_{j1}, C_{j2}, 0), \\ P_{2j,*} &= (0, 0, C_{j3}, 0, 0, \alpha C_{j2}/\beta), \\ P_{3j,*} &= (-\beta C_{j3}, \alpha C_{j1}, 0, 0, 0, 0), \end{aligned}$$

and define $K_{i\nu\ell} = \sum_{j=1}^3 D_{\nu+1,j} P_{ij,\ell+1}$. The eighteen derivative relations are

$$H_{6(i-1)+\nu} = \widehat{E}_i R_\nu - c_i^{\text{diag}} \sum_{\ell=0}^5 K_{i\nu\ell} R_\ell, \quad i = 1, 2, 3, \quad \nu = 0, \dots, 5.$$

The identities $\sum_{\ell} P_{ij,\ell+1} D_{\ell+1,\sigma(i)} = (CL)_{ji}$ cancel their degree-six parts, so $\deg H_n \leq 5$.

For $Q \in \{Q_1, Q_5, Q_6\}$, write

$$\partial_{\mathbf{R}_i} Q^{[4]} = \sum_k \widehat{p}_{\nu(i,k)} A_{i,k}, \quad Z_i = (c_i^{\text{diag}})^{-1} (\widehat{E}_i \mathbf{R}_i)^{[3]}.$$

The relation $R_{j(i,k)}^{[4]} = \epsilon_{i,k} \widehat{p}_{\nu(i,k)} Z_i$ shows that the nine polynomials

$$\widehat{E}_i Q - c_i^{\text{diag}} \sum_k \epsilon_{i,k} A_{i,k} R_{j(i,k)}$$

have degree at most five. The ordered triples $(\nu(i, k), j(i, k), \epsilon_{i,k})$ are

$$((12, 3, 1), (13, 4, 1)), \quad ((21, 2, 1), (23, 5, -1)), \quad ((31, 0, 1), (32, 1, -1)).$$

In each monomial of the derivative, divide by the first coordinate in its pair if that coordinate occurs, and otherwise by the second. This determines $A_{i,k}$ uniquely. Write $G_{i,j}^{\text{even}}$ for the resulting corrected derivative of Q_j , $j \in \{1, 5, 6\}$. The ordered list of thirty-seven relations is

$$\mathcal{F} = (Q_0, Q_1, Q_5, Q_6, R_0, \dots, R_5, H_0, \dots, H_{17}, (G_{i,j}^{\text{even}})_{i=1,2,3; j=1,5,6}),$$

with i ordered first and j second. Append the three pair relations

$$R_6 = \widehat{p}_{32} R_0 + \widehat{p}_{31} R_1, \quad R_7 = \widehat{p}_{23} R_2 + \widehat{p}_{21} R_5, \quad R_8 = \widehat{p}_{13} R_3 - \widehat{p}_{12} R_4.$$

The degree-five parts cancel by the sparse form of D , so $\deg R_i \leq 4$ for $i = 6, 7, 8$. They belong to the compatibility ideal. The ordered forty-row list is $(\mathcal{F}, R_6, R_7, R_8)$.

Put $\ell = LR$, with components numbered 1, 2, 3, and set $\iota(1) = 3$, $\iota(5) = 1$, $\iota(6) = 2$. The homogeneous parts satisfy

$$(H_n)_{32} = \sum_{j=1}^3 A_{n,j} \ell_j, \quad (Q_i)_{22} = c_i^{\text{even}} \ell_{\iota(i)}^2, \quad (Q_i)_{31} = \sum_{j=1}^3 B_{i,j} \ell_j.$$

On the critical chart with scalar coordinates s_d, t, B, C ,

$$\bar{p} = (1, tB/(s_d C), s_d, t, B, C), \quad \bar{\mathbf{R}} = (\alpha C, \beta B/s_d, 1),$$

the boundary restrictions at $\kappa = 0$ of the regularized derivative relations are $\sum_{j=1}^3 A_{n,j} e_{j-1}^{\text{alg}} + (H_n)_{41} + \delta(H_n)_{13}$; the even restrictions are $c_i^{\text{even}} (e_{\iota(i)-1}^{\text{alg}})^2 + \sum_{j=1}^3 B_{i,j} e_{j-1}^{\text{alg}} + (Q_i)_{40}$. Here every coefficient polynomial is evaluated on the displayed chart. The coefficient matrices are obtained by substituting into the compatibility polynomials and extracting their normal coefficients. When one $\mathbf{R}_i$ vanishes, the matrix of linear normal terms has columns for the

two active incidence variables, the scale, the critical parameter and the transverse incidence variable. Its scale column also contains $(H_n)_{50}$.

On $\hat{p}_{31} = \hat{p}_{32} = 0$, the leading transverse field is $Z_0 \partial_{R_3}$ up to a unit, with Z_0 independent of R_3 . Its successive actions are therefore $Z_0^j \partial_{R_3}^j$. The corresponding incidence relations are used on each of the fifteen nonempty supports.

For the canonical critical amplitudes use the coefficient algebra

$$K[b_0, c_0]/(b_0^2 + q, c_0^2 + q/\kappa_0), \quad K = \mathbb{Q}(a, \kappa_0, \eta, \zeta, s_d, z_d),$$

where $s_d = \bar{p}_{21}$ and $z_d = \bar{p}_{23}/\bar{p}_{32}$ are direction parameters. It has basis $1, b_0, c_0, b_0 c_0$. Both b_0 and c_0 are units, and the coefficient matrices are regarded as matrices over this algebra.

For implicit elimination let C be the Jacobian of the equations being solved, $d = \det C$, and R_j their coefficient in the normal direction j . The coefficient of a remaining equation F after elimination, multiplied by d , is

$$\hat{F}_j = dF_j - F_e \operatorname{adj}(C)R_j.$$

If L is the transverse line coordinate, eliminating it from the two remaining equations gives the coefficients

$$\hat{Q}_L \hat{H}_j - \hat{H}_L \hat{Q}_j, \quad j \in \{\kappa, \delta\}.$$

The nonvanishing on the six critical directions reduces to the following inequalities. Let $\omega^2 - \frac{13}{2}\omega + 1 = 0$. Both roots lie in $(0, 13/2)$, so

$$c = \frac{232135214 - 27335384\omega}{4404177} > 0, \quad d = \frac{63263572 + 27335384\omega}{4404177} > 0.$$

The squared direction parameter is respectively

$$X = \kappa_0, \quad X = -q, \quad X = -q/\kappa_0.$$

In the first two charts the determinant factor is $v^2 + cv + 1$, where respectively $v = a\kappa_0^2 > 0$ and $v = mq > 0$. In the third it is

$$(a - X)^2(X + 1)^2 - daX(a - X - 1) = (a - X)^2(X + 1)^2 + \frac{damq}{\kappa_0^2} > 0,$$

since $mq = b(a - 1)^2/[a(b - 1)^2] > 0$. On $b = a^2$, the two remaining even quadrics have nonzero Sylvester resultant, so that this hypersurface is excluded as well.

C.2 The relative projective boundary

We construct a proper modification of the weighted projective closure of the affine state fibres over a ratio base open set $\mathcal{U}$ containing $(\eta, \zeta) = (1, -1)$. The following criterion relates local exclusion or containment over this fibre to finiteness of the generic affine fibre. Here τ is the homogenizing coordinate, so the affine state locus is $\tau \neq 0$.

Lemma C.1 (A relative boundary criterion). *Let $\mathcal{B}$ be integral, let $Y \rightarrow \mathcal{B}$ be a dominant affine morphism of finite type, and let $\bar{Y} \rightarrow \mathcal{B}$ be a proper closure of Y . Let $\pi : \tilde{Y} \rightarrow \bar{Y}$ be a proper modification of the whole closure which is an isomorphism over Y , and fix $b_* \in \mathcal{B}$. Write $\partial\tilde{Y} = \tilde{Y} \setminus Y$. Suppose that every irreducible component C of $\partial\tilde{Y}$ dominating $\mathcal{B}$ has the following property at each of its points over b_* : either an identity in the local compatibility ideal rules out that point on the saturated closure, or a neighbourhood of that point in C lies in a specified relatively closed boundary family F which contains no component of $\partial\tilde{Y}$ dominating $\mathcal{B}$. Then the generic fibre Y_{gen} is finite.*

Proof. If the generic projective boundary were nonempty, one of its closures would dominate $\mathcal{B}$. Proper surjectivity of π gives an irreducible component of its inverse image dominating that closure. Enlarge it to a component C of $\partial\tilde{Y}$; this component still dominates $\mathcal{B}$. Since $C \rightarrow \mathcal{B}$ is proper, its image is closed and contains the generic point of $\mathcal{B}$, so C meets the fibre over b_* . The first alternative in the hypothesis is impossible. In the second, a nonempty open subset of the irreducible C lies in F , so its generic point lies in F , contradicting the stated generic exclusion. The generic boundary is therefore empty. The generic fibre equals its proper closure and is affine; an affine proper scheme of finite type over a field is finite [27, Lemma 29.45.11, Tag 01WN]. $\square$

Remark C.2 (Formal arcs and saturation). On each boundary chart, let I be generated by the specified original compatibility polynomials after substitution and exact division by the common exceptional monomials. These monomials are invertible on the affine locus, so the divided relations belong to the local ideal of the closure of the state image. That ideal is the contraction of its localization at τ and is therefore τ -saturated. The invariance and formal transfer arguments of Lemmas C.3 and C.4 apply to this saturated ideal. Thus $\tau \in \sqrt{I}$ implies $I : \tau^\infty = (1)$ and excludes the point from the closure. More generally, the absence of all nonconstant formal arcs satisfying the compatibility relations through that point with $\tau \neq 0$ gives the same set-theoretic exclusion: an irreducible component through the point not contained in $\tau = 0$ contains an algebraic curve through the point meeting $\tau \neq 0$; normalizing and completing that curve produces such an arc.

We use two facts about completed local rings. Initial forms whose common zero is the origin exclude nonconstant formal arcs when the remaining parameter belongs to the radical of the compatibility ideal. Implicit elimination preserves quadratic normal order when its Schur derivative vanishes identically in all base parameters. The precise statements and proofs are Lemmas C.6 and C.7 in Appendix C.5.

The quadratic dependence of the retained derivatives on the first state coordinates leads to the following weighted compactification.

Write $\hat{p} = (\hat{p}_{12}, \hat{p}_{13}, \hat{p}_{21}, \hat{p}_{23}, \hat{p}_{31}, \hat{p}_{32})$. Give the homogenizing coordinate τ weight one, each of the six entries of $\hat{p}$ weight one, and each of the three entries of $\mathbf{R}$ weight two. The homogeneous representatives of the affine fibre coordinates are

$$\bar{p} = \tau \hat{p}, \quad \mathbf{R} = \tau \mathbf{R}, \quad \mathbf{v} = \tau^2 \mathbf{R} = \tau \mathbf{R}.$$

Let W be this relative weighted projective space over $\mathcal{U}$, and let $\bar{Y} \subset W$ be the τ -saturated closure of Y . The second Veronese grading makes W projective over $\mathcal{U}$. We construct a modification of $\bar{Y}$ by two blowups of W . The first centre has ideal sheaf

$$\mathcal{J}_1 = (\tau, \widetilde{v_1, v_2, v_3}).$$

It is contained in $\tau = 0, \mathbf{v} = 0$, where some $\bar{p}_j$ is nonzero. On the chart $\bar{p}_j = 1$ this is the ordinary ideal (τ, v_1, v_2, v_3) . If a different entry of $\bar{p}$ is normalized, these coordinates change by $\tau' = \tau/c$ and $\mathbf{v}' = \mathbf{v}/c^2$ with c a unit, so the ideals agree on the overlap. On each chart $v_j \neq 0$ the ideal is the unit ideal.

Write $W_1 = \text{Bl}_{\mathcal{J}_1} W$, let E_1 be its exceptional divisor, and let D_1 be the strict transform of $\tau = 0$. The second centre is the scheme intersection $E_1 \cap D_1$, with ideal $\mathcal{J}_2 = \mathcal{I}_{E_1} + \mathcal{I}_{D_1}$, and put $W_2 = \text{Bl}_{\mathcal{J}_2} W_1$. The first blowup has a τ chart $\mathbf{v} = \tau \mathbf{R}$, on which D_1 is empty. Its other charts, indexed by an entry v_j , are

$$\mathbf{v} = \sigma \bar{\mathbf{R}}, \quad \tau = \sigma \kappa, \quad \bar{\mathbf{R}}_j = 1.$$

Here $E_1 = (\sigma = 0)$, $D_1 = (\kappa = 0)$ and $\mathcal{J}_2 = (\sigma, \kappa)$. Thus the second centre does not meet the ordinary τ chart. Its κ chart has $\sigma = \kappa \delta$; its σ chart has $\kappa = \sigma \beta_T$. The composite

projections to W are consequently

$$\begin{aligned} W_O : \quad \mathbf{v} &= \tau \mathbf{R}, \\ W_S : \quad \mathbf{v} &= \kappa \delta \overline{\mathbf{R}}, \quad \tau = \kappa^2 \delta, \\ W_T : \quad \mathbf{v} &= \sigma \overline{\mathbf{R}}, \quad \tau = \sigma^2 \beta_T. \end{aligned}$$

In W_O the remaining coordinates are $\bar{p}$, τ and $\mathbf{R}$. In W_S they are $\bar{p}$, κ , δ , $\overline{\mathbf{R}}$, and in W_T they are $\bar{p}$, σ , β_T , $\overline{\mathbf{R}}$; a selected entry is normalized to $\overline{\mathbf{R}}_j = 1$ in the last two charts. Their transition is

$$\sigma = \kappa \delta, \quad \kappa = \sigma \beta_T, \quad \delta \beta_T = 1.$$

The overlap of W_O and W_S is $R_j \neq 0$, or equivalently $\kappa \neq 0$, with

$$\kappa = R_j^{-1}, \quad \delta = \tau R_j^2, \quad \overline{\mathbf{R}} = \mathbf{R}/R_j.$$

Changing the selected entry from j to k , with $r = \overline{\mathbf{R}}_k$ a unit, changes the first blowup coordinates by

$$(\sigma, \kappa, \overline{\mathbf{R}}) \mapsto (\sigma r, \kappa/r, \overline{\mathbf{R}}/r).$$

This preserves (σ, κ) and induces the same second blowup on both charts. These transition maps and the changes of $\bar{p}$ normalization glue the displayed charts.

Let Y_1 and $\tilde{Y}$ be the successive strict transforms of $\bar{Y}$, that is, the closures of Y in W_1 and W_2 . Their maps are the blowups in the restricted centre ideals. In particular,

$$\tilde{Y} \longrightarrow Y_1 \longrightarrow \bar{Y} \longrightarrow \mathcal{U}$$

is a sequence of projective morphisms, and $\pi : \tilde{Y} \rightarrow \bar{Y}$ is a proper surjective modification which is an isomorphism over Y . Surjectivity follows because its closed image contains the dense affine locus Y . On the remaining weighted charts $v_j \neq 0$ both blowups are the identity. These charts include every point with $\bar{p} = 0$ at infinity. They have the standard finite quotient presentation $v_j = 1$, with the residual action $(\tau, \bar{p}) \mapsto (-\tau, -\bar{p})$ fixing the other v coordinates. On $\bar{p}_i \neq 0$, the normalization with $c = \bar{p}_i$ gives the overlap with the preceding charts. Together with the charts above, they cover the whole modification.

On a chart with a nonzero $\bar{p}$ entry, if $n = \text{ord } \tau > 0$ and $s_0 = \min_j \text{ord}(\mathbf{v})_j$, then W_O covers $s_0 \geq n$, W_S covers $n/2 \leq s_0 \leq n$, and W_T covers $0 \leq s_0 \leq n/2$. Indeed

$$\text{ord } \kappa = n - s_0, \quad \text{ord } \delta = 2s_0 - n, \quad \text{ord } \beta_T = n - 2s_0.$$

Every curve with generic point in Y lifts to $\tilde{Y}$ by properness. The displayed nonnegative orders describe the charts containing its lift. If every v_j vanishes identically, the lift lies in W_O with $\mathbf{R} = 0$. Otherwise a coordinate of minimal order supplies the selected entry $\overline{\mathbf{R}}_j = 1$. The critical rate $s_0 = n/2$ lies in the unit overlap of W_S and W_T . The partition is given in Appendix C.7.

For the local ideal calculations we also use the finite algebra generated by the auxiliary coordinates of Appendix C.4. On a normalized chart, including the finite presentations above, let A be the coordinate ring of an irreducible component of the closure of the affine locus, and set in its function field

$$\mathbf{e}^{\text{alg}} = (e_0^{\text{alg}}, e_1^{\text{alg}}, e_2^{\text{alg}})^{\top} = L(\bar{p})\mathbf{v}/\tau.$$

The three even relations give monic equations for $(e_0^{\text{alg}})^2, (e_1^{\text{alg}})^2, (e_2^{\text{alg}})^2$ whose remaining terms are linear in $\mathbf{e}^{\text{alg}}$ and have coefficients in A . Thus $A[e_0^{\text{alg}}, e_1^{\text{alg}}, e_2^{\text{alg}}]$ is generated as an A module by the eight squarefree monomials in $\mathbf{e}^{\text{alg}}$. Its spectrum is the closure of this rational graph, finite and surjective over the chart. Taking roots of the specified units gives further finite covers. The compatibility ideal pulls back to each of these covers; the

exceptional monomials used for exact division are units on Y . Any formal arc whose generic point lies in Y lifts to a finite cover after a finite extension of its complete discrete valuation ring: choose a point above its generic point and use the valuative criterion for the finite morphism. Orders are multiplied by the ramification index, so the valuation inequalities are unchanged. Consequently a local identity forcing τ into the radical on every chart above a point excludes every such arc through that point of $\tilde{Y}$. For containment in an ordinary boundary family, the closed image of the complement of the neighbourhoods above the point permits shrinking below; finite surjectivity then gives local containment downstairs. These give the local alternatives for the proper morphism π in Lemma C.1. The derivations below are restrictions of the state derivations to the state closure.

C.3 The induced derivations at infinity

The state equations constrain how a boundary component varies with η, ζ . We need their derivations to preserve each irreducible component of the boundary, so that their restrictions are well defined. When two restricted derivations have independent ratio components, that component cannot project into a proper algebraic subset of the ratio plane. This is the differential part of the boundary argument.

Lemma C.3 (Homogeneous derivations and boundary components). *Let $A = \mathbb{C}[\eta, \zeta, g^{-1}]$ and let I be an ideal in a relative state algebra over A . Suppose that I is preserved by derivations $\mathcal{V}_i$. On a selected state coordinate chart, a derivation congruent modulo the ideal is a derivation*

$$D_i = \mathcal{V}_i + \sum_{\alpha} f_{i\alpha} W_{i\alpha}, \quad f_{i\alpha} \in I,$$

where the coefficients of the $W_{i\alpha}$ are regular on that chart. It preserves the localized ideal I .

Choose positive integer fibre weights, and suppose that each D_i increases weighted degree by at most d_i^{der} . Its homogeneous regularization, followed by projective normalization on a chart where the chosen fibre coordinate is nonzero, preserves the saturated closure of I and has logarithmic derivative of the homogenizing coordinate τ . Consequently it preserves the radical boundary ideal and every minimal prime of that ideal. A finite root cover needed to normalize a coordinate of weight greater than one is allowed here: its root is a unit, so this cover is étale.

Let P be such a minimal prime. If two induced boundary derivations δ_1, δ_2 satisfy

$$\det \begin{pmatrix} \delta_1 \eta & \delta_1 \zeta \\ \delta_2 \eta & \delta_2 \zeta \end{pmatrix} \neq 0 \quad \text{in } \text{Frac}(B/P),$$

where B is the boundary chart algebra, then $P \cap A = (0)$. In particular this boundary component dominates the ratio base. If the determinant is nonzero at a closed boundary point, the conclusion holds for every irreducible component through that point.

Proof. For $f \in I$, the gauge correction applied to f is a sum of multiples of elements of I . Thus $D_i I \subset I$ after localization. For a weighted homogeneous coordinate $x_j^h = \tau^{w_j} x_j$, the regularization has component

$$\tilde{D}_i(x_j^h) = \tau^{d_i^{\text{der}} + w_j} D_i x_j, \quad \tilde{D}_i(\tau) = 0.$$

The degree bound makes every exponent of τ nonnegative; the same bound applies to the base coordinate components, of weight zero. This gives a regular derivation before projective normalization. It preserves the homogeneous closure because that ideal is the contraction of its localization at τ . Projective normalization rescales by a unit and adds a regular multiple of the Euler derivation, which preserves every homogeneous ideal and sends τ to a multiple of τ . If a root of the selected nonzero homogeneous coordinate is used, its derivative is

obtained by differentiating its defining power; the root and its integer exponent are units. This proves regularity on that etale chart as well.

In a characteristic zero ring a derivation preserves the nilradical: if $a^n = 0$, differentiating n times shows $n!(\delta a)^n \in (a)$, and hence δa is nilpotent. Thus the radical of a differential ideal is differential. A derivation also preserves each minimal prime of a reduced ring. Indeed, localization at a minimal prime is a field, and differentiation of $a/1 = 0$ shows that δa lies in that minimal prime.

Set $Q = P \cap A$. For $f \in Q$, stability of P gives

$$(\delta_i \eta) f_\eta + (\delta_i \zeta) f_\zeta = 0 \quad \text{in } \text{Frac}(B/P), \quad i = 1, 2.$$

Inverting the displayed determinant in that field yields $f_\eta, f_\zeta \in Q$. If $Q \neq 0$, clearing a denominator gives a nonzero polynomial in $Q \cap \mathbb{C}[\eta, \zeta]$ of minimal total degree. Its partial derivatives also belong to Q . Characteristic zero and minimality therefore force that polynomial to be a nonzero constant, contradicting propriety of Q . Hence $Q = 0$. $\square$

First integrals on the intermediate boundary. Fix the principal value ratios and work with $1 < \gamma < 2$, full derivative support and at least two nonzero amplitude pairs. After a cyclic relabeling normalize $\bar{p}_{12} = \bar{\mathbf{R}}_3 = 1$. The incidence relations give

$$\bar{p} = (1, b_0 z_d, s_d, c_0 z_d, s_d b_0, c_0), \quad \bar{\mathbf{R}} = (\alpha c_0, \beta b_0, 1), \quad b_0 c_0 \neq 0, \quad (s_d, z_d) \neq (0, 0).$$

Here $c_0 = \bar{p}_{32}$, $s_d = \bar{p}_{21}$ and $z_d = \bar{p}_{23}/c_0$; the coefficients α, β are those of Appendix C.1. Denote by D_1, D_2 the boundary derivations induced by the first and third coordinate gauges, respectively, as in Lemma C.3. The terms of bidegrees $(1, 1)$ and $(0, 1)$ in the gauged state equations give the principal frame and ratio components, respectively. On the weighted Rees chart divide these derivations by the selected nonzero homogeneous derivative coordinate. Substitution of the displayed incidence coordinates in the frame equations makes the principal frame components multiples of $\bar{p}$. Projective normalization therefore gives

$$D_i \bar{p}_{jk} = 0 \quad (j \neq k), \quad D_i s_d = D_i z_d = D_i c_0 = 0, \quad i = 1, 2.$$

On $s_d \neq 0$, recover $b_0 = \bar{p}_{31}/s_d$; on $z_d \neq 0$, recover $b_0 = \bar{p}_{13}/z_d$. Thus $D_i b_0 = 0$ on both charts. The ratio components are

$$\begin{pmatrix} D_1 \eta & D_1 \zeta \\ D_2 \eta & D_2 \zeta \end{pmatrix} = \begin{pmatrix} \frac{\eta c_0}{\eta + 1} & \frac{\eta c_0}{\eta + 1} \\ -\eta & -\frac{\eta \zeta}{\eta + 1} \end{pmatrix}.$$

The determinant is

$$\frac{\eta^2(\eta - \zeta + 1)c_0}{(\eta + 1)^2},$$

which equals $3c_0/4$ at $(\eta, \zeta) = (1, -1)$. Shrink $\mathcal{U}$ so that this determinant is a unit.

On an irreducible invariant boundary component in this family, s_d, z_d, b_0, c_0 belong to the common constant field of the induced derivations on its function field. The ratios η, ζ are algebraically independent over that constant field. Indeed, choose a nonzero polynomial relation of least total degree, if one exists. Applying D_1 and D_2 and inverting the displayed matrix shows that both partial derivatives also vanish on the component. Minimality forces these partial derivatives to be zero polynomials. In characteristic zero the original polynomial is then a nonzero constant, a contradiction. Consequently every coefficient of a polynomial relation in η, ζ over this constant field is zero.

Lemma C.4 (Transfer from a transverse fibre). *Let $B = \mathbb{C}[[b_1, b_2, x_1, \dots, x_N]]$, let I be an ideal of B , and let $\tau \in B$. Suppose that two regular derivations D_1, D_2 preserve I and satisfy $D_j \tau \in (\tau)$. If*

$$\det(D_j b_i)_{i,j=1,2}(0) \neq 0,$$

put $I_0 = (I + (b_1, b_2))/(b_1, b_2)$ and denote the image of τ by τ_0 . Then

$$\tau_0 \in \sqrt{I_0} \implies \tau \in \sqrt{I}.$$

Proof. Apply the two formal flows successively to the transverse fibre $b_1 = b_2 = 0$. The determinant assumption and the formal inverse function theorem make this a change of coordinates from $\mathbb{C}[[t_1, t_2, x_1, \dots, x_N]]$ to B . Choose a finite set of generators f of I . Since $D_j f = A_j f$ for matrices A_j over B , each flow transforms f by an invertible matrix with identity constant term. The pullback of I is therefore $I_0 \mathbb{C}[[t_1, t_2, x_1, \dots, x_N]]$. The equations $D_j \tau = h_j \tau$ likewise show that the pullback of τ is a unit times τ_0 . A power of τ_0 belongs to I_0 , and the same is consequently true of τ and I . $\square$

The paired chart. In the paired chart of Appendix C.8, normalize $\bar{p}_{12} = \bar{\mathbf{R}}_3 = 1$ and write $\bar{p}_{32} = \ell \neq 0$, $\tau = \kappa^2 \delta$ and $\sigma = \kappa \delta$. The coordinate gauges for the first and third state derivations, regularized by $\tau^2/\sigma = \kappa^3 \delta$, have regular lifts to this chart. Indeed, differentiation of the three equations defining the normal coordinates gives numerators divisible by κ ; the remaining denominators are units near the paired points. If A_j is the component of the regularized field along $\bar{p}_{12}$ before projective normalization, the lift satisfies $D_j \tau = -A_j \tau$. On the paired boundary family their ratio components are

$$\begin{pmatrix} D_1 \eta & D_1 \zeta \\ D_2 \eta & D_2 \zeta \end{pmatrix} = \begin{pmatrix} \frac{\eta \ell}{\eta + 1} & \frac{\eta \ell}{\eta + 1} \\ -\eta & -\frac{\eta \zeta}{\eta + 1} \end{pmatrix}.$$

Its determinant is $\eta^2(\eta - \zeta + 1)\ell/(\eta + 1)^2$, and equals $3\ell/4$ at $(\eta, \zeta) = (1, -1)$. These lifts preserve the vanishing ideal of the state image by contraction from $\kappa \delta \neq 0$. Lemma C.4 thus transfers a local inclusion of τ in the radical of that ideal on the fibre over $(\eta, \zeta) = (1, -1)$ to the completed local ring with both ratios variable.

For the coordinate derivations used here, the required degree bound holds for every rational weight $1 \leq \gamma \leq 2$. At intermediate weights, two available derivations have independent base components when the corresponding leading $\mathbf{R}$ coordinates are nonzero. The explicit degree bounds and minors are given in Appendix C.6. Apply Lemma C.3 to $\mathfrak{p}$ and its homogeneous closure on the weighted Rees model associated with the orders of a formal arc. The exceptional supports with fewer independent derivations are treated by local ideal identities.

C.4 Local ideals and formal arcs

Let A be a localized coefficient ring, and use the local compatibility ideal I of Appendix C.2.

Suppose $J = (g_1, \dots, g_e)$ is contained in the maximal ideal of a Noetherian local ring, the compatibility polynomials belong to J , and a selection satisfies $F = Ag$ with $\det A$ a unit. Adjugation gives $(F) = J$. More generally,

$$J^N \subset (F) + J^{N+1}$$

implies $J^N \subset (F)$ by Nakayama's lemma applied to $J^N/((F) \cap J^N)$. The coefficient ring includes the ratio and horizontal parameters throughout.

Implicit elimination with parameters uses the same principle. If u denotes coordinates to be eliminated and z residual normal coordinates, solve the chosen equations in the completed local ring to obtain $u = \varphi(z)$. Each residual polynomial is obtained by the substitution

$F(\varphi(z), z)$. Its leading term is the leading term of this substituted polynomial. All horizontal variables and principal value ratios remain in the coefficient ring. Faithful flatness of completion transfers ideal containments back to the local ring [27, Lemma 10.97.3, Tag 00MC].

For the plane parametrized by $\mathbf{v}$, the special four-point family is

$$\mathbf{v}_1^2 = -\frac{q(\zeta - \eta)^2}{\kappa_0 \zeta^2 (\eta + 1)^2}, \quad \mathbf{v}_2^2 = -\frac{q\eta^2 (\zeta - 1)^2}{\zeta^2 (\eta + 1)^2}, \quad \mathbf{v}_3 = 1.$$

The right sides are units on the inverted base. Hence this is a finite *etale* degree-four family. The local ideal of this family is reduced. If a projective scheme $\mathcal{X}$ contains a finite flat subscheme $\mathcal{X}_0$ and $\mathcal{X}_{b_*} = \mathcal{X}_{0,b_*}$ scheme-theoretically, properness and upper semicontinuity first make $\mathcal{X}$ finite near b_* [27, Lemma 37.44.2, Tag 02UP]. In $0 \rightarrow M \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}_0} \rightarrow 0$, flatness of $\mathcal{X}_0$ preserves the left injection after taking that fibre. Nakayama then gives $M = 0$ after shrinking. Thus equality on the fibre extends to equality near b_* .

The orders n, s_0 of Lemma 8.4 give

$$(\text{ord } \kappa, \text{ord } \delta) = (n - s_0, 2s_0 - n), \quad (\text{ord } \sigma, \text{ord } \beta_T) = (s_0, n - 2s_0).$$

The chart $\bar{p} = 0, \mathbf{v} \neq 0$ is an original weighted chart.

The finite cover defined by the auxiliary coordinates follows from the three even compatibility polynomials Q_1, Q_5, Q_6 . With $\ell = Lv$, their weighted homogenizations have exact expressions

$$Q_j^h = c_j^{\text{even}} \ell_{\iota(j)}^2 + \tau \sum_{k=1}^3 A_{jk} \ell_k + \tau^2 B_j, \quad \iota(1) = 3, \quad \iota(5) = 1, \quad \iota(6) = 2,$$

where

$$\begin{aligned} c_1^{\text{even}} &= \frac{2\zeta^2}{(\zeta - \eta)^2 (\zeta - 1)^2}, \\ c_5^{\text{even}} &= \frac{2\eta^2 \zeta^2}{(\zeta - \eta)^2 (\zeta - 1)^2}, \\ c_6^{\text{even}} &= \frac{2\eta^2 \zeta^2}{(\zeta - \eta)^2 (\eta + 1)^2}. \end{aligned}$$

The row $A_{j\bullet}$ contains both components $(Q_j)_{31}$ and $(Q_j)_{12}$, where the subscripts denote the degrees in the off-diagonal and remaining fibre coordinates, respectively. Write $(Q_j)_{31} = \mathbf{q}_j^\top \mathbf{v}$ and put

$$\begin{aligned} \mathbf{A}_j^{(p)} &= \frac{(\text{adj } L)^\top \mathbf{q}_j}{\det L}, \\ \mathbf{A}_j^{(v)} &= \left(\frac{[\bar{p}_{12}](Q_j)_{12}}{v_1}, \frac{[\bar{p}_{13}](Q_j)_{12}}{v_1}, \frac{[\bar{p}_{21}](Q_j)_{12}}{v_2} \right)^\top, \quad A_{j\bullet} = (\mathbf{A}_j^{(p)} + \mathbf{A}_j^{(v)})^\top. \end{aligned}$$

Here all the divisions are exact polynomial divisions, and

$$(Q_j)_{31} + (Q_j)_{12} = \sum_{k=1}^3 A_{jk} \ell_k, \quad B_j = \frac{Q_j^h - c_j^{\text{even}} \ell_{\iota(j)}^2 - \tau \sum_{k=1}^3 A_{jk} \ell_k}{\tau^2}$$

is the polynomial remainder. All three coefficients c_j^{even} are units in the stated localization. Setting $e_{k-1}^{\text{alg}} = \ell_k / \tau$, $1 \leq k \leq 3$, gives three monic quadratic equations, with remaining terms linear in the e_{k-1}^{alg} . Their algebra is generated as a module by the eight squarefree monomials in $e_0^{\text{alg}}, e_1^{\text{alg}}, e_2^{\text{alg}}$. This proves integrality of the auxiliary coordinates.

Order the principal values so that $1 < a < b$. Then

$$-1/a < \kappa_0 < 0, \quad q > 0, \quad m > 0, \quad aq + 1 > 0.$$

Let $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2$ be the cone amplitudes and put $\Lambda = \mathbf{t}_0 + \mathbf{t}_1 + \mathbf{t}_2$.

The normal matrix on the four-point cone. Put $f_0 = Q_0$ and $f_{\nu+1} = R_\nu$, $0 \leq \nu \leq 5$. Give the six off-diagonal coordinates weight one and the three coordinates R_i weight two. Form the ordered list

$$(f_j)_{j=0}^6, \quad (\hat{E}_i f_j)_{j=0,\dots,6; i=1,2,3}, \quad (\hat{E}_k \hat{E}_i f_j)_{j=0,\dots,6; i,k=1,2,3},$$

with the indices ordered from left to right. Denote its members by $F_0, \dots, F_{90}$, and put $d_n^{\text{cone}} = 5$ for $0 \leq n \leq 6$, $d_n^{\text{cone}} = 7$ for $7 \leq n \leq 27$, and $d_n^{\text{cone}} = 9$ otherwise. Define

$$F_n^h(\bar{p}, \mathbf{v}, \tau) = \tau^{d_n^{\text{cone}}} F_n(\bar{p}/\tau, \mathbf{v}/\tau^2).$$

The weighted degree bounds make these polynomials. Specialize their coefficients at $(\eta, \zeta) = (1, -1)$, after taking the indicated compositions of derivations.

Over $K = \mathbb{Q}(a, \kappa_0)$ choose units $s_1, s_2, c_{\text{cone}}$ satisfying

$$s_1^2 = -q/\kappa_0, \quad s_2^2 = -q, \quad c_{\text{cone}}^2 = s_1 s_2.$$

Use six normal coordinates $n_0, \dots, n_5$ and the formal functions

$$\mathbf{v}_1 = s_1(1 - n_4)^{1/2}, \quad \mathbf{v}_2 = s_2(1 - n_5)^{1/2}, \quad \mathbf{v}_3 = 1,$$

where both square roots have constant term one. Set

$$\begin{aligned} \bar{p} &= c_{\text{cone}}^{-1}(\mathbf{t}_0 + n_0, \mathbf{v}_2(\mathbf{t}_1 + n_1), \mathbf{t}_2 + n_2, \mathbf{v}_1 \mathbf{t}_1, \mathbf{v}_2 \mathbf{t}_2, \mathbf{v}_1 \mathbf{t}_0), \\ \tau &= c_{\text{cone}} n_3. \end{aligned}$$

The off-diagonal coordinates are ordered as $(12, 13, 21, 23, 31, 32)$. Consider the nonzero monomials $\tau^\ell \bar{p}^{(e_0, \dots, e_5)} \mathbf{v}^{(e_6, e_7, e_8)}$ in the constant and linear τ terms of F_n^h , so $\ell = 0$ or 1 . Put $d_{\hat{p}} = e_0 + \dots + e_5 + \ell$. Their common parity is

$$\begin{aligned} \epsilon_{n,1} &\equiv e_6 + e_3 + e_5 + \ell - (d_{\hat{p}} - 1)/2 \pmod{2}, \\ \epsilon_{n,2} &\equiv e_7 + e_1 + e_4 + \ell - (d_{\hat{p}} - 1)/2 \pmod{2}. \end{aligned}$$

with $\epsilon_{n,j} \in \{0, 1\}$. These monomials have odd $d_{\hat{p}}$, and the residues are independent of the monomial. For a zero row set both parities equal to zero. The normalized coefficient matrix is

$$(\mathbf{M}_{\text{cone}})_{n+1, \ell+1} = (\epsilon_n^{\text{cone}})^{-1} \partial_{n_\ell} \left\{ \frac{c_{\text{cone}} F_n^h(\bar{p}, \mathbf{v}, \tau)}{\mathbf{v}_1^{\epsilon_{n,1}} \mathbf{v}_2^{\epsilon_{n,2}}} \right\} \Big|_{n=0}, \quad 0 \leq n \leq 90, \quad 0 \leq \ell \leq 5.$$

Here $\epsilon_n^{\text{cone}} = 1$ for $n \geq 28$, and the first twenty-eight row units, in order, are

$$\begin{aligned} (\epsilon_0^{\text{cone}}, \dots, \epsilon_{27}^{\text{cone}}) &= (3/56, -3q/(16\kappa_0), -3q/(16\kappa_0), -\kappa_0/16, -3/32, 3/32, -3/32, \\ &\quad -3/448, 3\kappa_0/224, -57/448, 1/48, -3q/64, -1/16, -9q/(128\kappa_0), \\ &\quad -3q/(8\kappa_0), -1/16, \kappa_0/96, -23\kappa_0/384, 7\kappa_0/96, -43/768, -7/48, \\ &\quad 203/768, -43/768, 203/768, -7/48, 9/256, -\kappa_0/32, 3/16). \end{aligned}$$

The entries lie in $K[\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2]$. Thus the matrix has ninety-one rows of linear normal coefficients of the undifferentiated, once differentiated and twice differentiated relations.

On $\Lambda = 0$, elimination from the maximal minors leaves, apart from units, the two factors

$$\kappa_0^2 - \frac{1193}{27} \kappa_0 + \frac{292}{27}, \quad \kappa_0^2 + \kappa_0 - 1.$$

The first is positive and the second negative on $-1 < \kappa_0 < 0$. The coordinate axes and $b = a^2$ have a nonzero maximal minor.

The chart $t_2\Lambda \neq 0$. Put

$$\begin{aligned} S_0 &= \frac{3}{2}((M_{\text{cone}})_{3,5}(M_{\text{cone}})_{2,*} - (M_{\text{cone}})_{2,5}(M_{\text{cone}})_{3,*}), \\ S_1 &= \frac{\kappa_0}{q^2}((M_{\text{cone}})_{7,5}(M_{\text{cone}})_{4,*} - (M_{\text{cone}})_{4,5}(M_{\text{cone}})_{7,*}), \\ \mathbf{T}_{\text{cone}} &= \begin{pmatrix} (M_{\text{cone}})_{1,*} \\ S_0 \\ S_1 \\ (M_{\text{cone}})_{2,*} \\ (M_{\text{cone}})_{4,*} \end{pmatrix}. \end{aligned}$$

Let $(\mathbf{T}_{\text{cone}})_\perp$ consist of columns $(1, 2, 3, 5, 6)$ and let $(\mathbf{T}_{\text{cone}})_\tau$ be column 4. Their determinant satisfies

$$\det(\mathbf{T}_{\text{cone}})_\perp = \frac{2q}{3\kappa_0}\Lambda t_2^3.$$

There is consequently a unique vector $\mathbf{w}^{\text{cone}}$ with $w_4^{\text{cone}} = 1$ and $\mathbf{T}_{\text{cone}}\mathbf{w}^{\text{cone}} = 0$: its other five components are $-(\mathbf{T}_{\text{cone}})_\perp^{-1}(\mathbf{T}_{\text{cone}})_\tau$. Put $D_{\text{amp}} = t_2\Lambda$ and let $f_1^*, \dots, f_{84}^*$ be the nonzero polynomials in the ordered vector $D_{\text{amp}}\mathbf{M}_{\text{cone}}\mathbf{w}^{\text{cone}}$. Let $\mathbf{A}_{\text{coef}}$ be their coefficient matrix in the lexicographically ordered amplitude monomials, including the monomials of D_{amp} , and let $\mathbf{b}$ be the coefficient column of D_{amp} in that same order. Thus

$$(\mathbf{A}_{\text{coef}})_{\mu j} = [t^\mu]f_j^*, \quad \mathbf{b}_\mu = [t^\mu]D_{\text{amp}}.$$

There are 110 coefficient rows. In the ring

$$A = \mathbb{Q}[a, \kappa_0, (a\kappa_0qm(aq+1))^{-1}]$$

define the coefficient ideal

$$\mathfrak{d} = \{\delta \in A : \delta\mathbf{b} \in \text{im } \mathbf{A}_{\text{coef}}\} = \text{Ann}_A([\mathbf{b}] \in \text{coker } \mathbf{A}_{\text{coef}}).$$

Proposition C.5. *In this ring $\mathfrak{d} = (1)$. In particular, $D_{\text{amp}} = \sum_{j=1}^{84} n_j f_j^*$ for coefficients $n_j \in A$.*

Proof. Put $P = \mathbb{Q}[a, \kappa_0]$ and $K = \text{Frac } P$. For column j , let d_j be the monic least common denominator of its entries, g_j their monic polynomial gcd after multiplication by d_j , and c_j the positive rational gcd of the coefficients of $d_j(\mathbf{A}_{\text{coef}})_{*,j}/g_j$. Set

$$c_j^{\text{col}} = g_j c_j / d_j, \quad \mathbf{A}_{\text{coef}}^{\flat} = \mathbf{A}_{\text{coef}} \text{diag}((c_j^{\text{col}})^{-1}).$$

Both the numerator and denominator of every c_j^{col} are products of $a, \kappa_0, q, m, aq+1$ and nonzero rational constants. Thus all column factors are units in A .

The rank of $\mathbf{A}_{\text{coef}}^{\flat}$ is 67, and the complementary columns to

$$\mathcal{J}_{\text{free}} = (3, 22, 23, 24, 25, 26, 27, 28, 29, 30, 40, 43, 55, 61, 73, 76, 82)$$

are independent. Let $\mathcal{J}_{\text{dep}}$ be their increasing list, and let $\mathcal{I}_{\text{range}}$ be the lexicographically first list of 67 rows for which $\mathbf{C}_{\text{range}} = (\mathbf{A}_{\text{coef}}^{\flat})_{\mathcal{I}_{\text{range}}, \mathcal{J}_{\text{dep}}}$ is invertible. Define the vectors by

$$\begin{aligned} (\mathbf{x}_0)_{\mathcal{J}_{\text{free}}} &= 0, \quad (\mathbf{x}_0)_{\mathcal{J}_{\text{dep}}} = \mathbf{C}_{\text{range}}^{-1} \mathbf{b}_{\mathcal{I}_{\text{range}}}, \\ (\mathbf{h}_i)_{j\nu} &= \nu^i, \quad (\mathbf{h}_i)_{\mathcal{J}_{\text{dep}}} = -\mathbf{C}_{\text{range}}^{-1} (\mathbf{A}_{\text{coef}}^{\flat})_{\mathcal{I}_{\text{range}}, \mathcal{J}_{\text{free}}} (\nu^i)_{\nu=1}^{17}, \quad i = 1, 2. \end{aligned}$$

They satisfy $\mathbf{A}_{\text{coef}}^{\flat} \mathbf{x}_0 = \mathbf{b}$ and $\mathbf{A}_{\text{coef}}^{\flat} \mathbf{h}_i = 0$. The four rational solutions used below are

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_0 - \frac{(\mathbf{x}_0)_4}{(\mathbf{h}_1)_4} \mathbf{h}_1, \quad \mathbf{x}_2 = \mathbf{x}_0 - \frac{(\mathbf{x}_0)_4}{(\mathbf{h}_2)_4} \mathbf{h}_2, \\ \mathbf{x}_3 &= \mathbf{x}_0 - \frac{(\mathbf{x}_0)_{32}}{(\mathbf{h}_1)_{32}} \mathbf{h}_1. \end{aligned}$$

The three displayed denominators are nonzero. Let d_ν be the monic least common denominator of the entries of $\mathbf{x}_\nu$, and put $\mathbf{n}_\nu = d_\nu \mathbf{x}_\nu \in P^{84}$, $0 \leq \nu \leq 3$. Their polynomial identities, in all 110 coefficient rows, are

$$\mathbf{A}_{\text{coef}}^b \mathbf{n}_\nu = d_\nu \mathbf{b}, \quad d_\nu = a\kappa_0^2 q^4 m(aq + 1) D_\nu.$$

Here D_0 has bidegree $(34, 65)$ in (a, κ_0) , and D_1, D_2, D_3 have bidegree $(39, 73)$. These definitions fix both the four identities and their denominators from the coefficient matrix. Since the prefactor is a unit, each D_ν belongs to $\mathfrak{d}$.

For an independent variable v , let $G_\nu(v, \kappa_0)$ be the primitive integer polynomial proportional to

$$\kappa_0^{\epsilon_\nu} D_\nu(v/\kappa_0, \kappa_0), \quad (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) = (2, 4, 4, 4).$$

Their degrees in v are 34, 39, 39, 39. The three Sylvester determinants $R_\nu = \text{Res}_v(G_0, G_\nu)$ satisfy

$$\gcd(R_1, R_2, R_3) = c\kappa_0^{188} q^{218}, \quad c \in \mathbb{Q} \setminus \{0\}.$$

The Sylvester identities place R_ν in (G_0, G_ν) over $\mathbb{Q}[v, \kappa_0]$; substituting $v = a\kappa_0$ places them in $(D_0, D_\nu)A$. The univariate Bezout identity for their gcd therefore gives $1 \in (D_0, D_1, D_2, D_3)A \subset \mathfrak{d}$, as required. $\square$

It follows that the rank defect cannot occur on $D_{\text{amp}} \neq 0$.

On $\Lambda \neq 0$, $t_2 = 0$, and exactly one nonzero amplitude, the maximal minor ideal contains

$$-\frac{4q^4 t_0^2}{9\kappa_0^4}, \quad -\frac{4q^4 t_1^2}{9\kappa_0^3},$$

respectively. When both amplitudes are nonzero, its remaining zero locus is $t_0 t_1 = -1/c_{\text{hyp}}$, where

$$c_{\text{hyp}} = -\frac{4(a\kappa_0 + \kappa_0 + 2)}{a\kappa_0 q} > 0.$$

The hyperbola $t_2 = 0$. On $t_0 t_1 = -1/c_{\text{hyp}}$, substitute $t_0 = x$, $t_1 = -1/(c_{\text{hyp}} x)$ in $\mathbf{M}_{\text{cone}}$. Multiply each row by x to the negative of the lowest Laurent exponent appearing in that row, and denote the resulting polynomial matrix by $M^{\text{hyp}}(x)$. For a row set I write $\Delta_I = \det M_{I, \{1, \dots, 6\}}^{\text{hyp}}$. Define

$$\begin{aligned} g_0(x^2) &= x^{-2} \Delta_{(1,2,3,5,6,7)}, \\ g_1(x^2) &= x^{-2} \Delta_{(1,2,3,5,6,14)}, \\ g_2(x^2) &= x^{-2} \Delta_{(1,2,3,5,6,22)}. \end{aligned}$$

The right sides are polynomials in x^2 . The constants $b_0^{\text{hyp}}, b_1^{\text{hyp}}, b_2^{\text{hyp}}$ in the identity $\sum_j b_j g_j = 1$ are determined by coefficient comparison in these three polynomials. Their common denominator is

$$64aq^6(aq + 1)d_{12}(a, \kappa_0),$$

where

$$\begin{aligned} d_{12}(a, \kappa_0) &= a^4(297\kappa_0^8 + 1073\kappa_0^7 + 703\kappa_0^6 - 408\kappa_0^5 - 135\kappa_0^4) \\ &\quad + a^3(1073\kappa_0^7 + 3201\kappa_0^6 + 454\kappa_0^5 - 3204\kappa_0^4 - 1080\kappa_0^3) \\ &\quad + a^2(703\kappa_0^6 + 454\kappa_0^5 - 5648\kappa_0^4 - 9000\kappa_0^3 - 3240\kappa_0^2) \\ &\quad + a(-408\kappa_0^5 - 3204\kappa_0^4 - 9000\kappa_0^3 - 10608\kappa_0^2 - 4320\kappa_0) \\ &\quad - 135\kappa_0^4 - 1080\kappa_0^3 - 3240\kappa_0^2 - 4320\kappa_0 - 2160. \end{aligned}$$

For $1 < a < b$, use the positive parameters

$$u = \frac{b-a}{a(a-1)}, \quad v = \frac{b-a}{a}, \quad a = \frac{u+v}{u}, \quad \kappa_0 = -\frac{u}{1+u+v}, \quad b = a(1+v).$$

Under this substitution the numerator of d_{12} has forty-five strictly negative coefficients and its denominator has positive coefficients. Hence $d_{12} < 0$ and the displayed common denominator is nonzero.

The amplitude origin belongs to the four-point $\mathbf{v}$ family.

C.5 Two local criteria

C.5.1 Initial forms and formal arcs

Lemma C.6 (Initial forms of the compatibility ideal). *Let $(R, \mathfrak{m})$ be a completed local ring on a relative boundary chart, with residue field of characteristic zero. Let I be an ideal generated by original compatibility polynomials, and let $g_1, \dots, g_s \in \mathfrak{m}$ have positive integer weights $w_1, \dots, w_s$. Suppose finitely many such polynomials have exact presentations $f_i = P_i(g) \in I$ with no monomial of weighted degree less than d_i . After reduction of coefficients modulo $\mathfrak{m}$, let their degree- d_i parts be q_i . Assume that over an algebraic closure the common zero set of the q_i is just the origin, and that*

$$\tau^M \in I + (g_1, \dots, g_s)$$

for some $M > 0$. Then no local homomorphism $R \rightarrow L[[t]]$ annihilating I can have nonzero image of τ .

Proof. If some g_j has nonzero image, put

$$\nu = \min_j \frac{\text{ord}_t g_j}{w_j}.$$

After finite ramification, all $w_j \nu$ are integral. Let ξ_j be the coefficient of $t^{w_j \nu}$ in the image of g_j , taken as zero when its order is larger. At least one ξ_j is nonzero. Higher-weight monomials have order greater than $d_i \nu$, and a coefficient minus its residue value has positive order. Thus the coefficient of $t^{d_i \nu}$ in f_i is precisely $q_i(\xi)$. All these coefficients vanish, a contradiction. Hence all g_j have zero image, and the target identity forces τ^M , and therefore τ , to have zero image. $\square$

The ratio and horizontal parameters are elements of R .

For a linear polynomial presentation $f = Ag$, a full column rank residue matrix supplies the linear forms q_i . If the normal residuals lie in $(n)^2$, their quadratic initial forms can be used directly. Lemma C.6 applies in both cases.

Lemma C.7 (Implicit elimination with parameters). *Let $F(b, z, n) = 0$ be equations used for implicit elimination, satisfying $F(b, 0, 0) = 0$ identically in the parameters b , and assume $A = F_z(b, 0, 0)$ is locally invertible. Their implicit solution is $z = \varphi(b, n)$, with $\varphi(b, 0) = 0$. For another compatibility relation G vanishing on the same family, its restriction to this graph belongs to $(n)^2$ if*

$$\det(A)G_n - G_z \text{adj}(A)F_n = 0$$

identically on the whole family $z = n = 0$.

Proof. Implicit differentiation gives $\varphi_n = -A^{-1}F_n$. The displayed identity says that the reduced derivative $G_n + G_z \varphi_n$ is zero. Together with the zero constant term, the Taylor expansion in n , with all b retained as coefficients, gives the claimed ideal membership. $\square$

C.6 Degree bounds for the homogeneous derivations

Application to the state derivations. The coordinate gauges alter only the corresponding R_i component by a multiple of a necessary relation divided by one selected nonzero $\hat{p}$ coordinate. For a numerator monomial of bidegree $(d_{\hat{p}}, d_R)$, after this division the required degree inequalities are

$$\begin{aligned} d_{\hat{p}} + \gamma d_R - 2 &\leq \gamma, & \hat{p} \text{ components,} \\ d_{\hat{p}} + \gamma d_R - 1 - \gamma &\leq \gamma, & R \text{ components,} \\ d_{\hat{p}} + \gamma d_R - 1 &\leq \gamma, & \text{base components.} \end{aligned}$$

Each difference is affine in γ ; the inequalities at the endpoints $\gamma = 1, 2$ prove the bound for every rational $1 \leq \gamma \leq 2$. At the strict intermediate weights $1 < \gamma < 2$, the principal base minors are

$$\begin{aligned} &\frac{\eta \zeta^2 (\zeta - \eta - 1)}{(\zeta - \eta)(\zeta - 1)} R_1 R_2, \\ &- \frac{\zeta \eta^2 (\zeta - \eta - 1)}{(\zeta - \eta)(\eta + 1)} R_1 R_3, \\ &\frac{\eta \zeta (\zeta - \eta - 1)}{(\zeta - 1)(\eta + 1)} R_2 R_3. \end{aligned}$$

Their coefficients are units near $(\eta, \zeta) = (1, -1)$. The degree inequalities and these minors give the hypotheses of Lemma C.3. The leading incidence relations give two available rows on full R support and on two- R support except when only the row of the missing R coordinate is active. Those missing-row and single- R cases remain subject to their separate argument with the compatibility relations.

Remark C.8 (Weighted Rees models). To apply Lemma C.3 at an intermediate point of the proper boundary modification, take a curve through the point meeting the affine locus. Its orders $n = \text{ord } \tau$ and $s_0 = \min_j \text{ord}(\mathbf{v})_j$ give $\gamma = 2 - s_0/n \in (1, 2)$. If t is a uniformizing parameter of the normalized curve, the chart identities show that $t^n \hat{p}$ and $t^{2n-s_0} R$ are regular along it. Their leading coordinates are unit multiples of the corresponding $\bar{p}$ and $\bar{R}$ coordinates, so each group has a nonzero coordinate and retains its nonzero support. Using t as the homogenizing parameter gives a point of the weighted Rees closure of the state image with integer weights $(n, 2n - s_0)$. Its relative weight is $(2n - s_0)/n = \gamma$. Apply the lemma there. If two available coordinate fields give a nonzero horizontal minor, every minimal boundary component through that point dominates the base and contradicts the generic exclusion for that support. Thus the proper modification provides the curve, while the weighted Rees model provides regular derivations. The ordinary and primary endpoints retain their separate differential or local algebraic arguments.

C.7 The valuation partition of the boundary

The chart types W_O, W_S, W_T are defined in Appendix C.2. The strata are determined by the orders $n = \text{ord } \tau$ and $s_0 = \min_j \text{ord}(\mathbf{v})_j$. Table 3 gives the corresponding valuation regions.

Table 3: Valuation regions of the weighted boundary.

Region	Valuation condition
Weighted boundary	$s_0 = 0$
Supercritical	$0 < s_0 < n/2$
Critical	$s_0 = n/2$
Subcritical	$n/2 < s_0 < n$

Region	Valuation condition
Ordinary	$n \leq s_0 \leq \infty$

Support and coefficient strata. The leading incidence relations in Lemma 8.4 give 45, 21 and 7 nonempty amplitude supports when $\overline{\mathbf{R}}$ has one, two and three nonzero entries, respectively. The incidence relations give the same 73 possible coordinate supports in each of the three intermediate valuation regions. On a chart with a single nonzero derivative coordinate, $\overline{R}_3 = 1$, the amplitude coordinates and support numbers are those of Table 6. On a two-coordinate chart, cyclically arrange $\overline{R}_1 = 0$; the incidence relations give $\overline{p}_{23} = \overline{p}_{32} = 0$ and pair $\overline{p}_{21}$ with $\overline{p}_{31}$. The seven nonempty supports are the paired support, the two axes, the mixed support, and the paired support with either or both remaining amplitudes. On full $\overline{\mathbf{R}}$ support, the seven patterns are the nonempty selections of the three pairs (12, 32), (13, 23) and (21, 31).

The columns of $D(\overline{p})$ in Appendix C.1 have the disjoint supports (31, 32), (21, 23) and (12, 13). Consequently its rank is the number of nonempty groups, and rank zero would give $\overline{p} = 0$, outside an intermediate chart. On a fixed stratum, a nonzero minor of the linear normal matrix eliminates the corresponding normal coordinates. Substitution gives the residual initial forms used in Lemmas C.7 and C.6. For the pure axis, Proposition C.9 instead gives $\mathcal{I}_\perp^4 \subset I$.

Table 4 records the arguments for these support classes. The calculation nodes refer to the exact source interfaces in the accompanying Rank 3 calculation package.

Table 4: Intermediate support classes and their local arguments.

Support	Supercritical	Critical	Subcritical
One $\overline{R}_i$; 45 patterns	Unit leading coefficients (BC21)	Even equations and normal matrix (BC14)	Support table; axis and conic ideals (BC16–BC18, C9_ENDPOINT)
Two $\overline{R}_i$; 21 patterns	Unit coefficient combinations (BC21)	Full source and direction matrices; rank one endpoint equations (BC15, BC32)	Inactive support ideals and finite normal directions (BC19, BC32); horizontal derivations (BC06)
Three $\overline{R}_i$, one pair; 3 patterns	Paired normal matrix and kernel quadratic (BC21, BC33)	Original paired normal equations (BC21, BC33)	Canonical ideal and amplitude walls (BC20, BC33)
Three $\overline{R}_i$, two pairs; 3 patterns	Original Schur matrix (BC30)	Amplitude minors and four row determinant (BC30)	Generic support equations and two horizontal derivations (BC02, BC06, BC30, BC32)
Three $\overline{R}_i$, three pairs; 1 pattern	Torus normal block (BC21)	Amplitude rank ideal, exceptional directions and critical line (BC31); material resonance (BC14)	Generic support equations and two horizontal derivations (BC02, BC06, BC32)

On the selected fibre in W_T , normalize $\overline{R}_3 = 1$ and suppose that $\overline{R}_1 = 0$, $\overline{R}_2 \neq 0$. Along

an approaching arc, the affine coordinates have orders at least $-n$ and $s_0 - 2n$, respectively. The auxiliary identity gives

$$L(\widehat{p})R = \tau^{-2}e^{\text{alg}}.$$

The factorization $(R_\nu)_{31} = DCLR$ in Appendix C.1 and the factorization of $(H_j)_{32}$ therefore give

$$\text{ord}(R_\nu)_{31} \geq -4n, \quad \text{ord}(H_j)_{32} \geq -6n + s_0.$$

All remaining parts except $(R_\nu)_{12}$ and $(H_j)_{13}$ have these lower bounds, by the ordinary and weighted degree bounds. The two exceptional parts have orders at least $-5n + 2s_0$ and $-7n + 3s_0$. Since $2s_0 < n$, these are strictly smaller bounds. Multiplying the complete relations by $\tau^5/(\mathbf{v})_3^2$ and $\tau^7/(\mathbf{v})_3^3$, respectively, and taking residues gives the $(1, 2)$ and $(1, 3)$ equations evaluated at $(\bar{p}, \bar{\mathbf{R}})$. One leading compatibility relation, after multiplication by a unit, is

$$\bar{p}_{21}\bar{R}_2^3 = 0.$$

It gives $\bar{p}_{21} = \bar{p}_{31} = 0$ at the boundary point. The other leading relations give

$$(\bar{R}_2^2 + q)\bar{p}_{12} = (\bar{R}_2^2 + q)\bar{p}_{13} = 0.$$

Since $\bar{p} \neq 0$, they imply $\bar{R}_2^2 = -q$. The unit coefficient combinations in Appendix C.8.6 then force $\bar{p}_{12} = \bar{p}_{13} = 0$, contrary to $\bar{p} \neq 0$ (BC21).

Coefficient zero sets refine these supports in the full local coefficient ring. In a paired chart, put $\bar{p}_{12} = 1$ and $\ell = \bar{p}_{32}$. The amplitude walls are $D_\pm = m\ell^2 \pm k = 0$. They are disjoint because $D_+ - D_- = 2k$ is a unit. Each wall is intersected with the material resonance $k + q = 0$ and its complement. At the canonical paired intersection, the six-generator ideal of Appendix C.8 contains $\kappa\delta$, hence $\tau = \kappa^2\delta$ (BC20, BC33). In an axis chart, the identity $29D_{15} + 19D_9 = 960q$ makes the D_{15} pivot invertible when $D_9 = 0$ (BC19, BC32).

For critical full amplitude support, take $s_d = \bar{p}_{21}$ and $z_d = \bar{p}_{23}/\bar{p}_{32}$ in the normalization $\bar{p}_{12} = \bar{R}_3 = 1$. These parameters and the amplitudes belong to the common constant field by Appendix C.3. On $s_d z_d + s_d + z_d \neq 0$, the amplitude coefficient matrix with three columns, acting on $(1, \bar{p}_{32}^2, \bar{p}_{31}^2)^\top$, has kernel vector $(-\kappa_0, q, \kappa_0 q s_d^2)^\top$. After the unit column and row normalizations, the ideal where its rank drops on $s_d z_d \neq 0$ is generated by

$$\begin{aligned} (s_d - 1)^2(s_d^2 - \tfrac{13}{2}s_d + 1), \quad (s_d - 1)(z_d - 1)(z_d^2 - \tfrac{13}{2}z_d + 1), \\ (z_d - 1)^2(z_d^2 - \tfrac{13}{2}z_d + 1), \quad (s_d - 1)(z_d - 1)(s_d - z_d). \end{aligned}$$

Its zero set consists of $(1, 1)$ and $(1, \omega), (\omega, 1), (\omega, \omega)$ for the two roots $\omega^2 - \frac{13}{2}\omega + 1 = 0$. The amplitude matrix has rank zero at $(1, 1)$ and rank one at the other six points. At $(1, 1)$, two source minors give $\bar{p}_{31}^2 = -q$ and $\kappa_0 \bar{p}_{32}^2 = \bar{p}_{31}^2$. The remaining source has zero coefficient of δ and constant coefficient $-27q^2 \bar{p}_{31} \bar{p}_{32} / 32$ on the selected ratio fibre, a unit. For the other six points use the determinant factors in Appendix C.1. On $s_d z_d + s_d + z_d = 0$, the original coefficient equations are retained; write $s_d = -z_d/(z_d + 1)$, with $z_d(z_d + 1) \neq 0$. Order the principal values as in Appendix C.1 and put $N_b = a\kappa_0^2$, $\Lambda_b = mq$. Then

$$N_b > 0, \quad \Lambda_b > 0, \quad 1 - N_b - \Lambda_b = -\kappa_0(a + 1 + 2a\kappa_0) > 0.$$

After removing the direction factors $z_d, z_d + 1$, the resultant of the first and third even polynomials is, up to a nonzero rational factor,

$$\begin{aligned} & (\Lambda_b^2 - 1)(N_b^2 - 1)(N_b + \Lambda_b)^2(\Lambda_b - N_b)^4 N_b^6 \Lambda_b^6 \\ & \cdot (8N_b \Lambda_b + N_b + \Lambda_b)((N_b - \Lambda_b)^2 + N_b + \Lambda_b) \\ & \cdot \left[9(1 - N_b - \Lambda_b)(1 - (N_b - \Lambda_b)^2) + 8N_b \Lambda_b \right]. \end{aligned}$$

Every factor except $\Lambda_b - N_b$ is nonzero on this parameter region. Thus a common zero requires $N_b = \Lambda_b$, equivalently $b = a^2$. On that wall, $N_b = -\kappa_0(1 + \kappa_0) \leq 1/4$. Dividing the first two reduced polynomials on this wall by their remaining direction factor z_d gives resultant

$$64N_b^{16}(N_b - \frac{3}{2})(N_b + 9)(N_b + 1)^3(N_b - 1)^5(N_b^2 - \frac{1}{2}N_b + \frac{1}{2}).$$

It is nonzero for $0 < N_b \leq 1/4$, which excludes the material wall (BC31).

The conic is separated into $w(w^2 - k)(w^2 + k) \neq 0$, the two walls $w^2 = \pm k$, and the endpoint $w = 0$, with $w = \bar{p}_{23}$. On $w^2 = -k$, its equations give $\bar{p}_{12} = \bar{p}_{21} = 0$, so the point has support number 10. On $w = 0$ it has support number 7, or 13 after transposition; the exact relative ideal is given in Appendix C.9 (BC18, C9_ENDPOINT). Vanishing of another leading coordinate changes its support to one of the other listed patterns; that coordinate remains a parameter in the local equations.

The original weighted charts use the cone ideal and the four-point intersection ideal (BC10–BC13). The ordinary charts use the radical containments in $\mathcal{F}_{\text{in}}$, followed by the differential exclusion in Appendices C.10 and C.11. Their parameter intersections, including the additional τ wall, are described in Appendix C.12 (BC22–BC27, BC29).

C.8 Local ideals on the boundary strata

The local ideals below are generated by finitely many compatibility relations specified through (2.24) and their derivatives under the state derivations, after the indicated substitution. Division by an exceptional monomial is an exact polynomial division on the entire incidence chart. These local ideals are contained in the local ideal of the saturated closure of the state image, to which the formal transfer arguments apply. The center is $(\eta, \zeta) = (1, -1)$; the principal value ratios remain arbitrary.

C.8.1 Primary, ordinary and critical strata

Table 5 gives the local conclusions for these strata, with the supports to which each applies.

Table 5: Boundary conclusions on the primary and critical strata.

Chart	Conclusion	Stratum
Original weighted boundary $v \neq 0$	$\tau \in I$ locally	All one-, two- and three-coordinate v supports.
Original weighted boundary nonzero $\bar{p}$ cone	$\tau \in I$ locally	All nonzero amplitudes, including vanishing sums.
Ordinary exceptional family	Boundary contained in the ordinary boundary family	All points on a dominant component.
Critical single R	$\tau \in \sqrt{I}$ locally	Eight roots of the auxiliary equations; all fifteen $\bar{p}$ supports.
Critical two R	$\tau \in \sqrt{I}$ locally	Rank three, rank two and rank one coefficient strata.
Critical three R; at least two amplitude pairs	No invariant boundary component dominating the ratio plane	First integrals, amplitude coefficient equations and their rank strata.
Critical three R; one amplitude pair	$\tau \in \sqrt{I}$ locally	Paired normal equations and the hypersurface kernel quadratic.

The nonzero coefficients in the original weighted boundary chart are $G_0 = \eta\zeta - \eta + \zeta$ and $V_0 = (\eta + 2)(\zeta - 2)(2\eta + 1)(\zeta - 2\eta)(2\zeta - 1)(2\zeta - \eta)$, with values -3 and 729 at the center. Three nonzero $\mathbf{v}$ coordinates force either $\bar{p} = 0$ or $\mathbf{v}_1^2 = -q/\kappa_0$, $\mathbf{v}_2^2 = -q$, $\mathbf{v}_3^2 = 1$. Two nonzero coordinates have unit coefficients of the compatibility relations $-8\eta, 8, 8$ and again force $\bar{p} = 0$. One nonzero coordinate is covered by V_0 . The compatibility polynomials vanish on the remaining $\bar{p}$ cone. On the fibre $(\eta, \zeta) = (1, -1)$, $\mathbf{M}_{\text{cone}}$ has rank six at every nonzero amplitude. The amplitude origin is treated by the four-point local ideal below.

The four-point family. Use $F_0^h, \dots, F_{27}^h$ from the normal matrix construction, before specializing the ratio parameters. On $\mathbf{v}_3 = 1$ choose the four pairs of units v_1^0, v_2^0 with

$$(v_1^0)^2 = -\frac{q(\zeta - \eta)^2}{\kappa_0 \zeta^2(\eta + 1)^2}, \quad (v_2^0)^2 = -\frac{q\eta^2(\zeta - 1)^2}{\zeta^2(\eta + 1)^2}.$$

Put $y_i = 1 - (\mathbf{v}_i/v_i^0)^2$ for $i = 1, 2$, and introduce $\xi = (\xi_1, \dots, \xi_7)$ by

$$(\bar{p}, \tau) = (\xi_1, \mathbf{v}_2\xi_2, \xi_3, \mathbf{v}_1\xi_4, \mathbf{v}_2\xi_5, \mathbf{v}_1\xi_6, \mathbf{v}_1\mathbf{v}_2\xi_7).$$

In row n , divide by $\mathbf{v}_1^{\pi_{n1}}\mathbf{v}_2^{\pi_{n2}}$, where $\pi_{ni} \in \{0, 1\}$ is the common parity of its linear normal monomials. Let $N(y_1, y_2)$ be the resulting 28×7 coefficient matrix in ξ , extracted from the polynomials F_n^h . At $y_1 = y_2 = 0$ its kernel has basis

$$K_\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1/\alpha & 0 \\ 0 & 1 & 0 & \beta/\alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/\beta & 0 & 0 \end{pmatrix}.$$

Thus the local coordinates $z_1, \dots, z_4, \varepsilon_0, \varepsilon_1, \varepsilon_2$ are fixed by

$$\xi = K_\varepsilon^\top \varepsilon + (z_1, z_2, z_3, 0, 0, 0, z_4)^\top.$$

The four equations used for implicit elimination are precisely $F_0^h = F_1^h = F_2^h = F_3^h = 0$, that is, the homogenizations of Q_0, R_0, R_1, R_2 . Writing $N_0 = N(0, 0)$ and $N_i = \partial_{y_i} N(0, 0)$, form

$$\mathbf{C}_\varepsilon = ((N_0)_{*,(1,2,3,7)} \mid N_1 K_\varepsilon^\top \mid N_2 K_\varepsilon^\top).$$

Its rows $(1, 2, 3, 4, 5, 6, 7, 11, 12, 14)$ have determinant

$$\det(\mathbf{C}_\varepsilon)_{(1,2,3,4,5,6,7,11,12,14),*} \Big|_{(\eta,\zeta)=(1,-1)} = \frac{9q^{12}}{481036337152\kappa_0^9}.$$

The first four rows give a unit Jacobian in z . After solving them, the residuals of these twenty-eight relations vanish on both $\varepsilon = 0$ and $y_1 = y_2 = 0$: the former is the $\bar{p} = \tau = 0$ family, and the latter is the canonical kernel family. They therefore belong to

$$(y_1, y_2) \cap (\varepsilon_0, \varepsilon_1, \varepsilon_2) = (y_1 \varepsilon_i, y_2 \varepsilon_i : 0 \leq i \leq 2).$$

The displayed determinant gives an invertible coefficient matrix on these six generators after eliminating the four columns for the implicitly determined variables. Nakayama's lemma identifies the residual ideal with this intersection. The solved value of τ vanishes on the same two families, so τ belongs to the original local ideal.

The critical single-coordinate boundary. The leading equations reduce this case, on the $\overline{\mathbf{R}}_3 = \overline{p}_{13} = 1$ chart, to

$$w^2 = k, \quad xz = 10m, \quad 3(x^2 + z^2) = xz, \quad k = a\kappa_0, \quad m = 1 + k.$$

These equations have eight complex solutions. Use the exact incidence coordinates

$$\begin{aligned} \overline{p} &= \left(x, 1, z, w, \frac{tz - \kappa e_2^{\text{alg}}}{\beta}, \frac{xwt}{\beta} + \frac{\kappa(xe_1^{\text{alg}} - e_0^{\text{alg}})}{\alpha} \right), \\ \overline{\mathbf{R}} &= \left(\frac{\alpha wt}{\beta} + \kappa e_1^{\text{alg}}, t, 1 \right), \quad \tau = \kappa^2 \delta, \end{aligned}$$

where $\alpha = (\zeta - \eta)/(\zeta(\eta + 1))$ and $\beta = \eta(\zeta - 1)/(\zeta(\eta + 1))$. Define the regularized relations by

$$\mathcal{R}_0 = \kappa^8 \delta^4 R_0 \left(\frac{\overline{p}}{\kappa^2 \delta}, \frac{\overline{\mathbf{R}}}{\kappa^3 \delta} \right), \quad \mathcal{H}_2 = \kappa^{11} \delta^5 H_2 \left(\frac{\overline{p}}{\kappa^2 \delta}, \frac{\overline{\mathbf{R}}}{\kappa^3 \delta} \right).$$

The incidence substitution makes these polynomials regular. They vanish on the whole locus $t = \kappa = 0$, with both base variables and all seven other chart coordinates unrestricted. Thus they are exact linear combinations of t, κ in the relative local ring. On the candidate graph their normal matrix, with columns ordered as (t, κ) , has at $\eta = 1, \zeta = -1$ the form

$$\begin{pmatrix} 0 & -64w(x+z)^2 \\ \frac{\kappa_0}{2}zw^2(x+z) & * \end{pmatrix}.$$

Its determinant is $32\kappa_0zw^3(x+z)^3$. The candidate equations imply $(x+z)^2 = 70m/3$; consequently every factor of this determinant is a unit for every admissible triple of principal values and every candidate. The coefficient matrix is therefore invertible locally, over the coefficient ring of the base and amplitude parameters. Hence $(\mathcal{R}_0, \mathcal{H}_2) = (t, \kappa)$ and τ belongs to the local ideal of compatibility relations. This excludes every ordinary boundary arc with $\tau \neq 0$, regardless of ramification.

The critical missing-row boundary. It remains to treat $\overline{\mathbf{R}} = (0, t, 1)$ and $\overline{p} = (x, y, 0, 0, 0, 0)$, with $t \neq 0$ and $(x, y) \neq (0, 0)$. First take $x = 1$ and write $y = w$. The compatibility matrix in $(e_0^{\text{alg}}, e_1^{\text{alg}}, e_2^{\text{alg}}, \delta)$ has zero e_2^{alg} column. Since δ is a unit, the minors in columns $(e_0^{\text{alg}}, e_1^{\text{alg}}, \delta)$ and relation rows (R_3, Q_0, H_9) and (R_3, Q_0, H_{15}) must vanish. At $\eta = 1, \zeta = -1$ their inverted factors leave, with $v = t^2$,

$$\begin{aligned} f_A &= (-8v + 152q)w + t(-232v - 392q), \\ f_B &= (392v + 232q)w + t(-152v + 8q). \end{aligned}$$

The determinant of their coefficient matrix in (w, t) is $92160(v + q)^2$. As $t \neq 0$, it follows that $v = -q$. The first equation then gives $w = t$, so $w \neq 0$.

Put $E = e_2^{\text{alg}}/w$, $X = e_0^{\text{alg}}/w$ and $Y = e_1^{\text{alg}}/w$. The even equations and Q_0 give

$$E^2 = -64, \quad X^2 = 4a\kappa_0 E, \quad Y^2 = \frac{4a\kappa_0}{m} E, \quad Y = tX.$$

Thus E and X are nonzero, and $t^2 = 1/m$. Together with $t^2 = -q$, this implies $1 + mq = 0$, whereas $mq = (b/a)m^2 > 0$ for every triple of positive, pairwise distinct principal values.

The endpoint $x = 0, y \neq 0$ is covered by the relation and field covariance under interchange of labels 2 and 3. It exchanges the two available $\overline{p}$ coordinates and retains two nonzero $\overline{\mathbf{R}}$ coordinates. In the adapted variables

$$(\eta, \zeta) \mapsto (-\zeta, -\eta), \quad (\kappa_0, q, k, m) \mapsto \left(-\frac{\kappa_0}{q}, \frac{1}{q}, -\frac{k}{m}, \frac{1}{m} \right).$$

The cyclic section is fixed and mq becomes $1/(mq) > 0$. After projective normalization by units, the preceding argument applies. This excludes the entire missing-row projective line.

For two nonzero $\overline{\mathbf{R}}$ coordinates, the rank two paired chart is $\overline{p} = (0, 0, 1, 0, t, 0)$, $\overline{\mathbf{R}} = (0, \beta t, 1)$, with $t \neq 0$. The coefficient determinants of (R_0, R_2, Q_0) and (R_0, R_2, H_6) in $(e_0^{\text{alg}}, e_1^{\text{alg}}, \delta)$, after their nonzero monomial and unit factors are removed, are respectively

$$A_c = (-5\eta + 4\zeta)t^2 + q(-4\eta + 5\zeta), \quad B_c = 5t^2 + 4q.$$

They satisfy the Bézout identity $5A_c - (-5\eta + 4\zeta)B_c = 9q\zeta$. The rank one missing-row stratum was excluded above. For three nonzero $\overline{\mathbf{R}}$ coordinates, $D(\overline{p})$ has rank three when at least two amplitude pairs survive; the remaining one-amplitude chart has a determinant for the eliminated variables independent of its free kernel parameter, and its hypersurface kernel has a nonzero k^2 coefficient.

For the ordinary boundary the local conclusion is containment in $\mathcal{F}_{\text{in}}$. A boundary component dominating the ratio base meets the relative ordinary boundary family in a nonempty open set. Its generic point is already excluded, and irreducibility rules out that component.

C.8.2 The ordinary boundary family at the exceptional section

The family is defined in (8.3). We now specify the relative ideals used in its exclusion and in the local containment statement at $(\eta, \zeta) = (1, -1)$.

On $v = -2$, the relative ideals generated by the compatibility polynomials contain τ . Off $m\ell^2 = \pm k$ their nine generators are

$$(n_0 + n_3, n_1, n_2, \tau, un_{\perp}, (v+2)t, n_{\perp}^2, n_{\perp}t, t^2).$$

Here $t = n_3$ and $n_{\perp}$ is the remaining kernel variable. The coefficient matrices of the compatibility polynomials have unit minors on a nonempty base open set for every admissible set of principal values. The residual coefficient identities give a unit k in one block and a Bezout combination equal to one in the other; the extra factor $\kappa_0\ell^2 + q\zeta^2$ is covered by the extra- τ chart, treated in Section C.12. On the positive hypersurface, set $\mu = m\ell^2 - k$. The ten generators are

$$(n_0 + n_3, n_1, n_2, \tau, un_{\perp}, (v+2)t, n_{\perp}^2, n_{\perp}t, \mu t^2, t^3).$$

The identities of the compatibility relations retain arbitrary u . Their pure t^2 coefficients are multiples of μ . The remaining determinant polynomials have highest base coefficients containing mq and one, proving that they are not identically zero for any admissible set of principal values. On the negative hypersurface put $\nu = m\ell^2 + k$ and

$$y_0 = n_0 + n_3, \quad y_1 = n_1 + mt^2, \quad y_2 = n_2 + m\ell t^2, \quad y_3 = \tau.$$

The eleven generators are

$$(y_0, y_1, y_2, y_3, un_{\perp}, (v+2)t, n_{\perp}^2, n_{\perp}t, \nu t^2, ut^2, t^3).$$

All compatibility relations belong to this ideal on the whole relative chart. The two nontrivial coefficient blocks have cyclic determinants

$$\det B_0 = 1536k^2/m^3, \quad \det A_0 = \frac{2048\ell^5 qa^2(k+q)}{3m}.$$

If $k + q = 0$, use the same identities of the compatibility polynomials and evaluate at $\eta = 1, \zeta = -2$ instead. The determinants and remaining column unit are

$$\det A_0 = \frac{64512\ell^5 q^4}{25\kappa_0^2}, \quad \det B_0 = -\frac{155648q^2}{135\kappa_0^3}, \quad U_0 = \frac{20}{189}.$$

These expressions are nonzero for both amplitude roots and every admissible set of principal values satisfying the resonance relation. Thus the coefficient functions are nonzero. Consequently the original ideal of compatibility relations equals the stated generator ideal locally on a nonempty base open set, and contains τ . This excludes the ordinary boundary family over that base open set.

At the cyclic section, the only containment case is

$$u = 0, \quad v = -2, \quad \kappa_0 \ell^2 + q = 0.$$

For either complex value of ℓ , use the material constants $N_b = a\kappa_0^2$ and $\Lambda_b = mq$ of Appendix C.7. In local normal coordinates the boundary ideal generated by the compatibility polynomials has, off $\Lambda_b = N_b$, the radical of

$$J_8 = (n_0 + n_3, n_1, n_2, un_\perp, (v+2)t, n_\perp^2, n_\perp t, t^2), \quad t = n_3.$$

Its two eight-column determinants, up to fixed row signs, are

$$\begin{aligned} & \frac{\ell^5}{42\kappa_0^8 a^3 m^3} (\Lambda_b - N_b)^2 (N_b + \Lambda_b) (5\Lambda_b - 2N_b) (5\Lambda_b + 4N_b), \\ & - \frac{\ell^6}{168\kappa_0^7 a^3 q m^3} (\Lambda_b - N_b)^2 (N_b + \Lambda_b) (11\Lambda_b - 5N_b) (5\Lambda_b + 4N_b). \end{aligned}$$

At least one is a unit because $5(5\Lambda_b - 2N_b) - 2(11\Lambda_b - 5N_b) = 3\Lambda_b$. On $\Lambda_b = N_b$, equivalently $k + q = 0$, set

$$y_0 = n_0 + n_3, \quad y_1 = n_1 + mt^2, \quad y_2 = n_2 + m\ell t^2, \quad \nu = m\ell^2 + k.$$

The boundary ideal has the radical of

$$(y_0, y_1, y_2, un_\perp, (v+2)t, n_\perp^2, n_\perp t, \nu t^2, ut^2, t^3);$$

its ten-generator coefficient determinant is $-10240q^7/(7\kappa_0^6)$, a unit in the stated localization. Both radicals define $\mathcal{F}_{\text{in}}$ in the local boundary chart.

Thus the boundary of the state image is locally contained in $\mathcal{F}_{\text{in}}$ at this exceptional section. A lifted irreducible boundary component dominating the ratio plane that meets the section consequently has a nonempty open subset in $\mathcal{F}_{\text{in}}$; its generic point is in that family. The generic ordinary boundary family exclusion gives a contradiction.

C.8.3 Subcritical single-coordinate R chart

Normalize $\bar{\mathbf{R}}_3 = 1$. The original leading relation gives $\bar{p}_{31} = \bar{p}_{32} = 0$ for the leading coordinates. Put $(x, y, z, w) = (\bar{p}_{12}, \bar{p}_{13}, \bar{p}_{21}, \bar{p}_{23})$, and number a nonempty support by the sum of 1, 2, 4, 8 for the nonzero coordinates x, y, z, w , respectively. Cyclic permutations give the other two coordinate charts. Table 6 covers all fifteen nonempty supports and records the local ideal or the further equations needed in each case.

Table 6: Subcritical supports with one nonzero component.

Support	Result	Identity or remaining charts
1,4	$\tau^4 \in I$	Proposition C.9.
2,8	$\tau \in I$	Nine generators; selected determinants $-25qk^3/4032$ and $-544qk^3/10935$.

Support	Result	Identity or remaining charts
3,6,9,12	$1 \in I$	Units $8x^2y^2$, $16y^2z^2$, $16x^2w^2$, $8z^2w^2$.
5	$\tau \in I$	Eight roots; the unit factors $1+2z_0$ and $9+20z_0$ are complementary.
7,13	$\tau \in \sqrt{I}$	Both the conic and its complement.
10	$\tau \in \sqrt{I}$	$w^2 \neq \pm k$ and each hypersurface $w^2 = \pm k$.
11,14	$1 \in I$	The odd relation has unit coefficient of xyw or yzw .
15	$\tau \in \sqrt{I}$	The conic and its resonance hypersurfaces.

C.8.4 A coordinate axis on the boundary

The incidence chart and the moving horizontal family are

$$\alpha = \frac{\zeta - \eta}{\zeta(\eta + 1)}, \quad \beta = \frac{\eta(\zeta - 1)}{\zeta(\eta + 1)}, \quad s_h = -\frac{2\eta(\zeta + 1)}{\zeta(\eta + 1)}, \quad (\text{C.3})$$

$$e_0^{\text{alg}} = E + s_h \ell, \quad v = (\alpha + \kappa s_h) \ell, \quad \mathcal{I}_\perp = (y, E, e_2^{\text{alg}}). \quad (\text{C.4})$$

$$\bar{p} = \left(1, y, z, w, \frac{tz - \kappa e_2^{\text{alg}}}{\beta}, \frac{v - \kappa e_0^{\text{alg}}}{\alpha} \right), \quad \bar{\mathbf{R}} = (v, t, 1), \quad vy - \frac{\alpha}{\beta} tw - \kappa e_1^{\text{alg}} = 0, \quad (\text{C.5})$$

$$\hat{p} = \frac{\bar{p}}{\kappa^2 \delta}, \quad \mathbf{R} = \frac{\bar{\mathbf{R}}}{\kappa^3 \delta}, \quad \tau = \kappa^2 \delta, \quad \sigma = \kappa \delta. \quad (\text{C.6})$$

The order of the six homogeneous entries is $(\bar{p}_{12}, \bar{p}_{13}, \bar{p}_{21}, \bar{p}_{23}, \bar{p}_{31}, \bar{p}_{32})$. Substitution in the state relations and the affine $\mathbf{R}$ relations fixes their regular representatives. The four variables eliminated by the implicit function theorem are $(z, e_1^{\text{alg}}, \delta, w)$; all other variables stay in the coefficient ring.

The four equations used for implicit elimination have invertible Jacobian

$$-\frac{3qk}{224\kappa_0 m^2}.$$

Their implicit solution gives $\delta \in \mathcal{I}_\perp = (y, E, n_\perp)$, where $n_\perp = e_2^{\text{alg}}$. The relative Schur identity over the coefficient ring of η, ζ and the horizontal parameters places the other three residuals in $\mathcal{I}_\perp^2$. At the cyclic boundary point, invertible row operations identify their quadratic parts as

$$\begin{aligned} q_1 &= n_\perp^2 - 4yn_\perp + 64y^2, \\ h &= 4y^2 - yn_\perp, \\ q_2 &= E^2 - 64ky^2 - 4kyn_\perp - kn_\perp^2. \end{aligned}$$

Proposition C.9 (Exclusion on a coordinate axis). *Under these hypotheses on the compatibility relations, no formal arc satisfying the compatibility relations has $\tau = \kappa^2 \delta \neq 0$, for any allowed triple of principal values and any finite ramification.*

Proof. The common zero of h, q_1, q_2 is the origin, over any algebraically closed field of characteristic zero and for every k . Indeed, if $y \neq 0$, then $h = 0$ gives $n_\perp = 4y$, and $q_1 = 0$ gives $64y^2 = 0$. If $y = 0$, then $q_1 = 0$ gives $n_\perp = 0$, and $q_2 = 0$ gives $E = 0$. Lemma C.6 therefore makes every normal coordinate zero along a formal arc satisfying the compatibility equations. The implicit solution gives $\delta = 0$ and hence $\tau = 0$. $\square$

The ideal inclusion follows from the identities

$$\begin{aligned} q_1 - 4h &= n_\perp^2 + 48y^2, & -h &= yn_\perp - 4y^2, \\ q_2 + kq_1 - 8kh &= E^2 - 32ky^2, & yq_1 + n_\perp h &= 64y^3. \end{aligned}$$

They give $(y, n_\perp)^3 = 0$ in the quadratic quotient; replacing E^2 by $32ky^2$ then gives $\mathcal{I}_\perp^4 = 0$. Thus the residual multiplication matrix surjects onto $\mathcal{I}_\perp^4/\mathcal{I}_\perp^5$ at the point. Its unit minor and one application of Nakayama give $\mathcal{I}_\perp^4 \subset I$, hence $\tau^4 \in I$.

C.8.5 Supercritical and inactive two-coordinate $\overline{\mathbf{R}}$ charts

The columns of D have disjoint supports. Its rank is the number of nonempty coordinate groups. For two nonzero $\overline{\mathbf{R}}$ coordinates, the twelve exceptional configurations consist of three paired, six pure-axis and three mixed supports. In a mixed chart put $\bar{p} = (1, w, 0, 0, 0, 0)$ and $\overline{\mathbf{R}} = (0, t, 1)$, with $wt \neq 0$. Two coefficient minors have determinant $25/(4096 \cdot 49)$ and force $e_0^{\text{alg}} = e_1^{\text{alg}} = 0$. The two remaining equations in the variables being eliminated have a unit cyclic resultant $1024a^2\kappa_0^2$.

On the chart $\overline{R}_3 = 1$, put $\varrho = \overline{R}_2$. In a pure-axis chart the alternative pivots and quotient determinants are

$$D_9 = 29\varrho^2 + 49q, \quad D_{15} = q - 19\varrho^2, \quad -\frac{21\varrho q}{2D_9}, \quad \frac{\varrho(\varrho^2 + 16q)}{4D_{15}}. \quad (\text{C.7})$$

At $D_9 = 0$, $D_{15} = 960q/29$ and $\varrho^2 + 16q = 415q/29$ are units. The ideal contains $(\ell s, s^2)$; a coefficient determinant $6aq$ forces $s^2 = 0$, hence $\delta = 0$ on any reduced formal arc approaching the boundary. In the paired chart $\bar{p} = (1, 0, 0, 0, 0, \ell)$, $\overline{\mathbf{R}} = (\alpha\ell, 0, 1)$, $\ell \neq 0$, the determinant for the eliminated variables is $-9q^2\ell/(14336\kappa_0^2)$. Put $D_\pm = m\ell^2 \pm a\kappa_0$. Off both hypersurfaces the two residual coefficients $5m\ell^2 - 4a\kappa_0$ and $4m\ell^2 - 5a\kappa_0$ have unit determinant $9a\kappa_0$. On $D_- = 0$, the determinant after elimination is $-\ell^2/4$ and the residual determinant is $-3\ell/256$. On $D_+ = 0$, the ideal contains $(\varrho E, E^4)$, with fourth-order coefficient determinant $3/(1048576a\kappa_0)$. Thus $E = 0$ on every reduced formal arc.

C.8.6 The remaining intermediate strata

For full $\overline{\mathbf{R}}$ support the three original leading binomials pair $(\bar{p}_{12}, \bar{p}_{32})$, $(\bar{p}_{13}, \bar{p}_{23})$ and $(\bar{p}_{21}, \bar{p}_{31})$. At least two surviving pairs provide two coordinate horizontal derivations on the original closure. Their base minors are units in the stated localization times the corresponding nonzero $\overline{\mathbf{R}}$ entries. Thus a minimal prime through such a cyclic point dominates the base, and the generic support exclusion applies there. With exactly two nonzero $\overline{\mathbf{R}}$ entries, the twelve exceptional configurations have the local ideal identities established above; the remaining supports have two available horizontal derivations.

For the remaining subcritical full- $\overline{\mathbf{R}}$ paired chart put $\overline{\mathbf{R}} = (\alpha\ell + \kappa(E_0 + s_h\ell), \varrho, 1)$, $\ell\varrho \neq 0$, and use

$$\bar{p} = \left(1, \frac{(\alpha/\beta)s\varrho + \kappa e_1^{\text{alg}}}{\alpha\ell + \kappa(E_0 + s_h\ell)}, \frac{\beta w + \kappa e_2^{\text{alg}}}{\varrho}, s, w, \ell\right). \quad (\text{C.8})$$

In these incidence coordinates the normalized state is $\hat{p} = \bar{p}/(\kappa^2\delta)$ and $R = \bar{R}/(\kappa^3\delta)$. Let $\mathcal{F}_j$ denote member j of the ordered thirty-seven relations, with indices starting at zero. The regularized relations for the first twenty-eight members are

$$\mathcal{S}_0 = \kappa^7\delta^3Q_0, \quad \mathcal{S}_j = \kappa^8\delta^4\mathcal{F}_j \ (1 \leq j \leq 9), \quad \mathcal{S}_j = \kappa^{11}\delta^5\mathcal{F}_j \ (10 \leq j \leq 27),$$

where the stated normalized state is substituted on the right. Their regular extensions follow from $L\bar{R} = \kappa(E_0 + s_h\ell, e_1^{\text{alg}}, e_2^{\text{alg}})^\top$ and the exact factorizations of the leading relations. The pair relation used here is $\mathcal{P} = \kappa^9\delta^4R_7$ with the same substitution. We first compute on the fibre $(\eta, \zeta) = (1, -1)$. On the family $s = w = E_0 = e_1^{\text{alg}} = e_2^{\text{alg}} = \delta = 0$, these relations vanish identically, with κ, ℓ, ϱ free.

The matrix of linear normal coefficients of these relations can lose rank only at $m\ell^2 = \pm a\kappa_0$ or at $\ell^2 = -q/\kappa_0$, $(\varrho/\beta)^2 = -q$. On the positive hypersurface a maximal minor after implicit elimination is $8\varrho/(15m)$. On the negative hypersurface the additional residue has a three-minor $-96q\ell\varrho/(a^2\kappa_0^3)$. On the nonresonant canonical locus the residual ideal contains $\kappa\delta$, with coefficient minor $-25q^3\varrho/(9216\kappa_0^4)$.

At the resonance $a\kappa_0^2 = mq$, fix the principal value ratios. Then $\alpha = \beta = 1$, $s_h = 0$ and $a = -q/\kappa_0$. Put

$$W_\ell = \ell^2 + q/\kappa_0, \quad W_\varrho = \varrho^2 + q.$$

The four equations used for implicit elimination are exactly $\mathcal{S}_0 = \mathcal{S}_5 = \mathcal{S}_7 = \mathcal{S}_2 = 0$, corresponding to Q_0, R_1, R_3, Q_5 . Their Jacobian in $(w, E_0, e_1^{\text{alg}}, e_2^{\text{alg}})$ is invertible at the canonical point. Denote by $\mathcal{G} = (\mathcal{G}_1, \dots, \mathcal{G}_6)$ the residuals obtained from

$$(\mathcal{S}_1, \mathcal{S}_{17}, \mathcal{S}_{11}, \mathcal{S}_{19}, \mathcal{S}_{13}, \mathcal{P}) = (\tilde{Q}_1, \tilde{H}_7, \tilde{H}_1, \tilde{H}_9, \tilde{H}_3, \tilde{R}_7)$$

on this implicit graph. Define the ordered monomials

$$(g_1, \dots, g_6) = (sW_\ell, s\kappa, \delta W_\ell, \delta W_\varrho, \delta\kappa, s^2), \quad J = (g_1, \dots, g_6).$$

The coefficient matrix C_J has entries $[g_j]\mathcal{G}_i^{[2]}$, where degree is measured in $(s, \delta, W_\ell, W_\varrho, \kappa)$ after implicit elimination. Its determinant is

$$\det C_J = -\frac{5q\varrho}{1152\kappa_0^5}.$$

To identify the ideal generated by these residuals, work in the completed local ring of these five coordinates. The implicit graph is zero in $(w, E_0, e_1^{\text{alg}}, e_2^{\text{alg}})$ modulo (s, δ) . In each of the two quotient rings

$$\frac{\mathcal{O}}{(s^2, W_\ell, W_\varrho, \kappa)}, \quad \frac{\mathcal{O}}{(s^2, W_\ell, \kappa, \delta)}$$

it is exactly

$$(w, E_0, e_1^{\text{alg}}, e_2^{\text{alg}}) = -s(\varrho/\ell, 4, 2\varrho, 2\varrho/\ell).$$

Substitution in the regularized relations gives zero in all three quotient rings; in the first of the two displayed rings δ is unrestricted. Thus each $\mathcal{G}_i$ belongs to

$$\begin{aligned} (s, \delta) \cap (s^2, W_\ell, W_\varrho, \kappa) \cap (s^2, W_\ell, \kappa, \delta) \\ = (sW_\ell, s\kappa, \delta W_\ell, \delta W_\varrho, \delta\kappa, s^2) = J. \end{aligned}$$

The unit determinant of C_J gives $J = (\mathcal{G}_1, \dots, \mathcal{G}_6)$ by Nakayama's lemma. In particular $\tau = \kappa^2\delta \in J$ on this fibre. In each of these paired cases, pass to the ratio neighbourhood by applying Lemma C.4 to the differential ideal of the closure of the state image and the lifted coordinate fields. The exclusion therefore holds on a reduced formal arc approaching this boundary stratum.

Simultaneous vanishing of the elimination minors requires $a\kappa_0^2 = mq = 3/4$, whereas the resonance gives $a\kappa_0^2 = -\kappa_0(1 + \kappa_0) \leq 1/4$. Cyclic permutations give the three paired orientations.

In the supercritical full- $\overline{\mathbf{R}}$ inactive chart the canonical paired amplitudes obey $b_0^2 = -q$, $c_0^2 = -q/\kappa_0$. The three pivots for eliminating the auxiliary coordinates are $3/56$, $-5q^3/(24\kappa_0^2)$ and $-5q^2/(12\kappa_0)$. The residuals vanish on the whole line with $\sigma = s = w = E_0 = E_1 = E_2 = 0$, keeping the second blowup parameter free. Off $a\kappa_0^2 = mq$, six normals have a unit coefficient determinant. At that resonance elimination of five normal variables leaves a one dimensional kernel; its quadratic coefficient after implicit elimination is $-16q(\kappa_0^2 + \kappa_0 + 13/4)/\kappa_0$, which is nonzero since $\kappa_0^2 + \kappa_0 + 13/4 = (\kappa_0 + 1/2)^2 + 3$. The kernel vanishes on a reduced arc and the equations used for elimination force $\sigma = 0$. Cyclic label changes transport the two factors in the principal value ratios $mq \pm a\kappa_0^2$ explicitly; in an ordered original labeling $0 < a\kappa_0^2, mq < 1$, so the transformed factors have only the one already treated resonance.

For supercritical $\overline{\mathbf{R}}$ support of size two the remaining two amplitudes are multiplied, up to unit in the stated localization, by each of the following pairs: $(\zeta - 3\eta, \eta - 3\zeta)$, $(\zeta - 3, 3\zeta - 1)$ or $(\eta + 3, 3\eta + 1)$. Their respective unit combinations are $-(3A + B)/(8\eta) = 1$, $(B - 3A)/8 = 1$ and $(3A - B)/8 = 1$. Thus both amplitudes vanish. For support of size one the compatibility relations multiply the remaining $\overline{p}$ pairs by the six factors in V_0 and force $\overline{p} = 0$. None of these factors vanishes at the cyclic section.

For critical full derivative support with at least two amplitude pairs, the first integrals and coefficient equations exclude an invariant boundary component dominating the ratio plane (Appendices C.3 and C.7). On the strata treated by local normal ideals, their radical inclusions force $\tau = 0$ on every reduced formal arc through such a component, excluding an approach from the affine locus. At the ordinary exceptional family, local containment in $\mathcal{F}_{\text{in}}$ and its generic exclusion remove the remaining dominant component. The generic affine fibre therefore has no projective boundary. It is proper and affine, hence finite, and $\dim Y_{\text{gen}} = 0$ when $\dim \mathcal{B} = 2$.

C.9 Relative identities with one nonzero R coordinate

Work in the relative local ring of the single-coordinate boundary. Let I be the ideal of necessary regularized equations in this ring. Put $k = a\kappa_0$, $m = 1 + k$, $q = 1 + \kappa_0$, and

$$\alpha = \frac{\zeta - \eta}{\zeta(\eta + 1)}, \quad \beta = \frac{\eta(\zeta - 1)}{\zeta(\eta + 1)}.$$

On the $\overline{p}_{13} = \overline{\mathbf{R}}_3 = 1$ incidence chart, the exact coordinates are

$$\begin{aligned} \overline{p}_{12} &= x, \quad \overline{p}_{13} = 1, \quad \overline{p}_{21} = z, \quad \overline{p}_{23} = w, \\ \overline{p}_{31} &= \frac{tz - \kappa e_2^{\text{alg}}}{\beta}, \quad \overline{p}_{32} = \frac{xwt}{\beta} + \frac{\kappa(xe_1^{\text{alg}} - e_0^{\text{alg}})}{\alpha}, \\ \overline{\mathbf{R}}_1 &= \frac{\alpha wt}{\beta} + \kappa e_1^{\text{alg}}, \quad \overline{\mathbf{R}}_2 = t, \quad \overline{\mathbf{R}}_3 = 1, \quad \tau = \kappa^2 \delta. \end{aligned}$$

The incidence matrix gives $L(\bar{p})\overline{\mathbf{R}} = \kappa \mathbf{e}^{\text{alg}}$ exactly. Write $s = x + z$. The four boundary graph functions are

$$\begin{aligned} f_0 &= -2xw + \frac{4\eta\zeta - 2\eta - 4\zeta^2 + 2\zeta}{\eta\zeta} sw, \\ f_1 &= -\frac{2\zeta(\eta - 1)}{\eta(\zeta - 1)} w, \\ f_2 &= 2x + \frac{2\eta\zeta - 2\eta - 4\zeta^2 + 2\zeta}{\zeta} s, \\ d &= \frac{4\kappa_0(\eta + 1)^3}{q\eta^2} sw. \end{aligned}$$

Let $I_{\text{graph}} = (e_0^{\text{alg}} - f_0, e_1^{\text{alg}} - f_1, e_2^{\text{alg}} - f_2, \delta - d)$ denote their ideal. These functions are substituted into the compatibility relation polynomials.

Write $\mathcal{R}_\nu$, $\mathcal{H}_n$ and Q_j^{reg} for the regularized representatives with respective factors $\kappa^8\delta^4$, $\kappa^{11}\delta^5$, and $\kappa^7\delta^3$ for Q_0 or $\kappa^8\delta^4$ for Q_1, Q_5, Q_6 .

Away from $k + w^2 = 0$, put $c = (3k - w^2)/(k + w^2)$. The 28 normalized relations belong to

$$J_9 = (I_{\text{graph}}, \kappa s, ts, s^2, (t + 2\kappa)x, x^2 - cxs).$$

In the quotient over $\mathbb{Q}(k, \kappa_0, \eta, \zeta, w)$, multiplication in the quotient gives $\bar{p}_{31} = \bar{p}_{32} = 0$, $\delta^2 = \kappa\delta = 0$;

At the attachment $w^2 = -k$, regard k and w as independent parameters. Let C be the third even relation on the boundary graph. Its full polynomial is

$$\begin{aligned} C &= \frac{2k}{m^2}(x^3z + xz^3) + \frac{10x^2 - 4xz - 6z^2}{m}w^2 \\ &\quad + \frac{k}{m}(-6x^2 - 4xz + 10z^2). \end{aligned}$$

Modulo s^2 it is

$$\begin{aligned} C &= -\frac{4k}{m^2}x^4 + \frac{8k}{m^2}x^3s \\ &\quad + \frac{8(k + w^2)}{m}x^2 - \frac{8(3k - w^2)}{m}xs. \end{aligned}$$

The relative ideal is

$$J_{10} = (I_{\text{graph}}, \kappa s, ts, s^2, (t + 2\kappa)x, C, \kappa x^3).$$

Multiplying the congruence modulo s^2 by $m\kappa/8$ and reducing modulo (κs) gives

$$(k + w^2)\kappa x^2 \equiv \frac{m}{8}\kappa C + \frac{k}{2m}\kappa x^4 \pmod{(\kappa s)}.$$

Since C , κx^3 and κs belong to J_{10} , this gives $(k + w^2)\kappa x^2 \in J_{10}$. Replacing C by its displayed remainder modulo s^2 subtracts a multiple of the generator s^2 . The ideal remains J_{10} , and the coefficient matrix used in Nakayama's lemma is multiplied by an invertible elementary matrix.

The twenty-eight relations and the three pair relations vanish modulo J_{10} ; the third even relation reduces to C . The incidence pullback of each pair relation has numerator divisible by κ , so its regularized representative is polynomial. The derivative relations H_n retain their $(\hat{p}, \mathbf{R})$ components of bidegrees $(4, 1)$, $(5, 0)$ and $(1, 3)$ and the $(3, 2)$ incidence module; their remaining terms belong to $(\delta^2, \kappa\delta)$.

For the reverse inclusion, at the cyclic base, the four rows used for implicit elimination and six remaining relation classes give

$$\det_{10} = \frac{\kappa_0 q w^{10}}{4032(w^2 - 1)}, \quad \text{Norm}_{w^2=-k}(\det_{10}) = \frac{\kappa_0^2 q^2 k^{10}}{16257024m^2}.$$

This is a unit for every admissible triple of principal values and for both complex amplitude roots. The coefficient matrix has independent columns for the six classes modulo the maximal ideal times J_{10} ; Nakayama then gives equality with the necessary ideal of compatibility relations. Finally,

$$\tau = \kappa^2(\delta - d) + \frac{4\kappa_0(\eta + 1)^3}{q\eta^2} w\kappa(\kappa s) \in J_{10}.$$

The ideal inclusion applies to formal arcs over the base and w .

The conic and its derivative relations. The partial gauge with active coordinate $\hat{p}_{21}$ is

$$\mathcal{D}_{21} = \hat{p}_{21}\hat{E}_2 - c_2^{\text{diag}}R_2\partial_{R_2}.$$

By the definition of $\hat{E}_2$ in Appendix C.1,

$$\mathcal{D}_{21}Q_6 = \hat{p}_{21}\mathcal{V}_2Q_6 - (\hat{p}_{21}g_2Q_0 + c_2^{\text{diag}}R_2)\partial_{R_2}Q_6.$$

It vanishes on the actual state image, since Q_0, R_2, Q_6 belong to its invariant ideal. Let $\mathcal{G}_{21}$ be the polynomial extension of $\kappa^{13}\delta^6\mathcal{D}_{21}Q_6$ under the incidence substitution. For a part of ordinary degree d and R degree d_R , the exceptional factors are $\kappa^{13-2d-d_R}\delta^{6-d}$. The two negative κ exponents are $-1, -2$, and their substituted numerators are divisible by κ, κ^2 , respectively. After these exact divisions the polynomial satisfies

$$\mathcal{G}_{21} \in (t, \kappa).$$

On the graph I_{graph} with $t = \kappa = 0$, all $\mathcal{R}_\nu, Q_0^{\text{reg}}, \mathcal{H}_n$ vanish, and the even equations restrict to

$$\begin{aligned} Q_1^{\text{reg}} &= s^2 \left(-\frac{2xz}{m} - \frac{6w^2}{k} + 26 \right), \\ Q_5^{\text{reg}} &= s^2 \left(-\frac{2kxz}{m} + 26w^2 - 6k \right), \\ Q_6^{\text{reg}} &= C. \end{aligned}$$

Consequently these sources and $\mathcal{G}_{21}$ belong to

$$\mathcal{J}_{\text{conic}} = (I_{\text{graph}}, C, s^2, t, \kappa).$$

Here η, ζ are retained in the coefficient ring.

For the conic $x^2 = 2m(k + w^2)/k$, use the equation $C = 0$ to lift x to the dual algebra with $s^2 = 0$. If x_s denotes its first coefficient, it is determined by

$$x_s = -\frac{C_z}{C_x - C_z}, \quad z = -x + s,$$

where the derivatives are evaluated on the reduced conic. Over $\mathbb{Q}(k, \kappa_0, \eta, \zeta, w)$, the constant and linear coefficients of both regularized coordinate derivatives of Q_6 vanish. Hence those derivatives belong to the relative ideal of the nonreduced conic.

On the incidence chart, the two homogeneous parts of the differentiated equations of R degrees three and two pull back to exact multiples of κ^2 and κ , respectively. For the first field its denominator is $\bar{p}_{13} = 1$; for the second it is $\bar{p}_{21} = z$, a unit on the conic chart in question. At the cyclic ratio section, take the coefficient classes of

$$\mathcal{R}_2, \mathcal{R}_3, Q_0^{\text{reg}}, \mathcal{G}_{21}, \mathcal{H}_{14}, Q_1^{\text{reg}}, Q_5^{\text{reg}}, Q_6^{\text{reg}}$$

in $\mathcal{J}_{\text{conic}}/\mathfrak{m}\mathcal{J}_{\text{conic}}$, relative to the ordered generators

$$(e_0^{\text{alg}} - f_0, e_1^{\text{alg}} - f_1, e_2^{\text{alg}} - f_2, \delta - d, C, s^2, t, \kappa).$$

Here $\mathfrak{m}$ is the maximal ideal at the conic point. The norm of the determinant of this eight-column coefficient matrix over $x^2 = 2m(k + w^2)/k$ is

$$-\frac{9\kappa_0^2 q^2 w^2 m^7 (w^2 - k)^4 (w^2 + k)^{11}}{98k^{11}}.$$

It is nonzero for $w(w^2 - k)(w^2 + k) \neq 0$ at both roots. The selected relations belong to $\mathcal{J}_{\text{conic}}$, and their classes span this quotient. Nakayama's lemma gives $\mathcal{J}_{\text{conic}} \subset I$ in the relative local ring. Since $\kappa \in \mathcal{J}_{\text{conic}}$, this gives $\tau \in I$. The attachment $w^2 = -k$ was treated by J_{10} .

At $w^2 = k$, $x^2 = 4m$, the coefficient matrix of the 28 complete relations in the eight generators of $\mathcal{J}_{\text{conic}}$ has rank six. Its kernel has dimension two. The two full differentiated relations of Q_6 , restricted to a basis of this kernel, have coefficient matrix

$$\begin{pmatrix} -640kw & -16kw \\ -384kx & 0 \end{pmatrix}.$$

Its determinant $-6144k^2wx$ is nonzero at all four roots. The augmented normal matrix therefore has rank eight. These relations belong to $\mathcal{J}_{\text{conic}}$, so Nakayama's lemma gives $\mathcal{J}_{\text{conic}} \subset I$ in the local ring at this wall. In particular, $\tau \in I$.

The conic endpoint. Consider the point

$$\eta = 1, \quad \zeta = -1, \quad x = x_b, \quad z = -x_b, \quad x_b^2 = 2m,$$

with

$$w = e_0^{\text{alg}} = e_1^{\text{alg}} = \delta = t = \kappa = 0, \quad e_2^{\text{alg}} = 2x_b.$$

In the algebraic local ring at this point we prove

$$\kappa\delta \in I, \quad \tau = \kappa^2\delta \in I.$$

Proof. The complete regularized sources have the relative inclusions

$$\mathcal{R}_2, \mathcal{R}_3, Q_0^{\text{reg}}, \mathcal{H}_6, \mathcal{H}_{14}, \mathcal{H}_{16} \in (w, e_0^{\text{alg}}, e_1^{\text{alg}}, \delta).$$

Use the six equations

$$\mathcal{R}_2, \mathcal{R}_3, Q_0^{\text{reg}}, Q_1^{\text{reg}}, Q_6^{\text{reg}}, \mathcal{G}_{21}$$

as implicit equations for $(x, e_0^{\text{alg}}, e_1^{\text{alg}}, \delta, t, \kappa)$, with $s = x + z$, $v_{\text{end}} = e_2^{\text{alg}} - f_2$ and w fixed. Their Jacobian determinant at the point is

$$\frac{36}{7}k^2qm^2x_b \neq 0.$$

Thus the quotient of the completed local ring by these six equations is

$$\mathbb{C}[[v_{\text{end}}, s, w, \eta - 1, \zeta + 1]].$$

A tilde denotes restriction to this implicit graph. Set

$$\begin{aligned} \mathcal{B}_{\text{end}} &= Q_0^{\text{reg}} + \frac{27}{49mq}\mathcal{H}_6 - \frac{12}{49\kappa_0m}\mathcal{H}_{14} + \frac{39x_b}{98m}\mathcal{H}_{16}, \\ \mathcal{F}_{\text{end}} &= Q_1^{\text{reg}} - \frac{1}{k}Q_5^{\text{reg}}. \end{aligned}$$

For the first three implicit equations, the Jacobian in $(e_0^{\text{alg}}, e_1^{\text{alg}}, \delta)$ is $mq/3584$. Their solution at $w = 0$ is $e_0^{\text{alg}} = e_1^{\text{alg}} = \delta = 0$. The relative inclusions above and implicit uniqueness give

$$\tilde{\delta} \in (w), \quad \tilde{\mathcal{B}}_{\text{end}} \in (w).$$

Next restrict to $v_{\text{end}} = s^2 = 0$. The derivative of C in x at fixed s has value $-16kx_b/m \neq 0$. Solving $C = 0$, putting $e_i^{\text{alg}} = f_i$, $\delta = d$, $t = \kappa = 0$, gives a solution of the six implicit equations over this quotient ring, by their $\mathcal{J}_{\text{conic}}$ memberships. Uniqueness of the implicit solution gives

$$\tilde{\kappa}, \tilde{\mathcal{B}}_{\text{end}}, \tilde{\mathcal{F}}_{\text{end}} \in (v_{\text{end}}, s^2).$$

Multiplication by w is injective modulo (v_{end}, s^2) , since this quotient is free with basis $1, s$ over $\mathbb{C}[[w, \eta - 1, \zeta + 1]]$. Hence

$$(w) \cap (v_{\text{end}}, s^2) = w(v_{\text{end}}, s^2).$$

There is a matrix of formal power series $\mathbf{A}_{\text{end}}$ such that

$$\begin{pmatrix} \tilde{\mathcal{B}}_{\text{end}} \\ w\tilde{\mathcal{F}}_{\text{end}} \end{pmatrix} = w\mathbf{A}_{\text{end}} \begin{pmatrix} v_{\text{end}} \\ s^2 \end{pmatrix}.$$

Both source combinations have zero state differential at the endpoint. Restricting their Hessians to the tangent space of the implicit graph gives the quadratic terms, with the ratio parameters evaluated at their center,

$$\begin{aligned} \tilde{\mathcal{B}}_{\text{end}}^{(2)} &= -\frac{3x_b}{224m}v_{\text{end}}w, \\ \tilde{\mathcal{F}}_{\text{end}}^{(2)} &= \frac{1}{8}v_{\text{end}}^2 + \frac{7}{2}v_{\text{end}}s + 32s^2. \end{aligned}$$

Thus $(\mathbf{A}_{\text{end}})_{11}(0) = -3x_b/(224m)$, $(\mathbf{A}_{\text{end}})_{21}(0) = 0$ and $(\mathbf{A}_{\text{end}})_{22}(0) = 32$. It follows that

$$\det \mathbf{A}_{\text{end}}(0) = -\frac{3x_b}{7m} \neq 0.$$

Inverting this matrix gives

$$w(v_{\text{end}}, s^2) = (\tilde{\mathcal{B}}_{\text{end}}, w\tilde{\mathcal{F}}_{\text{end}}).$$

The two memberships for $\tilde{\kappa}$ and $\tilde{\delta}$ imply $\tilde{\kappa}\tilde{\delta} \in (\tilde{\mathcal{B}}_{\text{end}}, \tilde{\mathcal{F}}_{\text{end}})$. Lifting through the quotient by the six implicit equations puts $\kappa\delta$ in the completed necessary ideal. Faithful flatness of completion contracts this membership to the algebraic local ring [27, Lemma 10.97.3, Tag 00MC]. Multiplication by κ gives $\tau \in I$. Since $kmq \neq 0$ and $x_b^2 = 2m$, the two determinants are nonzero at either root x_b . $\square$

The horizontal conormal matrix. Put $K = \mathbb{Q}(k, \kappa_0, \eta, \zeta)$. Use the incidence chart

$$\begin{aligned} \bar{p} &= \left(1, y, z, w, \frac{tz - \kappa e_2^{\text{alg}}}{\beta}, \frac{v - \kappa e_0^{\text{alg}}}{\alpha} \right), \\ \bar{\mathbf{R}} &= (v, t, 1), \quad I = yv - (\alpha/\beta)tw - \kappa e_1^{\text{alg}}, \quad I_0 = yv - (\alpha/\beta)tw. \end{aligned}$$

Here $I = 0$ is the remaining incidence equation. For each relation F we substitute

$$(\hat{p}, \mathbf{R}) = \left(\frac{\bar{p}}{\kappa^2 \delta}, \frac{\bar{\mathbf{R}}}{\kappa^3 \delta} \right).$$

The representatives are determined by the incidence identities $L\bar{\mathbf{R}} = \kappa(e_0^{\text{alg}}, e_1^{\text{alg}}, e_2^{\text{alg}})^\top$ modulo (I) : the $(3, 1)$ part of R_ν is replaced by $(DC)_{\nu,*}(e_0^{\text{alg}}, e_1^{\text{alg}}, e_2^{\text{alg}})^\top$, the $(3, 2)$ part of H_n by $\sum_{i=1}^3 A_{n,i}e_{i-1}^{\text{alg}}$, the $(1, 2)$ part of Q_0 by $\sum_{i=1}^3 B_{0,i}e_{i-1}^{\text{alg}}$, and the $(2, 2)$ and $(3, 1)$ parts of Q_j by $c_j^{\text{even}}(e_{i(j)-1}^{\text{alg}})^2$ and $\sum_{i=1}^3 B_{j,i}e_{i-1}^{\text{alg}}$, respectively. Here $e^{\text{alg}} = (e_0^{\text{alg}}, e_1^{\text{alg}}, e_2^{\text{alg}})^\top$ and the coefficients $B_{0,i}$ are defined by $(Q_0)_{12} = \sum_{i=1}^3 B_{0,i}(LR)_i$. For a regularizing factor $\kappa^{d_{\text{reg}}} \delta^{e_{\text{reg}}}$,

each other part of bidegree $(d_{\widehat{p}}, d_R)$ is multiplied by $\kappa^{d_{\text{reg}}-2d_{\widehat{p}}-3d_R} \delta^{e_{\text{reg}}-d_{\widehat{p}}-d_R}$ and evaluated on the chart.

To specify the three differentiated relations, use the derivation

$$\mathcal{D}_{\text{axis}} = \widehat{p}_{12} \widehat{E}_1 - c_1^{\text{diag}} R_3 \partial_{R_1}, \quad c_1^{\text{diag}} = \frac{2(\eta - \zeta)}{\eta \zeta}.$$

Its coefficient derivatives include the ratio coordinates η, ζ . Let G_0, G_1, G_2 be the regularized representatives of $\mathcal{D}_{\text{axis}} Q_5, \mathcal{D}_{\text{axis}} R_7, \mathcal{D}_{\text{axis}} R_8$, with factors $\kappa^{13} \delta^6, \kappa^{14} \delta^6, \kappa^{14} \delta^6$, respectively. Their representatives are fixed as follows. Put $a_{\text{reg}} = 13$ for G_0 and 14 for G_1, G_2 . A part of ordinary degree d and R degree d_R has factor $\kappa^{a_{\text{reg}}-2d-d_R} \delta^{6-d}$. For a nonnegative exponent take its direct substitution. For a negative exponent start with its substituted polynomial P_0^{reg} and put

$$C_j = \frac{P_j^{\text{reg}}|_{\kappa=0}}{I_0}, \quad P_{j+1}^{\text{reg}} = \frac{P_j^{\text{reg}} - C_j I}{\kappa}.$$

The incidence identities make these quotients polynomial; the regularized part is $\delta^{6-d} P_{2d+d_R-a_{\text{reg}}}^{\text{reg}}$.

For a polynomial F independent of κ , denote by $\text{rem}(F)$ and $\text{quo}(F)$ the remainder and quotient in division by the monic polynomial I_0 , with leading monomial yv . Thus

$$F = \text{rem}(F) + I_0 \text{quo}(F),$$

and no monomial of $\text{rem}(F)$ is divisible by yv . Define

$$\begin{aligned} \lambda^W &= \left(1, -\frac{8m(\eta+1)}{3\eta}, \frac{8m\zeta(\eta+1)^2}{3\eta(\eta-\zeta)}, \frac{8m(\eta+1)}{3\eta} \right), \\ F^W &= (G_0, wG_1, vG_1, yG_2), \\ C_W &= \sum_{j=0}^3 \lambda_j^W \text{quo}(F_j^W|_{\kappa=0}), \\ W &= \frac{\sum_{j=0}^3 \lambda_j^W F_j^W - C_W I}{\kappa}. \end{aligned}$$

The numerator vanishes at $\kappa = 0$, so W is polynomial.

The seven relations for the second divided relation are

$$F^h = (t\mathcal{R}_5, \mathcal{H}_4, w\mathcal{H}_9, \mathcal{H}_{11}, y\mathcal{H}_{13}, v\mathcal{H}_{16}, t\mathcal{H}_{17}).$$

Let $\mathcal{M}$ be the finite set of monomials occurring in $\text{rem}(F_j^h|_{\kappa=0})$, and define the coefficient matrix

$$A_{\mu, j+1}^h = [\mu] \text{rem}(F_j^h|_{\kappa=0}), \quad \mu \in \mathcal{M}, \quad 0 \leq j \leq 6.$$

It has rank six over K , and its last six columns are independent. Consequently the equations

$$A^h \mathbf{c}^h = 0, \quad c_0^h = 1$$

define a unique vector $\mathbf{c}^h \in K^7$. Put

$$C_h = \sum_{j=0}^6 c_j^h \text{quo}(F_j^h|_{\kappa=0}), \quad h_0 = \frac{\sum_{j=0}^6 c_j^h F_j^h - C_h I}{\kappa}.$$

The defining kernel identity makes h_0 polynomial. For the horizontal family put

$$s_h = -\frac{2\eta(\zeta+1)}{\zeta(\eta+1)}, \quad e_0^{\text{alg}} = E + s_h \ell, \quad v = (\alpha + \kappa s_h) \ell.$$

Denote this substitution by Φ_{hor} . Its seven transverse coordinates are

$$n = (y, z, w, E, e_1^{\text{alg}}, e_2^{\text{alg}}, \delta),$$

and its coefficient ring is $T = K[t, \ell, \kappa]$. Set

$$C_h^{\text{hor}} = (\Phi_{\text{hor}}^* C_h)|_{n=0}, \quad h = \Phi_{\text{hor}}^* h_0 + \frac{C_h^{\text{hor}}}{\kappa} \Phi_{\text{hor}}^* I.$$

Here C_h^{hor} is divisible by κ in T . The ordered thirty relations are

$$\Phi_{\text{hor}}^*(\mathcal{R}_0, \dots, \mathcal{R}_5, Q_0^{\text{reg}}, \mathcal{H}_0, \dots, \mathcal{H}_{17}, Q_1^{\text{reg}}, Q_5^{\text{reg}}, Q_6^{\text{reg}}, W), \quad h.$$

They vanish at $n = 0$. Let $(\mathbf{N}_{\perp})_{ij}$ be the coefficient of the j th component of n in the i th relation modulo $(n)^2$, for $1 \leq i \leq 30$, $1 \leq j \leq 7$. The incidence conormal is

$$((\alpha + \kappa s_h)\ell, 0, -(\alpha/\beta)t, 0, -\kappa, 0, 0).$$

The products $w\mathcal{H}_9$ and $y\mathcal{H}_{13}$ have zero conormal because both factors vanish on the horizontal family.

Set

$$I = (4, 7, 26, 29), \quad R = (2, 5, 7, 3), \quad C = (1, 4, 6), \quad J = (27, 28, 30).$$

Write $A = (\mathbf{N}_{\perp})_{I,R}$ and $B = (\mathbf{N}_{\perp})_{I,C}$. The relative polynomial identity is

$$(\det A)(\mathbf{N}_{\perp})_{J,C} - (\mathbf{N}_{\perp})_{J,R} \text{adj}(A)B = 0.$$

At the horizontal origin the two base charts give

$$\det A|_{\eta=1, \zeta=-1} = -\frac{3qk}{224\kappa_0 m^2}, \quad \det A|_{\eta=2, \zeta=-1} = -\frac{32qk}{1701\kappa_0 m^2}.$$

Both determinants are units for every admissible triple of principal values. The coefficient denominators specialize to nonzero rational constants.

Transport to the chart with three nonzero coordinates. Let M be the ordinary boundary 37×6 matrix of linear normal terms. For $1 \leq i \leq 28$, the original rows, before adding derivative rows, transform to the full-R chart by

$$N_i(\kappa) = d_i(\kappa)M_i(l, \varrho/\kappa, 1/\kappa)J(\kappa), \quad d_i = \begin{cases} \kappa, & i = 1 \text{ or } i \geq 11, \\ 1, & 2 \leq i \leq 10. \end{cases}$$

Here J is the Jacobian of the six transverse coordinate changes. Each transformed entry is regular at $\kappa = 0$. The matrix of linear normal terms and its variation are $N(0)$ and $\partial_{\kappa}N(0)$.

The two short odd factors and the even determinant factor are

$$P = \frac{\varrho^2}{\beta^2} - 5\kappa_0 l^2 - 4q, \quad Q = \frac{\varrho^2}{\beta^2} + 4\kappa_0 l^2 + 5q, \quad (ml^2 - k)(ml^2 + k).$$

They satisfy $Q - P = 9(\kappa_0 l^2 + q)$. On the two amplitude hypersurfaces work in the quadratic algebras $l^2 = \pm k/m$. Elimination of the columns corresponding to the solved variables in $N(0)$ gives the Schur blocks, and $\partial_{\kappa}N(0)$ gives the transverse variation.

C.10 Algebraicity on the ordinary boundary family

The two Rees derivations on the ordinary projective closure annihilate ℓ and have base components

$$\begin{aligned} X_0\eta &= \ell\eta(v-2\eta)/(\eta+1), & X_0\zeta &= \ell\eta(v+2)/(\eta+1), \\ X_2\eta &= -v\eta, & X_2\zeta &= -\eta\zeta(v+2)/(\eta+1). \end{aligned}$$

Their determinant is $-\ell\eta^2(v+2)\Theta/(\eta+1)^2$. The sheet $\Theta = 0$ is transverse: there $X_2\Theta = 4(\zeta-1)\eta(\eta+1)/(\zeta-\eta-1)$. The ordinary projective closure uses weight one for $\bar{p}$ and $\mathbf{R}$. Lemma C.3 therefore gives derivations of its invariant boundary of the state image.

The horizontal connection also has the first integral

$$\mathcal{J}_{\text{in}} = \frac{u^2}{\beta^2(v+2)(v-2\eta)}.$$

The quantities Θ and β are those of Section 8.2.

We give the coefficient identities used in Lemma 8.7. All matrices are formed from the relations and their normal coefficients over the rational field in a, κ_0, η, ζ . The extra factor $G_0 = \eta\zeta - \eta + \zeta$ is nonzero at the cyclic section. It can be inverted on the base open set used here. The amplitude ℓ remains an indeterminate, and $\ell \neq 0$.

Near the family (8.3) use the normal coordinates

$$\bar{p} = (1, n_0, n_1, n_2, n_3, \ell), \quad \mathbf{R} = (\lambda_{\text{in}}(v)\ell + n_{\perp}, u, v), \quad \mathbf{n} = (n_0, n_1, n_2, n_3, n_{\perp}, \tau). \quad (\text{C.9})$$

All forty homogenized compatibility relations vanish identically on this family, with η, ζ, ℓ, u, v retained as parameters. Consequently each belongs to $(\mathbf{n})$ in the completed relative local ring. If a six-column minor of the matrix of linear normal terms is a unit, the corresponding six relations generate $(\mathbf{n})$: write their column of polynomials as $A(\mathbf{n})\mathbf{n}$, and invert A , whose constant determinant is that minor. In particular the compatibility ideal contains τ , contrary to saturation of the closure of the state image. Every such minor therefore vanishes at the generic point of a boundary component of the state image in this family.

At the locus $u = 0$, the linear coefficient of $n_{\perp}$ is divisible by u in the parameter ring. Thus the relations belong to

$$J_7 = (n_0, n_1, n_2, n_3, \tau, un_{\perp}, n_{\perp}^2).$$

Indeed every normal monomial of degree at least two either contains one of the first five generators or is a multiple of $n_{\perp}^2$. A unit seven-column matrix for these generators gives $J_7 \subset I$ by Nakayama's lemma, and again $\tau \in I$. Thus its seven-column minors also vanish on the saturated boundary. The coupled coefficients give necessary equations for the boundary component.

The two normalized horizontal derivations are obtained by inverting the displayed 2×2 base-component matrix of X_0, X_2 . They commute and annihilate ℓ and $\mathcal{J}_{\text{in}}$. Their ζ components are

$$D_{\zeta}v = \frac{(v-2\eta)(v+2)}{\Theta}, \quad D_{\zeta}\ell = 0, \quad D_{\zeta}(u/\beta) = \frac{v+1-\eta}{\Theta}(u/\beta). \quad (\text{C.10})$$

The connection satisfies $[D_{\eta}, D_{\zeta}] = 0$ and $D_{\eta}\mathcal{J}_{\text{in}} = D_{\zeta}\mathcal{J}_{\text{in}} = 0$. The last identity in (C.10) also follows from $D_{\zeta}\mathcal{J}_{\text{in}} = 0$ on a dense open set and hence as a rational identity on the whole family. It specializes to $D_{\zeta}(u/\beta) = -(u/\beta)/(2\eta)$ on $v = 2\eta$.

Away from $u = 0$ and $v = -2$, two six-column normal minors have the factorizations

$$\Delta_j = c_j u(v+2)\ell^{4-j}P_j(\ell, u, v), \quad j = 0, 1,$$

where c_j are units in the stated base localization. The factor u comes from the normal column whose coefficient is

$$\frac{\zeta^2 G_0}{7(\zeta - 1)^2(\zeta - \eta)} u.$$

After replacing u^2 by $\beta^2 \mathcal{J}_{\text{in}}(v + 2)(v - 2\eta)$, the coefficient of v^5 in P_0 is

$$\begin{aligned} & \frac{km^2\eta^5(\zeta - 1)^4}{\eta + 1} \mathcal{J}_{\text{in}} \ell^2 (m\ell^2 - k)(m\ell^2 + k) \{A_0(\kappa_0\ell^2 + q) + B_0(\mathcal{J}_{\text{in}} + q)\}, \\ A_0 &= -(7\eta^2\zeta + 7\eta^2 + 17\eta\zeta + 6\eta + 6\zeta + 2), \\ B_0 &= 14\eta^2\zeta + 14\eta^2 + 13\eta\zeta + 15\eta + 3\zeta + 4. \end{aligned}$$

The coefficient determinant $(-7)3 - 14(-6) = 63$ shows that the final brace cannot vanish identically in η, ζ unless $\kappa_0\ell^2 + q = \mathcal{J}_{\text{in}} + q = 0$. On $m\ell^2 = k$, two coefficient minors at the cyclic section have primitive factors

$$f_1 = 21622N - 989(k + q), \quad f_2 = 14006N + 377(k + q), \quad N = a\kappa_0^2.$$

The identity $377f_1 + 989f_2 = 22003428N$ prevents simultaneous vanishing. On $m\ell^2 = -k$, the constant coefficient in v is $-565248k^4 \mathcal{J}_{\text{in}}$, which is nonzero. At the remaining possibility $\kappa_0\ell^2 + q = \mathcal{J}_{\text{in}} + q = 0$, the v^4 coefficient of P_1 is

$$\begin{aligned} & -\frac{2aq^3}{\kappa_0^2} (\zeta - 1)^4 \eta^4 (7\eta\zeta + 7\eta + 3\zeta + 4)(k + q)m^2 \\ & \cdot \{(-8N - 5(k + q))\eta + (24N + 18(k + q))(\zeta - 1)\}. \end{aligned}$$

The two coefficients in braces satisfy

$$18(-8N - 5(k + q)) + 5(24N + 18(k + q)) = -24N.$$

If $k + q = 0$, this remaining possibility is already the negative amplitude hypersurface. Thus at least one normal minor gives a nonzero polynomial equation for v over the constant field with η, ζ adjoined, for every admissible triple of principal values.

When $u = 0$, retain the two coupled columns associated with $un_{\perp}$ and $n_{\perp}^2$, as well as the five remaining normal columns. Their pivots are respectively

$$\frac{\zeta^2 G_0}{7(\zeta - 1)^2(\zeta - \eta)}, \quad \frac{2\zeta^2 \eta^2}{(\zeta - \eta)^2(\zeta - 1)^2}.$$

The seven-column determinants factor into these pivots, a 2×2 normal minor, and a 3×3 normal minor. The two 2×2 minors cannot both vanish identically: off $m\ell^2 = -k$ the first suffices, while on that hypersurface the remaining factor is $-48k^3(4\eta\zeta - 4\eta + 3\zeta - 1)$. The two 3×3 factors have the form $c_j(v + 2)\ell^{3-j}A_j(v)$, where A_j has degree four and leading coefficients

$$m\eta^4(3\eta + 2)\ell^2(5\kappa_0\ell^2 + 4q), \quad 3m\eta^4\zeta\ell^2(4\kappa_0\ell^2 + 5q).$$

They cannot vanish simultaneously, since

$$5(4\kappa_0\ell^2 + 5q) - 4(5\kappa_0\ell^2 + 4q) = 9q.$$

All four pairings of the two 2×2 factors with the two 3×3 factors occur as seven-column minors, multiplied by the two displayed pivots. At least one such product is therefore a nonzero polynomial equation for v . This proves the algebraicity assertion at $u = 0$ away from $v = -2$.

On $v = 2\eta$ and $u \neq 0$, expand the six-column minors along the $n_{\perp}$ pivot, a unit multiple of u . The remaining five-column minors have constant terms in u equal to the products

of the preceding 2×2 and 3×3 minors. Their selected rows have zero first $n_\perp$ coefficient identically, so this is an expansion of the normal minors. After removal of their unit factors, the relevant three-column polynomial has highest ζ coefficient

$$-16mq(\eta + 1)^5 \ell^2.$$

It is nonzero for every admissible triple of principal values and nonzero amplitude. The minors therefore give a nonzero equation for u , and hence for u/β . At $u = 0$ the seven-column factorization using the two $n_\perp$ pivots and these same constant terms has a unit minor; the J_7 argument above excludes the component directly.

C.11 Relative ideals on the ordinary boundary

Let $\mathfrak{n}$ be the normal ideal. If $\mathfrak{n}^d \subset J$, the class of a compatibility polynomial modulo J is determined by its normal terms of degree less than d . Indeed, the remaining terms belong to $\mathfrak{n}^d$. This also holds in the completed local ring over the ratio and horizontal parameters.

Use the normal coordinates (C.9) and the function λ_{in} defined in (8.3). For $F \in \mathcal{F}$, let F^{h} be its ordinary homogenization and define its normal coefficient matrices by

$$M_{F,j} = [(\mathbf{n})_j]F^{\text{h}}, \quad 1 \leq j \leq 6, \quad C_{F,\gamma} = [\mathbf{n}^\gamma]F^{\text{h}}, \quad |\gamma| = 2,$$

after substitution of (C.9). Append the three pair relations to obtain the forty-row matrices. Their constant terms vanish on the ordinary boundary family.

Put $t = n_3$, $w = v + 2$, $k = a\kappa_0$, and $m = 1 + k$. On the ordinary chart near the locus where the matrix of linear normal terms has a two dimensional kernel, use the four variables eliminated by the implicit function theorem

$$y = (n_0 + n_3, n_1, n_2, \tau).$$

The relevant ideal is

$$J_9 = (y, un_\perp, wt, n_\perp^2, n_\perp t, t^2). \quad (\text{C.11})$$

Every normal monomial of degree at least two is in this ideal. The linear normal coefficients on the entire family supply the remaining divisibilities: the coefficient of $n_\perp$ is divisible by u , and the coefficient of t is divisible by w . These divisibilities hold in the polynomial parameter ring in u and w . The quadratic normal coefficients determine the coefficient matrix of the displayed generators at their common zero. A maximal minor of that matrix gives the local equality of the ideals generated by the compatibility polynomials.

C.11.1 The two amplitude hypersurfaces

On the positive hypersurface put $\mu = m\ell^2 - k$ and retain

$$J_{10} = (y, un_\perp, wt, n_\perp^2, n_\perp t, \mu t^2, t^3). \quad (\text{C.12})$$

For the ten selected compatibility relations, the pure t^2 coefficients at $w = 0$, with arbitrary u , are 8μ in row 3, $-48\ell\mu$ in row 30, and zero in the other selected rows. Every term of normal degree at least three is already in J_{10} . Together with the whole first-coefficient divisibilities, these exact identities prove membership of the compatibility relations in J_{10} . The cubic columns are $[t^3]F^{\text{h}}$ after the same substitution.

For the negative hypersurface put $\nu = m\ell^2 + k$. The quadratic change of coordinates is

$$y_0 = n_0 + n_3, \quad y_1 = n_1 + mt^2, \quad y_2 = n_2 + m\ell t^2, \quad y_3 = \tau.$$

Use

$$J_{11} = (y_0, y_1, y_2, y_3, un_\perp, wt, n_\perp^2, n_\perp t, \nu t^2, ut^2, t^3). \quad (\text{C.13})$$

Let M_{i,n_1}, M_{i,n_2} be the original linear normal coefficients and $q_{i,tt}$ the pure kernel-square coefficient at $w = 0$. The shifted square coefficient is exactly

$$g_i(\ell, u) = q_{i,tt} - mM_{i,n_1} - m\ell M_{i,n_2}. \quad (\text{C.14})$$

For all forty relations, its remainder modulo ν contains only terms divisible by u . Hence $g_i \in (\nu, u)$ in the parameter ring. All other terms are covered by the displayed linear, mixed, square and cubic generators.

The coefficient of νt^2 at $u = 0$ is obtained by exact polynomial division by ν , in the polynomial ring. Likewise the coefficient of ut^2 is the linear u coefficient of the remainder. The cubic coefficient is computed after the same quadratic change of coordinates.

At the specified hypersurface centre, with $u = w = 0$ and respectively $\mu = 0$ or $\nu = 0$, the ten- or eleven-column matrix represents the selected relations modulo the maximal ideal times the target ideal. A nonzero determinant makes this matrix a unit in the local ring. Equivalently, Nakayama's lemma applied to the finitely generated quotient $J/(I \cap J)$ gives $J \subset I$; compatibility membership gives the reverse containment for the selected ideal. In particular $\tau \in I$. This is an ideal inclusion over the coefficient ring of u, w, μ or ν and the base variables.

C.11.2 Coordinate axes and the additional normal hypersurface

Let $M_j(\ell, u, v)$ and $C_\gamma(\ell, u, v)$ denote the reconstructed first and second coefficients, and let N be the six ordinary boundary normals. At the axis, ℓ is itself a normal and $n_\perp = \mathbf{R}_1 - \lambda_{\text{in}}(v)\ell$. Consequently its Taylor polynomial of degree two is

$$\sum_j (M_j(0, u, v) + \ell \partial_\ell M_j(0, u, v)) N_j + \sum_{|\gamma|=2} C_\gamma(0, u, v) N^\gamma.$$

The omitted terms have total axis-normal degree at least three. The pure ℓ coefficients vanish because the relations vanish on the ordinary boundary family.

For the additional normal hypersurface, write

$$\ell^2 = -\frac{(a-b)\hat{h}_3^2}{b(a-1)}, \quad \kappa = \frac{\zeta - 1}{\zeta}.$$

In this quadratic coefficient algebra substitute

$$N = (-t + \kappa\ell\tau, 0, 0, t, n_\perp, \tau)$$

in the reconstructed quadratic form. The matrix of linear normal terms annihilates this substitution. Include the nine mixed coefficients obtained by differentiating the matrix of linear normal terms with respect to (ℓ, u, v) . These quadratic and mixed coefficients determine the fifteen-column coefficient matrix.

C.11.3 Nonvanishing for every triple of principal values

The matrix for J_{11} splits, after the unit $un_\perp$ column is removed, into a four-column and a six-column block. The latter has at $(\eta, \zeta) = (1, -1)$ the determinant

$$\ell^5 \frac{2048}{3} \frac{qa^2}{m} (k + q).$$

The factor $k + q$ defines a hypersurface in the principal value parameter space. On that entire hypersurface, the base point $(\eta, \zeta) = (1, -2)$ gives

$$\det A = \ell^5 \frac{64512}{25} \frac{q^4}{\kappa_0^2}, \quad \det B = -\frac{155648q^2}{135\kappa_0^3}, \quad U = \frac{20}{189}.$$

The norm over the quadratic amplitude algebra is nonzero, so these matrices are invertible on a nonempty base open set.

C.12 An additional hypersurface on the ordinary boundary

We exclude the additional τ hypersurface over an open set of the ratio base. Fix positive, pairwise distinct principal values and initially order them as $1 < a < b$. The conclusion holds over a nonempty open set of the two base variables for every such fixed triple of principal values.

Use the 37 necessary relations described above, with normal variables ordered as $(\bar{p}_{13}, \bar{p}_{21}, \bar{p}_{23}, \bar{p}_{31}, n_\perp, \tau)$. Put

$$\kappa_0 = \frac{a-b}{a(b-1)}, \quad q = 1 + \kappa_0, \quad \eta = \kappa_0 \hat{h}_2, \quad \zeta = \frac{\kappa_0 \hat{h}_3}{q},$$

With λ_{in} from (8.3), their boundary family is

$$\bar{p} = (1, 0, 0, 0, 0, \ell), \quad \mathbf{R} = (\lambda_{\text{in}}(v)\ell, u, v), \quad \tau = 0.$$

The hypersurface under consideration is

$$u = 0, \quad v = -2, \quad \ell_0^2 = d = -\frac{(a-b)\hat{h}_3^2}{b(a-1)} \neq 0. \quad (\text{C.15})$$

Work over the quadratic coefficient extension containing either ℓ_0 . Set $\kappa = (\zeta - 1)/\zeta$, $g = \ell^2 - d$, and $w = v + 2$. The three hypersurface coordinates are $W = (g, u, w)$; g is a local coordinate because $\ell_0 \neq 0$. Introduce the remaining normal coordinates $n = (k_\perp, n_\perp, \tau)$ and coordinates to be eliminated ρ_0, ρ_1, ρ_2 by

$$\begin{aligned} \bar{p}_{13} &= \rho_0 - k_\perp + \kappa\ell\tau, & \bar{p}_{21} &= \rho_1, \\ \bar{p}_{23} &= \rho_2, & \bar{p}_{31} &= k_\perp, \\ n_\perp &= \mathbf{R}_1 - \lambda_{\text{in}}(v)\ell. \end{aligned} \quad (\text{C.16})$$

The linear module on the entire family. Let M be the 37×6 matrix of linear normal coefficients of the relations along the entire ordinary boundary family. Direct coefficient extraction and polynomial multiplication give, for every row j ,

$$\begin{aligned} M_{j5} &= uA_j, \\ M_{j4} - M_{j1} &= wB_j, \\ M_{j6} + \kappa\ell M_{j1} &= uC_j + wD_j + gE_j, \end{aligned} \quad (\text{C.17})$$

where A_j, B_j, C_j, D_j, E_j are polynomials in ℓ, u, v over $\mathbb{Q}(a, b, \hat{h}_2, \hat{h}_3)$. These identities hold on the whole family, before setting $W = 0$. For example, the last identity is obtained by polynomial division first by u, w and then by $\ell^2 - d$; multiplication recovers the original coefficient exactly. Consequently the part linear in n belongs to the module generated by

$$J = (un_\perp, wk_\perp, u\tau, w\tau, g\tau). \quad (\text{C.18})$$

Use rows 2, 18, 6 of the compatibility matrix to eliminate the three variables. At the hypersurface, let

$$A = M_{(2,18),(2,3)}, \quad d_A = \det A, \quad c = M_{6,1}, \quad \Delta = d_A c.$$

The sparse elimination block has $M_{2,1} = M_{18,1} = M_{6,2} = M_{6,3} = 0$. Writing $(C_{j1}, C_{j2}) = (M_{j2}, M_{j3}) \text{adj } A$, its undivided cokernel identity is

$$\Delta M_j - cC_{j1}M_2 - cC_{j2}M_{18} - d_A M_{j1}M_6 = 0. \quad (\text{C.19})$$

This identity holds in the coefficient algebra $\ell^2 = d$ at $u = 0, v = -2$. The norm of Δ is nonzero on the specified base open set. The formal implicit function theorem therefore solves $(\rho_0, \rho_1, \rho_2) = R(W, n)$, with $R(W, 0) = 0$. The identities on the entire family (C.17) give

$$R_{\text{linear}} = -A_{\text{range}}(W)^{-1}B_{\text{range}}(W)J, \quad R \in (W)(n) + (n)^2.$$

Substituting this expression in every other relation preserves the same five-generator linear module, by the analytic Schur complement formula.

The module of quadratic normal coefficients and its nonvanishing conditions.

Take the terms of normal degree two after implicit elimination. Substitute $k_{\perp} = \ell_0 y$, $du = \ell_0 U$, $dv = \ell_0 V$, $d\ell = D$ and use columns

$$(y^2, yn_{\perp}, y\tau, n_{\perp}^2, n_{\perp}\tau, \tau^2, Dy, Dn_{\perp}, D\tau, \\ Uy, Un_{\perp}, U\tau, Vy, Vn_{\perp}, V\tau).$$

The corresponding powers of ℓ_0 are

$$(2, 1, 1, 0, 0, 0, 1, 0, 0, 2, 1, 1, 2, 1, 1).$$

Reduction modulo $\ell_0^2 - d$ gives rank three on columns $(1, 3, 4, 6, 12)$ and rank six on columns $(2, 5, 9, 11, 13, 15)$; the other four columns vanish. In the original coordinates the quadratic relations have independent generators J and

$$Q = (n_{\perp}^2, k_{\perp}n_{\perp}, n_{\perp}\tau, P_{\text{hom}}(k_{\perp}, \tau)), \\ P_{\text{hom}}(k_{\perp}, \tau) = \tau^2 P\left(\frac{k_{\perp}}{\ell_0\tau}\right). \quad (\text{C.20})$$

Here P is the monic quadratic spanning the even block of the residual coefficient matrix in the displayed column basis.

The maximal minors used in the preceding reduction are nonzero polynomials of $(\hat{h}_2, \hat{h}_3)$ for every admissible triple of principal values. Their highest base coefficients are nonzero rational constants times powers of

$$a, \quad b, \quad a-1, \quad b-1, \quad a-b.$$

For the quadratic leading coefficient and discriminant these coefficients are respectively nonzero constants times $a(a-b)^2$ and $ab(a-b)^4$. Thus the reduction and the separability of P hold on a common nonempty base open set. Cyclic relabeling preserves this conclusion.

Choose nine analytic combinations of the equations remaining after implicit elimination whose degree two coefficient matrix is the identity in the ordered basis (J, Q) . Their exact linear-normal coefficient matrix $B(W)$ satisfies $B(0) = (I_5, 0)^{\top}$. Adjoining the four constant columns $(0, I_4)^{\top}$ gives an invertible analytic 9×9 matrix, with determinant one at $W = 0$. Its inverse puts the nine equations in the form

$$J_i + \mathcal{E}_i^{\text{rem}} = 0 \quad (1 \leq i \leq 5), \\ Q_j + \mathcal{E}_{5+j}^{\text{rem}} = 0 \quad (1 \leq j \leq 4), \\ \mathcal{E}_i^{\text{rem}} \in (W)(n)^2 + (n)^3. \quad (\text{C.21})$$

Indeed, (C.17) removes the entire linear-normal part. The remaining quadratic coefficients vary by $(W)(n)^2$, and the higher normal terms belong to $(n)^3$.

The third order equations. Consider any formal or Puiseux arc in a fixed base fibre through (C.15), with τ not identically zero. Let ν be its valuation and put

$$t = \nu(\tau) < \infty, \quad d_0 = \min\{\nu(k_\perp), \nu(n_\perp), t\}, \quad s_0 = \min \nu(W) > 0.$$

An infinite valuation is allowed for a zero component. The error terms in (C.21) have valuation at least $\min(s_0 + 2d_0, 3d_0) > 2d_0$. If $d_0 < t$, the equation with leading term $n_\perp^2$ forces $\nu(n_\perp) > d_0$. Then $\nu(k_\perp) = d_0$, and the nonzero $k_\perp^2$ term of P_{hom} is its unique term of lowest valuation, a contradiction. Hence $k_\perp, n_\perp = O(\tau)$.

The three equations with leading terms $u\tau, w\tau, g\tau$ next imply $s_0 \geq 2t$: if $s_0 < 2t$, a component with valuation s_0 has leading term of valuation $s_0 + t$, strictly below both possible error orders. Equivalently these equations have the form $(I + \tau A)W = \tau^2 b$ with bounded coefficients; the coefficient matrix is a unit. The $n_\perp \tau$ equation gives $n_\perp = O(\tau^2)$. Finally,

$$P\left(\frac{k_\perp}{\ell_0 \tau}\right) = O(\tau).$$

At the residue root y_0 of P , the derivative $P'(y_0)$ is a unit. Thus

$$\begin{aligned} k_\perp &= \ell_0 y_0 \tau + O(\tau^2), & n_\perp &= O(\tau^2), \\ (\ell - \ell_0, u, v + 2) &= O(\tau^2), & P(y_0) &= 0. \end{aligned} \tag{C.22}$$

The valuation argument applies after arbitrary finite ramification.

Along such an arc the linear normal matrix satisfies

$$M(\ell, u, v) = M(\ell_0, 0, -2) + O(\tau^2).$$

Its variation multiplied by the third order correction to the eliminated variables is $O(\tau^5)$. Thus the coefficient of that correction in the third order system is $M(\ell_0, 0, -2)$, which is annihilated by the row combination in (C.19). The variation of M multiplied by the first order normal term instead contributes at order three. These terms give the columns for the second order corrections of ℓ, u, v . In particular,

$$g\tau = 2\ell_0(\ell - \ell_0)\tau + O(\tau^5).$$

The mixed coefficients in the fifteen-column matrix above supply these contributions to the third order system.

The five bounded quotients obtained by dividing the corrections in (C.22) by τ^2 have arbitrary residues. The variables eliminated by the implicit function theorem have their prescribed quadratic correction plus $O(\tau^3)$. Dividing each compatibility relation by τ^3 , cancelling the ambient third order correction to the eliminated variables by (C.19), and taking residues therefore gives the third linear system in these residue variables.

An undivided determinant and its quadratic norm. Only four rows of the necessary third system are needed: 1, 8, 12, 14, formed from their third order coefficients and rows 2, 18, 6 used for implicit elimination. Over the quadratic algebra $P(y) = 0$, their $k_{\perp,2}$ and du_2 columns vanish identically. Use the remaining unknowns $(E_2, L_2, dv_2, 1)$, where $n_{\perp,2} = \ell_0 E_2$ and $dL_2 = \ell_0 L_2$. Let R_j denote these third order rows. Form a 3×3 matrix, with $j = 1, 12, 14$ and $c = E_2, dv_2, 1$, by

$$O_{jc} = R_{8,L_2} R_{j,c} - R_{j,L_2} R_{8,c}. \tag{C.23}$$

Every lift satisfies $O(E_2, dv_2, 1)^\top = 0$, and hence $\det O = 0$. The determinant condition follows by cross multiplication.

Write the monic quadratic as $P = y^2 + P_1 y + P_0$ and reduce $\det O \equiv f_0 + f_1 y \pmod{P}$. Its quadratic-algebra norm is

$$\mathcal{N} = f_0^2 - P_1 f_0 f_1 + P_0 f_1^2. \tag{C.24}$$

If P and $\det O$ vanish at any common root, $\mathcal{N} = 0$. It remains to show that $\mathcal{N}$ is a nonzero rational function of the base for each admissible triple of principal values.

Laurent expansions of the norm. Use the principal value ratios a, b . In the original coefficient functions substitute $\hat{h}_2 = T$, $\hat{h}_3 = \pm T$ and $\epsilon = T^{-1}$. The two expansions of the same norm are

$$\mathcal{N}(T, \pm T) = N_{\pm}(a, b)\epsilon^{-30} + O(\epsilon^{-29}). \quad (\text{C.25})$$

Their leading coefficients are

$$\begin{aligned} N_- &= \frac{9216}{49} \frac{a^2(b-1)^4(b-a)^{14}}{(a-1)^{18}b^{17}} C_- \\ &\quad \times \frac{(7ab - a - 6b)^2(11ab - 3a - 8b)^4}{(5ab - a - 4b)^6(2ab - a - b)^2}, \end{aligned} \quad (\text{C.26})$$

$$\begin{aligned} N_+ &= \frac{9216}{49} \frac{a^2(b-1)^4(b-a)^{12}}{(a-1)^{18}b^{17}} C_+ \\ &\quad \times \frac{(5ab + a - 6b)^2(5ab + 3a - 8b)^4}{(3ab + a - 4b)^6}, \end{aligned} \quad (\text{C.27})$$

where

$$\begin{aligned} C_- &= 2352a^3b^2 + 961a^2b^3 - 672a^3b \\ &\quad - 3970a^2b^2 - 1984ab^3 + 48a^3 \\ &\quad + 577a^2b + 1664ab^2 + 1024b^3, \\ C_+ &= 1200a^3b^2 + 1089a^2b^3 + 480a^3b \\ &\quad - 2946a^2b^2 - 2112ab^3 + 48a^3 \\ &\quad - 575a^2b + 1792ab^2 + 1024b^3. \end{aligned}$$

Nonvanishing conditions in all three cyclic labelings. Besides $a, b, a-1, b-1, b-a$ and C_- , the factors for the negative ray are

$$F_{n,c} = nab - ca - (n-c)b,$$

$$(n, c) \in \{(2, 1), (5, 1), (7, 1), (11, 3), (53, 13), (65, 17)\}.$$

Put $s = a - 1 > 0$, $t = b - a > 0$. In the identity orientation,

$$F_{n,c}(a, b) = ns(s+1+t) + ct > 0,$$

and direct expansion gives

$$\begin{aligned} C_-(a, b) &= 3313s^2(s+1)^3 \\ &\quad + st(s+1)^2(7587s + 610) \\ &\quad + t^2(s+1)(5235s^2 + 548s + 49) \\ &\quad + (31s-1)^2t^3 > 0. \end{aligned}$$

The same negative ray covers the second cyclic relabeling of the principal value ratios $(1/b, a/b)$, since

$$b^2 F_{n,c}(1/b, a/b) = -(n-c)a(b-1) - c(b-a) < 0,$$

and

$$\begin{aligned} b^5 C_-(1/b, a/b) &= 1024s^5 + 2048s^4t + 1024s^3t^2 \\ &\quad + 4800s^4 + 9536s^3t + 4736s^2t^2 \\ &\quad + 6528s^3 + 13504s^2t + 6977st^2 \\ &\quad + 2752s^2 + 6016st + 3313t^2 > 0. \end{aligned}$$

For the first cyclic relabeling of the principal value ratios $(b/a, 1/a)$, use the positive-ray expansion, whose additional factors are C_+ together with

$$G_{n,c} = nab + ca - (n + c)b,$$

$$(n, c) \in \{(3, 1), (5, 1), (5, 3), (27, 13), (31, 17)\}.$$

For this orientation,

$$a^2 G_{n,c}(b/a, 1/a) = ca(b - 1) + n(b - a) > 0,$$

and

$$\begin{aligned} a^5 C_+(b/a, 1/a) &= 48s^5 + 144s^4t + 144s^3t^2 + 48s^2t^3 \\ &\quad + 145s^4 + 866s^3t + 1297s^2t^2 + 576st^3 \\ &\quad + 146s^3 + 1234s^2t + 2816st^2 + 1728t^3 \\ &\quad + 49s^2 + 512st + 2752t^2 > 0. \end{aligned}$$

Every factor in the original principal value ratios is nonzero under both cyclic substitutions. The displayed identities make $N_{\pm}$ nonzero for each ordered triple of positive, pairwise distinct principal values.

Cyclic covariance and exclusion. The even cyclic frame permutation acts by

$$(a, b, \hat{h}_2, \hat{h}_3) \mapsto \left(\frac{b}{a}, \frac{1}{a}, \frac{\hat{h}_3}{\hat{h}_2}, \frac{1}{\hat{h}_2} \right),$$

$$\hat{p} \mapsto (\hat{p}_{23}, \hat{p}_{21}, \hat{p}_{32}, \hat{p}_{31}, \hat{p}_{12}, \hat{p}_{13}) / \hat{h}_2,$$

$$\mathbf{R} \mapsto (\mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_1) / \hat{h}_2.$$

It has order three. Exact compatibility identities preserve the selected generating ideal, and the directional derivations transform by the chain rule modulo a vertical derivation multiplied by Q_0 . Consequently the differential ideal is invariant, and each transported relation vanishes on every solution arc.

On the ordinary projective compactification the map is

$$[\bar{p} : \mathbf{R} : \tau] \mapsto [\text{permuted } \bar{p} : \text{permuted } \mathbf{R} : \hat{h}_2 \tau].$$

Changing to the relevant nonzero ordinary boundary chart multiplies coordinates by a unit. Since $\ell_0^2 = d \neq 0$, this operation preserves both $\tau \neq 0$ and its valuation order. The base change is birational with inverted denominators, so it also preserves nonzero rational base functions.

For each fixed original triple of principal values, the negative ray in orientations zero and two and the positive ray in orientation one prove that the three transported norms are nonzero rational base functions. Intersect their nonvanishing sets with the finitely many open sets defined by the elimination minors, quadratic coefficient minors and discriminants established above. This intersection is nonempty on the irreducible base. On it any lift through an additional τ hypersurface would satisfy (C.23) and hence force its norm to vanish, a contradiction. This proves exclusion over a nonempty base open set in all three orientations.

Combining this exclusion with the relative ideals (C.11)–(C.13) gives the ordinary boundary conclusion.

C.13 Exceptional ratio fibres

We now consider the characteristic pencils, the balanced line, and the projective direction and amplitude fibres used in the argument for an algebraic ratio curve. Throughout, $a, b > 0$, and $1, a, b$ are pairwise distinct. Put

$$\Delta_a = a(b-1), \quad \Delta_b = b(a-1), \quad \Delta_{ab} = a-b, \\ \kappa_0 = \Delta_{ab}/\Delta_a, \quad q = \Delta_b/\Delta_a = 1 + \kappa_0.$$

In particular $a, \kappa_0, q, 1 + a\kappa_0$ are nonzero. Their signs are determined by the ordering of the principal values. The dimensionless ratio coordinates are

$$\eta = \kappa_0 \hat{h}_2, \quad \zeta = \kappa_0 \hat{h}_3/q, \quad \varsigma = d\zeta/d\eta, \quad N = d\varsigma/d\eta, \\ L = \eta\varsigma - \zeta, \quad K = 1 + \eta - \zeta.$$

When η is used as a local coordinate, “outside finitely many values” means after fixing the triple of principal values and any constant of the particular ratio curve.

The first tangencies $E_i\zeta = \varsigma E_i\eta$ give the coordinate graph

$$\begin{aligned} \hat{p}_{12} &= w_3, & \hat{p}_{13} &= w_2, & \hat{p}_{23} &= w_1, \\ \hat{p}_{21} &= \eta w_3 + \frac{((\eta+1)\varsigma - \zeta)m_3}{2\varsigma}, \\ \hat{p}_{31} &= -\varsigma w_2 - \frac{(1 + \eta\varsigma - \zeta)m_2}{2\eta}, \\ \hat{p}_{32} &= -\frac{\varsigma w_1}{\eta} - \frac{(1 - \varsigma)m_1}{2\eta}, \\ R_1 &= \frac{(\varsigma - \eta\varsigma)m_1}{\eta\varsigma}, & R_2 &= \frac{\varsigma m_2}{\varsigma}, & R_3 &= -\frac{m_3}{\eta}. \end{aligned} \tag{C.28}$$

Ratio curve equations. The seven necessary ratio-curve equations are obtained from the reconstructed state fields by

$$E_i(E_j\zeta) - \varsigma E_i(E_j\eta) - N(E_i\eta)(E_j\eta) = 0, \\ (i, j) = (1, 2), (1, 3), (2, 3), (1, 1), (2, 2), (3, 3),$$

together with the original quartic Q_0 . The first tangencies are inserted through the invertible coordinate graph (C.28). The six additional cubic equations are obtained from the original quartics $Q_{15}, Q_{18}, Q_{13}, Q_{19}, Q_{10}, Q_{11}$ by subtracting multiples of the three diagonal tangencies. Their pre-substitution normal forms are

$$Q_\nu = R_\nu + \epsilon \hat{p}_{jk}(E_i R_i)/k_i, \\ k_1 = -\frac{2(\Delta_a \hat{h}_3 - \Delta_b \hat{h}_2)}{\Delta_{ab} \hat{h}_2 \hat{h}_3}, \quad k_2 = \frac{2(\Delta_b - \Delta_{ab} \hat{h}_3)}{\Delta_{ab} \hat{h}_3}, \quad k_3 = \frac{\Delta_a + \Delta_{ab} \hat{h}_2}{\Delta_b \hat{h}_2}.$$

Here R_ν has degree at most three. In row order, the triples $(i, \hat{p}_{jk}, \epsilon)$ are

$$(1, \hat{p}_{12}, 1), (1, \hat{p}_{13}, 1), (2, \hat{p}_{21}, 1), \\ (2, \hat{p}_{23}, -1), (3, \hat{p}_{31}, 1), (3, \hat{p}_{32}, -1).$$

The denominators are products of the spectral, ratio and noncharacteristic factors specifying this chart. The three additional quartics are Q_1, Q_5, Q_6 ; the equations after cancellation of their highest weighted terms include their lower degree components.

For homogeneous boundary equations the nonzero pair weights are

$$c_1 = (a\kappa_0)^{-1}, \quad c_2 = 1, \quad c_3 = (1 + a\kappa_0)^{-1}.$$

After division by these explicit weights, differentiation of every coefficient with respect to a and κ_0 is zero. Hence the normalized coefficients are independent of the principal value ratios.

C.14 Characteristic pencils and the singular affine system

For the first characteristic pencil write $\hat{h}_3 = q\hat{h}_2 + c$, $T = \kappa_0 c/q$ and $\eta = \kappa_0 \hat{h}_2$; thus $T(T-1) \neq 0$. In the unnormalized state put

$$L_p = p_{33} - qp_{22} - cp_{11}, \quad B_p = p_{33}p_{23} + qp_{22}p_{32}.$$

The first tangency gives $B_p = 0$. If $\hat{p}_{23} = 0$, the opposite pair vanishes and the zero-pair argument applies. Otherwise set $(u, v, x, s, y) = (\hat{p}_{12}, \hat{p}_{13}, \hat{p}_{21}, \hat{p}_{23}, \hat{p}_{31})$. Let Ψ_p be the substitution into the state given by

$$p_{11} = 1, \quad p_{22} = \eta/\kappa_0, \quad p_{33} = q(\eta + T)/\kappa_0, \quad p_{32} = -(\eta + T)s/\eta,$$

with the other five off-diagonal entries (u, v, x, s, y) . Its six τ_j are the unique solution of the three cubic compatibility relations together with $E_2 L_p = E_3 L_p = E_1 B_p = 0$. The determinant of these six equations is

$$\frac{4qc^2(1 + \kappa_0 \hat{h}_2)(1 - \kappa_0 \hat{h}_3/q)(1 - \kappa_0 c/q)^2 s}{\hat{h}_2}.$$

Its u_0, u_1 are then the unique solution of $E_2^2 L_p = E_3^2 L_p = 0$, whose determinant is $-2q^2(1 - T)^2$. All derivatives here are taken before the substitution. The remaining compatibility polynomials are

$$\begin{aligned} f_p &= \frac{s\Psi_p(E_2 E_1 L_p)}{[uvx]\{s\Psi_p(E_2 E_1 L_p)\}}, \\ g_p &= \frac{s\Psi_p(E_3 E_1 L_p)}{[u^2x]\{s\Psi_p(E_3 E_1 L_p)\}}, \\ r_p &= s\Psi_p(E_3 E_2 L_p). \end{aligned}$$

The two normalizing coefficients are nonzero rational functions of the pencil parameters. Define their coefficients by the identities

$$\begin{aligned} T_0 &= ux + \lambda_{\text{pencil}}vy + N_p, \\ f_p &= vT_0 + s^2(Av + By) + s(Cu + Dx), \\ g_p &= uT_0 + s^2(Eu + Fx) + s(Gv + Jy). \end{aligned}$$

For example $A = [vs^2]f_p$, $D = [xs]f_p$, $E = [us^2]g_p$, $J = [ys]g_p$ and $N_p = [v]f_p = [u]g_p$; the displayed monomials determine all ten coefficients uniquely. The third residual has degree at most one in each of x, y and satisfies

$$J_{xy} = [xys]r_p = -\frac{4qT}{(T-1)\kappa_0}, \quad \partial_x \partial_y r_p = J_{xy}s.$$

Thus the first two equations are the affine system

$$M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (N_p + As^2)v + Cus \\ (N_p + Es^2)u + Gvs \end{pmatrix} = 0, \quad M = \begin{pmatrix} uv + Ds & \lambda_{\text{pencil}}v^2 + Bs^2 \\ u^2 + Fs^2 & \lambda_{\text{pencil}}uv + Js \end{pmatrix}.$$

Its determinant is $s\delta$, where

$$\delta = -Bs u^2 - \lambda_{\text{pencil}}Fsv^2 + (J + \lambda_{\text{pencil}}D)uv + sDJ - BF s^3.$$

For each fixed permitted triple of principal values and T ,

$$\lambda_{\text{pencil}}ABDEFGJ(J + \lambda_{\text{pencil}}D)J_{xy} \neq 0$$

outside finitely many η . This follows from its explicit polynomial coefficients in η , including the nonzero constant coefficients $2T(1 - T), T(T - 1), T(1 - T)$, the monic cubic coefficient of G , and

$$\eta^2(J + \lambda_{\text{pencil}}D)\Big|_{\eta=0} = \frac{2T^2(1 + a\kappa_0)}{a(q - 1)^2}.$$

Where a quadratic leading coefficient $T - 2$ or $T + 1$ vanishes, the corresponding constant or linear coefficient is nonzero. The coefficient polynomials are therefore nonzero for every permitted T .

The singular locus $\delta = 0$ of the affine system has fibre dimension at most one, including zero columns and $M = 0$. To see this, work over $\mathbb{C}(s)$, the homogeneous conic matrix of δ has determinant

$$(B\lambda_{\text{pencil}}Fs^2 - (J + \lambda_{\text{pencil}}D)^2/4)s(DJ - BF s^2) \neq 0.$$

Consequently δ is absolutely irreducible; its coefficient $J + \lambda_{\text{pencil}}D$ also makes it primitive over $\mathbb{C}[s, s^{-1}]$. Writing $\mathbf{b}_p$ for the constant column in the affine system, its first Cramer numerator divided by s is

$$X = s^{-1}\{M_{12}(\mathbf{b}_p)_2 - M_{22}(\mathbf{b}_p)_1\}.$$

Comparison of its cubic coefficients shows that divisibility by δ would require

$$(J + \lambda_{\text{pencil}}D)G + \lambda_{\text{pencil}}F(E - A)s^2 = 0,$$

which is impossible. The compatible base (u, v, s) therefore has dimension at most one. When both columns of M are nonzero, the third mixed equation cuts the affine solution line in finitely many points: its quadratic coefficient is $J_{xy}sn_xn_y \neq 0$, where (n_x, n_y) spans the kernel. If the first column vanishes, write $u = \epsilon s$, $\epsilon^2 = -F$, $v = -D/\epsilon$; consistency has a nonzero degree five coefficient $B\epsilon E$. The second-column case similarly has coefficient $-F\epsilon A$. Hence those compatible bases are finite even when the third equation vanishes on a solution line. If $M = 0$, the equations $u^2 = -Fs^2$, $\lambda_{\text{pencil}}v^2 = -Bs^2$, $uv = -Ds$ leave at most four base points; the remaining equation has nonzero xy coefficient. These rank cases give the dimension bound throughout the singular locus, including the coordinate hyperplanes in u, v, x, y .

On $\det M \neq 0$, put

$$Y = s^{-1}\{M_{21}(\mathbf{b}_p)_1 - M_{11}(\mathbf{b}_p)_2\}.$$

The affine equations give $x = X/\delta$, $y = Y/\delta$. Define

$$P = \frac{a^4\kappa_0^{11}(T - 1)^4\eta^{11}(\eta + T)^6}{qT^3(1 + a\kappa_0)^2} \frac{\delta^2}{s} r_p|_{x=X/\delta, y=Y/\delta},$$

$$Q = \frac{a^5\kappa_0^{12}(T - 1)^5\eta^{12}(\eta + T)^6}{T^2(1 + a\kappa_0)^2} \delta^2 \Psi_p(Q_0)|_{x=X/\delta, y=Y/\delta}.$$

These polynomials have degrees eight and nine in (u, v, s) . The coefficient of u^6 in P is a unit times $s\mathcal{B}_0\mathcal{B}_1\mathcal{C}_0$, where

$$\begin{aligned} \mathcal{B}_0 &= (T - 2)\eta^2 + (3T - 4)\eta + T(T - 1), \\ \mathcal{B}_1 &= 2(T - 1)\eta^2 + (2T^2 - 4T + 1)\eta + T(1 - T), \\ \mathcal{C}_0 &= \eta^3 + (-2T^2 + 4T)\eta^2 + (-3T^3 + 6T^2 - 2T)\eta - T^4 + 2T^3 - T^2. \end{aligned}$$

Each factor is a nonzero polynomial in η for every allowed T . Work in $\mathbb{C}[v, s, s^{-1}][u]$. Since the leading coefficient of P is a unit, every nonunit factor of P has positive u -degree and retains this degree under $v = 0, s = 1$. The restricted polynomials are cubics in $U = u^2$. We

take their Sylvester resultant in the order P, Q , with both degrees fixed at three. A nonzero resultant therefore proves that P, Q are coprime on the whole chart.

The resultant is nonzero by the following four cases. The general η^{118} coefficient leaves precisely

$$T = 2, \quad aq + 1 = 0, \quad \Psi = (q - 1)T(aq + 1) - 4 = 0.$$

More explicitly, that coefficient is

$$\begin{aligned} & -1024T^6q^8(T-2)^{13}(T-1)^{13}a^{15} \\ & \times (q-1)^{40}(aq+1)^5\Psi^2. \end{aligned}$$

We consider these hypersurfaces in turn.

The hypersurface $T = 2, aq + 1 \neq 0$. Write $\ell = aq + 1$, $t = 1/\eta$, and divide the first specialized cubic by the unit factor $\eta + 1$. The expansions of the resulting cubics, after multiplication by t^{15} and t^{20} , begin with

$$\begin{aligned} F_0(U) &= 16a^2q\kappa_0^6U(U-\ell)(2\kappa_0U-\kappa_0\ell+4), \\ G_0(U) &= -16a^3q^2\kappa_0^7U(U-\ell)^2. \end{aligned}$$

Their cubic leading coefficients are nonzero; hence no common algebraic root has negative t -valuation, and its limiting value must be 0 or ℓ . Positivity gives $1 + \kappa_0\ell = q(1 + a\kappa_0) = aq^2/b > 0$. Thus the F -root at ℓ is simple, and the second residual coefficient in G is $-16qa^3\kappa_0^9\ell^3 \neq 0$. At zero, the G -root is simple and its first two coefficients are $-8/\kappa_0$, $24/\kappa_0$. The first residual coefficient in F is

$$-128qa\kappa_0^3\ell H_p, \quad H_p = a^2q\kappa_0^3 + a\kappa_0(\kappa_0^2 - 12\kappa_0 - 8) - 8q.$$

If M_2 denotes the second residual coefficient, elimination gives

$$\text{Res}_a(H_p, M_2) = -8388608q^6\kappa_0^{12}(q^2 - 4q + 1)(5q^2 + 508q + 27).$$

On the first quadratic locus the nonzero linear remainder forces $a = q - 4$, hence $\ell = 0$, contrary to the chart. On the second it forces $a = 259q/270 + 4871/50$. In its quadratic number field the third residual coefficient has norm

$$\frac{107979287999067998650368}{15625} \neq 0.$$

The linear remainder coefficients in both fields have norm 45164265472. These nonzero norms give the exclusion in both embeddings, also on the intersection with $\Psi = 0$.

The hypersurface $aq + 1 = 0$, including its intersections. Positivity gives $q < 0$, $a = -1/q$, $b = -q$. Multiply the specialized cubics by q^2, q^3 ; their resultant is multiplied by q^{15} , a unit in the stated localization. For their resultant $R(\eta)$, put $c_j = [t^j]t^{117}R(1/t)$, and set

$$\mathcal{P}_{\text{lin}} = (q^2 + 6q + 1)T - 4\kappa_0^2, \quad \mathcal{P}_{\text{quad}} = (T - 1)\mathcal{P}_{\text{lin}} - 8qT.$$

The first four coefficients vanish, and

$$c_4 = 16384T^6q^8(T-1)^9(T-2)^{13}\kappa_0^{25}\mathcal{P}_{\text{lin}}\mathcal{P}_{\text{quad}}^4.$$

Below $\doteq$ denotes equality up to a nonzero rational constant. On $\mathcal{P}_{\text{lin}} = 0$, one has $(q^2 + 6q + 1) \neq 0$, $\mathcal{P}_{\text{quad}} = -8qT \neq 0$, and

$$\begin{aligned} c_5 \bmod \mathcal{P}_{\text{lin}} &\doteq \frac{q^{12}\kappa_0^{49}}{(q^2 + 6q + 1)^{35}}(q^2 - 14q/3 + 1)^8(q^2 - 10q + 1)^{14} \\ &\times (q^4 - 140q^3 + 150q^2 - 140q + 1). \end{aligned}$$

On $\mathcal{P}_{\text{quad}} = 0, (q^2 + 6q + 1) \neq 0$, the norm of the remainder of c_8 is

$$\text{Res}_T(\mathcal{P}_{\text{quad}}, c_8 \bmod \mathcal{P}_{\text{quad}}) \doteq \frac{q^{38} \kappa_0^{90}}{(q^2 + 6q + 1)^{39}} (q^4 - 4q^3/27 + 722q^2/27 - 4q/27 + 1)^4.$$

All displayed nonmonomial factors have positive coefficients after $q = -s, s > 0$, and have positive constant coefficients. At $(q^2 + 6q + 1) = \mathcal{P}_{\text{quad}} = 0$, one has $T = 4/3$; a Bézout identity for $(q^2 + 6q + 1)$ and $c_8(q, 4/3)$ has right hand side one. Hence the latter polynomial is nonzero at both roots of $(q^2 + 6q + 1)$. Finally, at $T = 2$, the scaled Sylvester determinant $t^{102}R(1/t)$ has zero constant and linear coefficients and quadratic coefficient

$$274877906944q^8\kappa_0^{23}(q^2 - 10q + 1)(q^2 - 4q + 1) \\ \times (q^2 - 18q + 1)^4 \neq 0.$$

The collision hypersurface. On $\Psi = 0, T \neq 2$, three resultant coefficients P_1, P_2, P_3 satisfy the polynomial identity

$$g^4 = A_1P_1 + A_2P_2 + A_3P_3, \\ g = qT(T-1)(T-2)(q-1)((q-1)T-4)(T+4).$$

The factors in this identity are units, since on this hypersurface $(q-1)T-4 = -aq(q-1)T$ and $T+4 = q(1+a(q-1))T$. The relations $aq+1=0$ and $\Psi=0$ are incompatible. Thus for each fixed triple of principal values and pencil constant the resultant is a nonzero polynomial in η . The regular and singular parts of the affine system give the required fibre bound. Cyclic relabelling covers the other two characteristic pencils.

C.15 The ratio line $\zeta = \eta + 1$

We now bound the dimension on the ratio line $K = 1 + \eta - \zeta = 0$. Use the spectral differences defined above and the normalized curl matrix $\hat{\mathbf{p}}$, whose diagonal is $(1, \hat{h}_2, \hat{h}_3)$. Define

$$\mathcal{B}_1^{\text{bal}} = \hat{h}_3\hat{p}_{23} + q\hat{h}_2\hat{p}_{32}, \\ \mathcal{B}_2^{\text{bal}} = \hat{p}_{31} + (\kappa_0/q)\hat{h}_3\hat{p}_{13}, \\ \mathcal{B}_3^{\text{bal}} = \hat{h}_2\hat{p}_{12} - \kappa_0^{-1}\hat{p}_{21}.$$

The normalized fields give, modulo (K) ,

$$E_1K = -2(\Delta_{ab}/\Delta_b)\mathcal{B}_1^{\text{bal}}, \quad E_2K = 2\mathcal{B}_2^{\text{bal}}, \quad E_3K = 2(\Delta_{ab}/\Delta_a)\mathcal{B}_3^{\text{bal}}.$$

Thus $K = 0$ on a solution open set forces all three $\mathcal{B}_i^{\text{bal}}$ to vanish there. Equivalently,

$$\text{diag}(\Delta_{ab}, \Delta_a/\hat{h}_2, -\Delta_b/\hat{h}_3)\hat{\mathbf{p}}$$

is symmetric.

First we prove that an opposite pair cannot remain zero on this line. By a signed cyclic relabeling it suffices to consider $\hat{p}_{13} = \hat{p}_{31} = 0$. Write $y = \hat{p}_{12}$ and $w = \hat{p}_{23}$. Then $\hat{p}_{21} = \Delta_{ab}\hat{h}_2y/\Delta_a$ and $\hat{p}_{32} = -\Delta_a\hat{h}_3w/(\Delta_b\hat{h}_2)$. The four coordinate tangencies in directions one and three have coefficient determinant $\Delta_b/(2\Delta_a)$ in the normalized variables $(\hat{\tau}_0, \hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_4)$. Substitution in the remaining tangencies gives

$$y^2 = Y(\hat{h}_2, \hat{h}_3) = -\frac{N_Y}{4\hat{h}_2b\Delta_{ab}},$$

where

$$N_Y = ab(ab-1) - 2ab(b-1)\hat{h}_2 - b\Delta_{ab}\hat{h}_2^2 + 2ab(a-1)\hat{h}_3 - 3a\Delta_{ab}\hat{h}_3^2.$$

The same compatibility equations give

$$\begin{aligned}
E_3 \hat{h}_2 &= \frac{2\hat{h}_2(1 - \Delta_{ab}\hat{h}_3/\Delta_b)}{\Delta_a} y(\Delta_a + \Delta_{ab}\hat{h}_2), \\
E_3 \hat{h}_3 &= \frac{2\hat{h}_2(1 - \Delta_{ab}\hat{h}_3/\Delta_b)}{\Delta_a} y\Delta_{ab}\hat{h}_3, \\
E_3 y &= \left(1 - \frac{2\Delta_{ab}^2 \hat{h}_2 \hat{h}_3}{\Delta_a \Delta_b}\right) y^2 + \frac{N_3}{4b\Delta_a}, \\
N_3 &= ab(ab-1) + 2ab(b-1)\hat{h}_2 + 3b\Delta_{ab}\hat{h}_2^2 \\
&\quad - 2ab(a-1)\hat{h}_3 + a\Delta_{ab}\hat{h}_3^2.
\end{aligned}$$

Where $y \neq 0$, differentiating $y^2 = Y$ consequently gives

$$2 \left(1 - \frac{2\Delta_{ab}^2 \hat{h}_2 \hat{h}_3}{\Delta_a \Delta_b}\right) Y + \frac{N_3}{2b\Delta_a} - \frac{2\hat{h}_2(1 - \Delta_{ab}\hat{h}_3/\Delta_b)}{\Delta_a} \{(\Delta_a + \Delta_{ab}\hat{h}_2)Y_{\hat{h}_2} + \Delta_{ab}\hat{h}_3 Y_{\hat{h}_3}\} = 0.$$

Multiplication of this identity by $8(b-a)\hat{h}_2(b-1)(a-1)ab^2$ clears its denominators. On $K=0$, namely $\hat{h}_3 = \Delta_b(\Delta_a + \Delta_{ab}\hat{h}_2)/(\Delta_a\Delta_{ab})$, the numerator becomes a nonzero polynomial in $\hat{h}_2$: its constant coefficient is

$$\frac{4a^3(b-1)^3(a-1)b^2}{b-a} \neq 0.$$

Hence $\hat{h}_2$ is locally constant, contrary to the displayed nonzero $E_3\hat{h}_2$. If $y=0$ on an open set, there are two opposite zero pairs. The pressure identities then force the third pair to vanish: for example $d\phi_2 = -h_1\hat{p}_{23}\alpha_1$ and $d\phi_3 = h_1\hat{p}_{32}\alpha_1$, whose transverse exterior derivatives are $-h_1\hat{p}_{23}h_1$ and $h_1\hat{p}_{32}h_1$. When all off-diagonal entries vanish, the diagonal flatness equations are $h_i^2 - (h_j - h_k)^2 = 0$, which imply $h_2h_3 = 0$ and contradict the assumption that all three abnormalities are nonzero. Together with the $y \neq 0$ case, this excludes a persistent zero opposite pair. Cyclic signed relabeling covers the other two opposite pairs.

It remains to work on the dense open set with all three opposite pairs nonzero. Write

$$\hat{p} = \begin{pmatrix} 1 & u/\Delta_{ab} & v/\Delta_{ab} \\ \hat{h}_2 u/\Delta_a & \hat{h}_2 & \hat{h}_2 \omega/\Delta_a \\ -\hat{h}_3 v/\Delta_b & -\hat{h}_3 \omega/\Delta_b & \hat{h}_3 \end{pmatrix}, \quad \hat{h}_3 = \frac{\Delta_b(\Delta_a + \Delta_{ab}\hat{h}_2)}{\Delta_a\Delta_{ab}}, \quad uv\omega \neq 0.$$

The three cubics and the three tangencies $E_i \mathcal{B}_i^{\text{bal}} = 0$ reconstruct all six normalized second jets. Their coefficient determinant is

$$-\frac{4\Delta_b\hat{h}_2^2(\Delta_a + \Delta_{ab}\hat{h}_2)^2 uv\omega}{\Delta_a^5 \Delta_{ab}},$$

which is a unit in the stated localization on this chart. Define the two remaining polynomials by

$$f_0 = uv\omega E_2 \mathcal{B}_1^{\text{bal}}|_{\text{graph}}, \quad f_1 = uv\omega E_3 \mathcal{B}_1^{\text{bal}}|_{\text{graph}}.$$

They have degree five in (u, v, ω) . Put

$$\begin{aligned}
\epsilon &= b-1, \quad \varphi = a-1, \quad c_b = \frac{b\hat{h}_2(\Delta_a + \Delta_{ab}\hat{h}_2)}{2a^2\Delta_{ab}^3\epsilon^3}, \\
U_b &= \Delta_{ab}\epsilon u^2, \quad V_b = \Delta_{ab}\varphi v^2, \quad W_b = \epsilon\varphi\omega^2.
\end{aligned}$$

The leading forms are

$$\begin{aligned} \text{in}(f_0) &= \frac{c_b}{\varphi} u \{(V_b - W_b)^2 - U_b(V_b + W_b)\}, \\ \text{in}(f_1) &= -\frac{c_b}{\epsilon} v \{(U_b + W_b)^2 - V_b(U_b - W_b)\}. \end{aligned}$$

Every displayed prefactor is nonzero for every admissible triple of principal values and ratio value on this line. The first quadratic in (U_b, V_b, W_b) has symmetric-matrix determinant -1 , hence is irreducible; the second has a nonzero U_b^2 coefficient and is not a scalar multiple. Their common zero set has dimension one. The map $(u, v, \omega) \mapsto (U_b, V_b, W_b)$ is finite, so the pulled-back quartics have no common polynomial factor. The extra factors u, v are also coprime to the opposite quartic, as is seen by setting $u = 0$ or $v = 0$. Thus the two displayed degree five forms are coprime, and the polynomials f_0, f_1 are coprime as well.

For each fixed permitted $\hat{h}_2$ their common zero set in three variables has dimension at most one. Allowing $\hat{h}_2$ gives dimension at most two, and the rational jet reconstruction recovers the complete state from this set. Hence this branch has state rank at most two. On a hypothetical rank three chart, $K \neq 0$ may therefore be selected on a dense open subset.

Tangency and flatness at zero pair coordinates. The tangencies of a coordinate are used on a patch where that coordinate vanishes identically. In the pressure identities above, evaluation on (e_2, e_3) eliminates derivatives of the two coefficients. With all six off-diagonal entries zero, the Koszul formula, with the nine formal derivatives $D_i h_j$, gives

$$2 \text{Ric}_{ii} = h_i^2 - (h_j - h_k)^2.$$

Flatness and $h_1 \neq 0$ imply $h_2 \neq h_3$. Subtracting the last two equations gives $h_1 = h_2 + h_3$, and the first then gives $4h_2h_3 = 0$. This is incompatible with the branch under consideration. For the nonzero pair coordinates, the two opposite axis restrictions of the leading quartics are $W_b(W_b + V_b)$ and $W_b(W_b - U_b)$. Thus the two prefactors u, v are coprime to the opposite quartics.

C.16 Direction supports and finite amplitude fibres

Put $L = \eta\varsigma - \zeta$ and $K = 1 + \eta - \zeta$. We first consider the noncharacteristic chart

$$\eta\zeta(\eta - \zeta)(\zeta - 1)(\eta + 1)K(\varsigma - 1)(1 + \eta\varsigma - \zeta)((\eta + 1)\varsigma - \zeta) \neq 0.$$

Write the leading mixed equations as $v_i^\top B_{ij} v_j = 0$, where $v_i = (w_{i+1}, m_{i+1})^\top$, $0 \leq i < j \leq 2$. Their coefficient matrices B_{ij} are nonzero. With $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, an invertible block induces a projective map between its two directions. The composition around the three pairs is the monodromy. Three invertible blocks admit a positive-dimensional direction fibre only when the monodromy is scalar. Two invertible blocks give at most two directions. With at most one invertible block, a positive-dimensional fibre requires the corresponding product to have zero trace, including the case of a nonzero nilpotent product.

Scalar monodromy is equivalent to $B_{12} = \lambda_{\text{mon}} B_{01}^\top J B_{02}$. The exceptional Cramer factors are

$$\begin{aligned} L_1 &= \eta(\eta - 1)\varsigma + 2\eta + \zeta, \\ L_2 &= (\eta + 2\zeta)\varsigma - \zeta(\zeta + 1), \quad (B_{01}^\top J B_{02})_{00}. \end{aligned}$$

The projection polynomials. Let

$$\mathfrak{g} = \eta\zeta(\eta - \zeta)(\zeta - 1)(\eta + 1)(1 + \eta - \zeta)(\varsigma - 1)(1 + \eta\varsigma - \zeta)((\eta + 1)\varsigma - \zeta).$$

For a nonzero polynomial g and a rational function with rational coefficients, let ng_g denote its reduced numerator after all irreducible factors dividing g have been removed. Normalize

the remaining polynomial to have leading coefficient one in graded lexicographic order $\eta > \zeta > \varsigma > N$. This specifies the operation uniquely. All factor families below consist of monic irreducible factors; repeated occurrences in different projections retain their projection index.

Put $D_{\text{mix}} = B_{01}^\top J B_{02}$ and

$$M_{\text{mon}} = J B_{02} J B_{12}^\top J B_{01}^\top.$$

The four equations $(B_{12})_{ij} - \lambda_{\text{mon}}(D_{\text{mix}})_{ij} = 0$ are ordered by $(i, j) = (0, 0), (0, 1), (1, 0), (1, 1)$. Their coefficient matrix on $(1, \lambda_{\text{mon}}, N, \xi_{\text{mon}})$, where $\xi_{\text{mon}} = \lambda_{\text{mon}} N$, is

$$\mathcal{L}_{r,:} = ((B_{12})_{ij}|_{N=0}, -(D_{\text{mix}})_{ij}|_{N=0}, \partial_N(B_{12})_{ij}, -\partial_N(D_{\text{mix}})_{ij}).$$

Its coefficients are independent of N ; $\mathcal{L}_{r2} = 0$ for $r = 0, 1, 2$, and $\mathcal{L}_{32} = -2\eta\zeta(\eta - \zeta)$. On the regular Cramer chart define

$$\lambda_{\text{mon}} = -\frac{\mathcal{L}_{00}}{\mathcal{L}_{01}}, \quad \xi_{\text{mon}} = -\frac{\mathcal{L}_{10} + \mathcal{L}_{11}\lambda_{\text{mon}}}{\mathcal{L}_{13}}, \quad N = -\frac{\mathcal{L}_{30} + \mathcal{L}_{31}\lambda_{\text{mon}} + \mathcal{L}_{33}\xi_{\text{mon}}}{\mathcal{L}_{32}}.$$

The two remaining equations are the reduced numerators of

$$\mathcal{L}_{20} + \mathcal{L}_{21}\lambda_{\text{mon}} + \mathcal{L}_{23}\xi_{\text{mon}}, \quad \mathcal{L}_{32}\xi_{\text{mon}} + \mathcal{L}_{31}\lambda_{\text{mon}}^2 + \mathcal{L}_{33}\lambda_{\text{mon}}\xi_{\text{mon}} + \mathcal{L}_{30}\lambda_{\text{mon}},$$

after this substitution. Reduce with respect to $\mathfrak{g}L_1L_2(D_{\text{mix}})_{00}$, where L_1, L_2 are the displayed Cramer factors. Substitute $\varsigma = (\zeta + \ell)/\eta$ into these two polynomials and reduce with respect to η , obtaining $p_0, p_1 \in \mathbb{Q}[\eta, \zeta, \ell]$. The scalar-monodromy projection family is specified by the nonvertical irreducible factors of $\text{Res}_\ell(p_0, p_1)$ other than the factors of $\eta\zeta(\eta - \zeta)(\zeta - 1)(\eta + 1)(1 + \eta - \zeta)$.

Two singular mixed blocks. For $a \in \{01, 02, 12\}$ set

$$d_a = \text{ng}_{\mathfrak{g}}(\det B_a).$$

For the three ordered pairs

$$(a, b) = (01, 02), (01, 12), (02, 12),$$

write $d_a = A_a N + C_a$ and set $N_a = -C_a/A_a$. The coefficient A_a is a nonzero unit times L_1 for $a = 01$, and a nonzero unit times L_2 for $a = 02$. On its nonzero chart put

$$p_{ab} = \text{ng}_{\mathfrak{g}L_a}(d_b(\eta, \zeta, \varsigma, N_a)), \quad t_{ab} = \text{ng}_{\mathfrak{g}L_a}(\text{tr } M_{\text{mon}}(\eta, \zeta, \varsigma, N_a)),$$

where $L_{01} = L_1$ and $L_{02} = L_2$. Let $\mathcal{P}_{ab}$ be the monic irreducible factors f of $\text{Res}_\varsigma(p_{ab}, t_{ab})$ satisfying

$$\begin{aligned} \partial_\zeta f &\neq 0, \\ f &\nmid \eta\zeta(\eta - \zeta)(\zeta - 1)(\eta + 1)(1 + \eta - \zeta). \end{aligned}$$

Factors depending only on η cannot contain a curve on which η is a coordinate. For $f \in \mathcal{P}_{ab}$, with $d = \deg_\varsigma p_{ab}$, the tangency polynomial is

$$T_{ab,f} = (\partial_\zeta f)^d p_{ab} \left(\eta, \zeta, -\frac{\partial_\eta f}{\partial_\zeta f} \right) \pmod{f}.$$

For each such factor, $\text{gcd}_{\mathbb{Q}(\eta)[\zeta]}(f, T_{ab,f}) = 1$. The zero charts of A_a are the separate L_1, L_2 pivot cases.

For a scalar-monodromy projection factor f , homogenize p_0, p_1 at $\ell = -\eta f_\eta / f_\zeta - \zeta$ on $f_\zeta \neq 0$ and reduce their cleared numerators modulo f . If the results are T_0, T_1 , then $\text{gcd}_{\mathbb{Q}(\eta)[\zeta]}(f, T_0, T_1) = 1$. The compatibility and tangency equations on the three exceptional

Cramer divisors exclude a persistent ratio curve there. It follows that the mixed direction fibre has at most two points over the generic point of a nonconstant ratio curve.

For one nonzero pair, the two pure cubic coefficients have the unit combinations

$$(\zeta - \eta)(\varsigma - 1), \quad (\zeta - 1)(\eta\varsigma - \zeta + 1), \quad (\eta + 1)((\eta + 1)\varsigma - \zeta),$$

respectively. Thus none of the three projective pair lines is a boundary component. For two nonzero pairs, an active bilinear block determines one direction from the other. A positive dimensional direction fibre has a nonconstant projection to at least one of the two projective direction lines. In that orientation, the coefficient minors below vanish for infinitely many directions and hence vanish identically as binary forms. Their values at $(1, 0)$ therefore vanish.

Let $\mathcal{C}_0^{\text{amp}}, \dots, \mathcal{C}_5^{\text{amp}}$ be the six normalized homogeneous cubic equations, in the order of the quartic equations $(Q_{15}, Q_{18}, Q_{13}, Q_{19}, Q_{10}, Q_{11})$. Their odd pair indices are $(0, 0, 1, 1, 2, 2)$. For a missing pair m , let (a, b) be the remaining indices, first in increasing and then in decreasing order; use $B = B_{ab}$ in the first orientation and $B = B_{ba}^\top$ in the second. On the dense direction chart substitute

$$v_a = x(t_0, t_1)^\top, \quad v_b = yJB^\top(t_0, t_1)^\top, \quad v_m = 0.$$

The four active cubics, after removal of their odd amplitudes, define a 4×2 matrix $M(t_0, t_1)$ on (x^2, y^2) . Its six two-row minors are homogeneous binary forms of degree six. A positive-dimensional direction set has nonconstant projection to at least one of the two projective lines. In that orientation the minors vanish at infinitely many directions and therefore vanish identically. They consequently vanish at $(t_0, t_1) = (1, 0)$, even if $JB^\top(1, 0)^\top = 0$. For a block of rank one, the two orientations together cover its direction components. Thus the endpoint equations below are necessary for a positive-dimensional direction fibre.

At $(t_0, t_1) = (1, 0)$, substitute

$$v_a = x(1, 0)^\top, \quad v_b = yJB^\top(1, 0)^\top, \quad v_m = 0.$$

Keep the four cubics whose odd pair is a or b , in their stated order. If a cubic has odd pair a , write its pullback as $x((M_{\text{pair}})_{r1}x^2 + (M_{\text{pair}})_{r2}y^2)$; if it has odd pair b , write it as $y((M_{\text{pair}})_{r1}x^2 + (M_{\text{pair}})_{r2}y^2)$. These coefficient extractions define the endpoint matrix $M_{\text{pair}} = M(1, 0)$.

Take its six two-row minors in the order $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$, reduce their nonzero numerators by $\mathfrak{g}$, and keep those independent of N . For each of the six choices of missing pair and orientation, this leaves three polynomials. Exactly one of them has three distinct irreducible factors; denote it by q_* , and denote the other two, in their stated order, by q_1, q_2 . For each irreducible factor u of q_* define

$$G_{m, \text{or}, u} = \gcd(\text{Res}_\varsigma(u, q_1), \text{Res}_\varsigma(u, q_2)) \in \mathbb{Q}[\eta, \zeta],$$

with the same monic normalization. Their factor families consist of the irreducible nonvertical factors of $G_{m, \text{or}, u}$ outside the plane localization factors. For such a factor f , apply the same homogenized tangency substitution $\varsigma = -f_\eta/f_\zeta$ to u, q_1, q_2 . If the three resulting polynomials are T_0, T_1, T_2 , the polynomial identity is

$$\gcd_{\mathbb{Q}(\eta)[\zeta]}(f, T_0, T_1, T_2) = 1.$$

For the six factors of highest degree, primitive integer representatives of these polynomials and reductions preserving their degrees establish this coprimality in characteristic zero.

For fixed directions of two nonzero pairs, quartic combinations give $\pm 4x^2y^2$. Their twelve axis configurations are excluded by unit coefficients except on $\varsigma = N = 0$ and $L = N = 0$. On

these families the highest weighted terms cancel, leaving quadratic necessary equations. For constant ζ , their coefficients in η admit a Bezout combination equal to a nonzero constant. For $\zeta = c\eta$, one coefficient is $32(1 - c)$. The localization gives $c \neq 1$, so the quadratic equations remain nonzero for every permitted curve constant, including $c = -1$.

For three nonzero pairs, let $\mathbf{A}_{\text{amp}}$ be the 6×3 matrix of coefficients in their squared amplitudes, after division of each cubic by its single odd amplitude. Restricting the cubics to the i th pair gives two forms $w_i m_i (B_{i,r1} w_i + B_{i,r2} m_i)$, $r = 1, 2$; this defines B_i . Up to units,

$$\det B_1 = L(L + \varsigma + 1), \quad \det B_2 = \varsigma(L + \varsigma - 1), \quad \det B_3 = L - \varsigma + 1.$$

The axis equations reduce rank zero to these five hypersurfaces. The three remaining direction charts have representatives

$$\begin{aligned} &((1, \theta_1^{\text{dir}}), (0, 1), (0, 1)), \\ &((0, 1), (1, \theta_2^{\text{dir}}), (1, 0)), \\ &((1, 0), (1, 0), (1, \theta_3^{\text{dir}})), \quad \theta_i^{\text{dir}} = -B_{i,11}/B_{i,12}. \end{aligned}$$

The pure cubic equations make each denominator and each θ_i^{dir} a unit on the corresponding hypersurface. Tangency to the five hypersurfaces gives respectively $\eta N, (\eta + 1)N, N, (\eta + 1)N, (\eta - 1)N = 0$. On the last hypersurface, $\eta = 1$ would force $\zeta = 1$, which is inverted. Thus $N = 0$ on each persistent branch. Table 7 gives the coefficients that exclude rank zero on the forced linear directions.

Table 7: Coefficients on the exceptional direction hypersurfaces.

Hypersurface	Nonzero coefficient or consequence
$L = 0$	$(\mathbf{A}_{\text{amp}})_{31} = 2(\eta - \zeta)$.
$\varsigma = 0$	$(\mathbf{A}_{\text{amp}})_{12} = 2(\zeta - 1)$.
$L - \varsigma + 1 = 0$	The mixed coefficient is $-4\eta^2((\eta - 1)\varsigma + 1)^2/[\varsigma(\varsigma - 1)]$; here $(\eta - 1)\varsigma + 1 = \zeta$.
$L + \varsigma + 1 = 0$	The mixed coefficient is $4\eta(\eta + \zeta)^2/(\zeta - 1)$. Its vanishing gives $\zeta = -\eta$, $\varsigma = -1$, and $(\mathbf{A}_{\text{amp}})_{31} = 4\eta$.
$L + \varsigma - 1 = 0$	The mixed coefficient is $-4\eta^4(\zeta + 1)^2/(\eta - \zeta)$. Its vanishing gives $\zeta = -1$, $\varsigma = 0$, and $(\mathbf{A}_{\text{amp}})_{12} = -4$.

Rank at least two leaves finitely many projective amplitudes. At rank one the kernel is a projective line, and a positive-dimensional amplitude fibre requires that line to divide all three quartics. If its defining linear form has a zero coefficient, the pure quartic coefficients $xy(x + y)^2$ and $xy(x^2 + y^2)$ force $xy = 0$. For the six configurations, let $i \in \{1, 2, 3\}$ and $\epsilon \in \{0, 1\}$, and set $(w_i, m_i) = (\epsilon, 1 - \epsilon)$. Take column i of $\mathbf{A}_{\text{amp}}$ after this substitution. For each other pair j , the two rows with odd amplitude in pair j are linear in (w_j, m_j) ; write them as

$$B_{i,\epsilon;j} \begin{pmatrix} w_j \\ m_j \end{pmatrix}, \quad d_{i,\epsilon;j} = \det B_{i,\epsilon;j}.$$

Order the other pairs increasingly as $j_0 < j_1$. The determinant $d_{i,\epsilon;j_\epsilon}$ is linear in N with unit coefficient. Define

$$n_{i,\epsilon} = -\frac{d_{i,\epsilon;j_\epsilon}|_{N=0}}{\partial_N d_{i,\epsilon;j_\epsilon}}, \quad P_{i,\epsilon} = \text{ng}_{\mathfrak{g}} \text{ pp num } d_{i,\epsilon;j_{1-\epsilon}}(N = n_{i,\epsilon}).$$

Here pp takes the primitive polynomial numerator, and $\text{ng}_{\mathfrak{g}}$ removes the factors that are units on the displayed direction chart. Apart from the factors ς and L , which give the two straight branches, $P_{i,\epsilon}$ has a unique irreducible factor $F_{i,\epsilon}$. Put

$$\mathcal{D}_{i,\epsilon} = \partial_{\eta} + \varsigma \partial_{\zeta} + n_{i,\epsilon} \partial_{\varsigma}.$$

Over the localization of $\mathbb{Q}(\eta)[\zeta, \varsigma]$ at the displayed direction factors, the following polynomial identities hold:

$$\Gamma_{i,\epsilon} \in (F_{i,\epsilon}, \mathcal{D}_{i,\epsilon} F_{i,\epsilon}, \mathcal{D}_{i,\epsilon}^2 F_{i,\epsilon}),$$

where, in the order $(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)$,

$$\begin{aligned} \Gamma_{1,0} &= \eta\varsigma - \zeta + 1, \\ \Gamma_{1,1} &= (\varsigma - 1)\zeta, \\ \Gamma_{2,0} &= \varsigma - 1, \\ \Gamma_{2,1} &= (\eta\varsigma - \zeta + 1)\zeta, \\ \Gamma_{3,0} &= (\eta - \zeta)((\eta + 1)\varsigma - \zeta)^3 \zeta^3, \\ \Gamma_{3,1} &= (\zeta - 1)((\eta + 1)\varsigma - \zeta)^3 \zeta^3. \end{aligned}$$

Each $\Gamma_{i,\epsilon}$ is a unit in this localization. Clearing the coefficient denominators multiplies the right side by a nonzero polynomial in η and a product of the displayed direction factors. Since η is nonconstant on $\mathcal{B}$, these identities exclude all six configurations at its generic point. The amplitude line with three nonzero coefficients is treated by $\mathcal{P}_1, \mathcal{P}_2$ in Appendix C.20.

The distinguished direction with full support is

$$\begin{aligned} (w_1, m_1) &= \left(\frac{\varsigma}{2\zeta}, 1 \right), \quad N = \varsigma L / (\eta\zeta), \\ (w_2, m_2) &= (w_3, m_3) = \left(-\frac{L}{2\eta\zeta}, 1 \right). \end{aligned}$$

For these amplitude conics, put $k_{\text{dir}} = \eta\varsigma/\zeta$. The three amplitude conics have determinants

$$\begin{aligned} &-4(k_{\text{dir}} - 2)(k_{\text{dir}} - \tfrac{1}{2})k_{\text{dir}}(k_{\text{dir}} - 1)^2, \\ &-4(k_{\text{dir}} - 1)(k_{\text{dir}} - \tfrac{1}{2})(k_{\text{dir}} + 1)k_{\text{dir}}^2, \\ &2(k_{\text{dir}} - 2)(k_{\text{dir}} - 1)k_{\text{dir}}(k_{\text{dir}} + 1). \end{aligned}$$

The y^2, z^2 coefficient minor of the first two conics is $-2(k_{\text{dir}} - 1)^2$. Thus a nondegenerate conic and a nonproportional second conic exist for every $k_{\text{dir}} \neq 0, 1$, including $k_{\text{dir}} = 2, \frac{1}{2}$. Their common projective zero set is finite, as is its inverse image under the squaring map on pair amplitudes. At $\varsigma = 0$ the cubic amplitude row is $(0, (\zeta - 1)/(4\eta^3), 0)$; at $L = 0$ it is $((\eta - \zeta)/(4\eta^3), 0, 0)$. Both contradict full support. For the seven patterns with at least one $m_i = 0$, the necessary equations are the homogeneous cubic and quartic equations. On their straight branches the amplitude rows are

$$\left(0, -\frac{(\zeta - 1)\zeta^2}{\zeta - \eta - 1}, 0 \right), \quad \left(-\frac{(\eta - \zeta)\zeta^2}{\eta^3(\zeta - \eta - 1)}, 0, 0 \right).$$

On each of $\varsigma = 0$ and $L = 0$, the distinguished direction and the direction with a vanishing m_i satisfy the mixed equations. The latter has $m_2 = 0$ and $m_1 = 0$, respectively, so these directions are distinct. The mixed direction bound makes them exhaustive. The displayed amplitude rows exclude full support on both graphs.

C.17 The quadratic hypersurface

Write the quadratic eliminant in the form

$$\mathcal{P}_0 = \eta\zeta(\eta^2 - 1)\varsigma^2 + 2(\zeta^2 - \eta^2)\varsigma - \eta\zeta(\zeta^2 - 1).$$

The polynomial $Q(w)$ has roots

$$\alpha = -\frac{L}{2\eta\zeta}, \quad \beta = \frac{(\varsigma - 1)(L + \varsigma)}{2\zeta K}, \quad \beta - \alpha = \frac{L^2 + \eta\varsigma^2 - \zeta}{2\eta\zeta K}.$$

If $M(w)$ is invertible, its equations give $U = V = Q(w) = 0$. At $w = \beta$, the mixed equations force $L^2 + \eta\varsigma^2 - \zeta = 0$, so $\beta = \alpha$. On $\mathcal{P}_0 = 0$ the determinant at α is zero. Thus every admissible direction lies on the determinant zero set, also when the two roots coincide.

The four cubic direction equations are ordered as the cubic part of Q_0 , followed by the diagonal second tangencies $(1, 1)$, $(2, 2)$ and $(3, 3)$. We use the row normalizations defining M in Appendix C.19. Let $\mu_{12}, \mu_{13}, \mu_{23}$ denote the quadratic mixed rows obtained from (C.32), after the direction substitution and $m_1 = m_2 = m_3 = 1$, normalized so that their coefficient of N is -1 . Set

$$f_0 = \mu_{13} - \mu_{12}, \quad f_1 = \mu_{23} - \mu_{12}.$$

Thus the six direction equations, in their specified order, are f_0, f_1 and the four cubic rows.

For $\eta \neq \pm 1$, let

$$\mathbb{E}_{\mathcal{P}_0} = \mathbb{Q}(\eta, \zeta)[\varsigma]/(\mathcal{P}_0).$$

For $u, v \in \mathbb{Q}(\eta, \zeta)$ its norm is

$$\text{Nm}_{\mathcal{P}_0}(u + v\varsigma) = u^2 - \frac{2(\zeta^2 - \eta^2)}{\eta\zeta(\eta^2 - 1)}uv - \frac{\zeta^2 - 1}{\eta^2 - 1}v^2.$$

We use num for the numerator after cancellation and pp for its primitive polynomial part.

Put

$$a_M = [w^2] \det M(w), \quad b_M = [w] \det M(w).$$

On $a_M \neq 0$, the second determinant root is

$$\gamma_M = -\frac{b_M}{a_M} - \alpha$$

since $\det M(\alpha) = 0$.

The two plane polynomials arising from the leading coefficient and the separation of the determinant roots are

$$P_{\text{c}} = \text{pp} \left(\frac{\text{num Nm}_{\mathcal{P}_0}(a_M)}{\eta^{11}\zeta(\eta - 1)^4(\eta + 1)^8(\zeta - \eta)^3(\zeta - 1)^3(\eta\zeta - \eta + \zeta)} \right),$$

$$P_{\text{sep}} = \text{pp} \left(\frac{\text{num Nm}_{\mathcal{P}_0}(\gamma_M - \alpha)}{(\zeta - \eta)(\zeta - 1)} \right).$$

Both displayed quotients are polynomials. Their total degrees are 19 and 20, respectively, and both have bidegree $(12, 14)$.

At $w = \alpha$, define the matrix

$$\mathbf{M}_\alpha = M(\alpha)_{\{1,2,3,4\}, \{1,2,3\}}.$$

Its columns correspond to (UV, U, V) . For each $I \subset \{1, 2, 3, 4\}$ with $|I| = 3$, put

$$d_{\alpha, I} = \det(\mathbf{M}_\alpha)_{I, \{1,2,3\}},$$

$$c_\alpha = \gcd_{|I|=3} \text{num}(d_{\alpha, I}),$$

$$p_{\alpha, I} = \frac{\text{num}(d_{\alpha, I})}{c_\alpha}.$$

The common numerator factor c_α is a nonzero rational multiple of

$$(\zeta - \eta)(\zeta - 1)\zeta^2\eta^2(\eta + 1)^2.$$

Reducing each $p_{\alpha,I}$ modulo $\mathcal{P}_0$, write

$$\begin{aligned} p_{\alpha,I} &\equiv a_{\alpha,I} + b_{\alpha,I}\zeta \pmod{\mathcal{P}_0}, \\ \Pi_{\alpha,I} &= \text{num Nm}_{\mathcal{P}_0}(a_{\alpha,I} + b_{\alpha,I}\zeta). \end{aligned}$$

The degree nine polynomial in the rank drop argument is

$$P_\alpha = \text{pp} \left(\frac{\gcd_{|I|=3} \Pi_{\alpha,I}}{(\zeta - \eta)^3(\zeta - 1)^3} \right).$$

The quotient is a polynomial of total degree nine.

For the other determinant root use $w_* = w - \alpha$, as in (C.30). Express the six direction equations in these coordinates and denote them, in order, by

$$(\psi_1, \dots, \psi_6) = (f_0, f_1, F_1^{\text{mom}}, F_2^{\text{mom}}, F_3^{\text{mom}}, F_4^{\text{mom}}),$$

where $F_1^{\text{mom}}, \dots, F_4^{\text{mom}}$ are the four cubic rows used in M_* . Define the six by four matrix $\mathbf{M}_{\text{six}}(w_*)$ by

$$(\mathbf{M}_{\text{six}})_{i,*} = \left(\partial_U \partial_V \psi_i|_{U=V=0}, \partial_U \psi_i|_{U=V=0}, \partial_V \psi_i|_{U=V=0}, \frac{\psi_i(0, 0, w_*)}{w_*} \right), \quad 1 \leq i \leq 6.$$

The last entry is a polynomial, since every ψ_i vanishes at $U = V = w_* = 0$. The direction equations are

$$\mathbf{M}_{\text{six}}(w_*)(UV, U, V, w_*)^\top = 0.$$

Let

$$d_{\text{six}}(w_*) = \det(\mathbf{M}_{\text{six}}(w_*)_{\{1,2,3,4\}, \{1,2,3,4\}}), \quad P_{\text{six}} = \text{pp} \left(\frac{\text{num Nm}_{\mathcal{P}_0}(d_{\text{six}}(\gamma_M - \alpha))}{(\zeta - 1)(\zeta - \eta)^2} \right).$$

Its norm numerator has bidegree (60, 69), and the displayed quotient is a polynomial. Its primitive part P_{six} has bidegree (58, 66) and constant coefficient content when considered as a polynomial in ζ over $\mathbb{Q}[\eta]$.

For a plane polynomial f , put

$$\begin{aligned} \Theta_{\mathcal{P}_0}(f) &= \eta\zeta(\eta^2 - 1)f_\eta^2 - 2(\zeta^2 - \eta^2)f_\eta f_\zeta \\ &\quad - \eta\zeta(\zeta^2 - 1)f_\zeta^2. \end{aligned}$$

On a nonvertical irreducible curve $f = 0$, differentiation gives $\varsigma = -f_\eta/f_\zeta$ at its generic point. Substitution in $\mathcal{P}_0 = 0$ and multiplication by f_ζ^2 give $\Theta_{\mathcal{P}_0}(f) \equiv 0 \pmod{f}$. A Bézout identity for f and $\Theta_{\mathcal{P}_0}(f)$ with right hand side 1 over $\mathbb{Q}(\eta)[\zeta]$ then excludes such a curve with nonconstant η .

The norms of the leading coefficient and root separation of the determinant of the coefficient matrix have, besides localization factors, the factor $(\eta + 1)\zeta - \eta$, P_{lc} and P_{sep} . For $f = P_{\text{lc}}$ or P_{sep} , $\gcd_{\mathbb{Q}(\eta)[\zeta]}(f, \Theta_{\mathcal{P}_0}(f)) = 1$. Their primitive integer coefficients retain their degrees under reduction, so Bezout identities imply coprimality over $\mathbb{Q}(\eta)$. The remaining factors $\zeta + 1$ and $\zeta + \eta$ give $\varsigma = 0$ and $L = 0$, respectively. The finite levels $\eta = \pm 1$ are excluded over the function field of a nonconstant ratio curve.

On the Möbius curve the derivatives are

$$\zeta = \frac{\eta}{\eta + 1}, \quad \varsigma = \frac{1}{(\eta + 1)^2}, \quad N = -\frac{2}{(\eta + 1)^3}.$$

On the full support chart $m_1 = m_2 = m_3 = 1$, the seven specialized direction equations generate the unit ideal over $\mathbb{Q}(\eta)$. Clearing denominators gives a polynomial identity whose right side is a nonzero polynomial of degree twelve in η . It excludes a nonconstant ratio curve on this branch.

At $w = \alpha$, a nonzero auxiliary vector requires a rank drop of a 4×3 matrix. After removal of localization factors, the polynomial P_α is coprime to $\Theta_{\mathcal{P}_0}(P_\alpha)$ over $\mathbb{Q}(\eta)$. The distinguished direction gives $N = \varsigma L/(\eta\zeta)$. Differentiation of $\mathcal{P}_0 = 0$ therefore yields

$$F_{\mathcal{P}_0} = \eta\zeta((\mathcal{P}_0)_\eta + \varsigma(\mathcal{P}_0)_\zeta) + \varsigma L(\mathcal{P}_0)_\varsigma = 0.$$

The resultant is

$$\begin{aligned} \text{Res}_\varsigma(\mathcal{P}_0, F_{\mathcal{P}_0}) = & -12(\zeta - \eta)(\zeta - 1)(\zeta + 1)(\eta - 1)(\eta + 1) \\ & \times (\eta + \zeta)\zeta^5\eta^5(\eta^2 + \zeta^2 + 1). \end{aligned}$$

The last factor is positive for real η, ζ . The other nonunit factors give finite η -levels or the two straight branches already treated. The differentiated equation holds on the discriminant as well.

For the other determinant root, put $w_* = w - \alpha \neq 0$ and use the six compatibility rows in (UV, U, V, w_*) . The polynomial P_{six} is coprime to both $(P_{\text{six}})_\zeta$ and $\Theta_{\mathcal{P}_0}(P_{\text{six}})$ over $\mathbb{Q}(\eta)$. The corresponding Bezout identities hold under a reduction preserving the degree 66 in ζ , and Gauss's lemma gives characteristic zero coprimality. Thus the other root cannot carry a persistent direction.

The remaining amplitude lines have three nonzero coefficients and are subject to the eliminants below. A finite family of algebraic branch conditions permits restriction to a persistent branch. Since η is nonconstant, a nonzero polynomial in η cannot vanish on such a branch.

C.18 The direction equations

The localization and the preceding branches cover the factors that may vanish identically on $\mathcal{B}$. On the remaining curve chart we may invert

$$\begin{aligned} g_* = & \eta\zeta(\eta - \zeta)(\zeta - 1)(\eta + 1)(1 + \eta - \zeta) \\ & \cdot (\zeta - 1)(\eta\varsigma - \zeta + 1)((\eta + 1)\varsigma - \zeta)\varsigma(\eta\varsigma - \zeta). \end{aligned} \tag{C.29}$$

The compatibility relations constrain the generic fibre of $Y \rightarrow \mathcal{B}$. Since η is nonconstant on the algebraic curve, every derivative $\varsigma = d\zeta/d\eta, \varsigma', \varsigma'', \dots$ used by these relations belongs to $\mathbb{C}(\mathcal{B})$. On the nonempty solution open set where $d\eta \neq 0$, the chain rule identifies these with the derivatives of the ratio functions. The curve reconstruction expresses R_1, R_2, R_3 rationally in the six off-diagonal coordinates by (C.28). Each necessary relation therefore holds on that open set and, by analyticity, on the connected solution chart. After clearing denominators it belongs to the invariant prime $\mathfrak{p}$. Consequently the generic fibre Y_{gen} is contained in the necessary variety with six coordinates over $\mathbb{C}(\mathcal{B})$. The argument below makes its projective boundary finite. Rational reconstruction transfers the dimension bound to all nine fibre coordinates of the algebraic closure, as required by Lemma 8.3.

At the projective boundary, each nonzero pair (w_i, m_i) has direction $(w_i : m_i) \in \mathbb{P}^1$. After representatives of the three directions are chosen, their scalar amplitudes are defined up to a common multiplier. Full support means $(w_i, m_i) \neq (0, 0)$ for all three pairs. The seven patterns with at least one $m_i = 0$ are treated separately in Appendix C.16. On the remaining charts set $t_i = w_i/m_i$ for the direction coordinates. The derivatives $\hat{\tau}_j$ are the state coordinates in (3.2). The distinguished direction is

$$t_1 = \frac{\varsigma}{2\zeta}, \quad t_2 = t_3 = -\frac{L}{2\eta\zeta}.$$

The cubic and quartic equations give finitely many projective amplitudes for this direction. The corresponding conics, and all vanishing pair or m_i patterns, are treated in Appendix C.16. We may therefore concentrate on the remaining directions on the charts $m_1 m_2 m_3 \neq 0$. Define U, V, w_* by the invertible affine change

$$t_1 = \frac{V - \eta w_*}{L} + \frac{\varsigma}{2\zeta}, \quad t_2 = \frac{U + w_*}{\varsigma} - \frac{L}{2\eta\zeta}, \quad t_3 = w_* - \frac{L}{2\eta\zeta}. \quad (\text{C.30})$$

Thus the distinguished direction has $U = V = w_* = 0$. With $c_m = L/(2\eta\zeta)$ and $w_* = w + c_m$, set

$$\begin{aligned} Q_*(w_*) &= 2\zeta w_* \{2\eta\zeta K w_* - (L^2 + \eta\varsigma^2 - \zeta)\}, \\ Q(w) &= (2\eta\zeta w + L) \{2\zeta K w - (\varsigma - 1)(L + \varsigma)\}. \end{aligned} \quad (\text{C.31})$$

Then $Q(w) = Q_*(w + c_m)$. This change makes the six direction equations linear in the auxiliary coordinates $(UV, U, V, Q_*(w_*))$. These coordinates also satisfy the quadratic relation $Q_*\xi_2\xi_3 - \xi_1\xi_4 = 0$, where $\xi = (UV, U, V, Q_*)$. Five linear combinations form the 5×4 matrix $M_*(w_*)$; the four trilinear equations form $M(w)$ after this translation. In polynomial matrix formulas a single indeterminate w denotes the argument of either matrix. The auxiliary vector of any other direction is nonzero, including at a double root of Q_* . The coefficient extraction and the translation identity are given in Appendix C.19, equations (C.33)–(C.34). The row factors in these formulas determine the matrices uniquely.

On an algebraic ratio curve with nonconstant η , use $\varsigma = d\zeta/d\eta$, $N = d\varsigma/d\eta$ and (C.28). Differentiating the tangencies $\mathcal{V}_j\zeta = \varsigma\mathcal{V}_j\eta$ gives

$$\mathcal{V}_i(\mathcal{V}_j\zeta) - \varsigma\mathcal{V}_i(\mathcal{V}_j\eta) - N(\mathcal{V}_i\eta)(\mathcal{V}_j\eta) = 0, \quad (\text{C.32})$$

for $(i, j) = (1, 2), (1, 3), (2, 3), (1, 1), (2, 2), (3, 3)$. Together with $Q_0 = 0$, these equations determine the homogeneous direction equations after substitution of the coordinate graph. On full support, choose representatives with $m_1 = m_2 = m_3 = 1$. After multiplication by units, each mixed row has coefficient -1 at N . Its two differences eliminate N . These differences and the four cubic rows, including Q_0 , give six polynomials independent of a, κ_0, N . Their coefficient contents are units in the stated localization.

Use w_* from (C.30). Each direction equation has the form

$$A(w_*)UV + B(w_*)U + C(w_*)V + D(w_*)w_*.$$

For the four cubic rows, D is divisible by Q_*/w_* . The same holds for $(L + 1)$ times the first mixed difference plus $(\varsigma - 1)$ times the second. These five rows are $\text{coeff}_\xi(F_i^{\text{mom}})$ in (C.33). Their polynomial coefficients and row factors are given in Appendix C.24.

C.19 The linear representation

Use L, K, c_m, Q_*, Q from (C.31). The coefficient matrices satisfy

$$M(w) = (M_*(w + c_m))_{1:4, 1:4}, \quad Q(w) = Q_*(w + c_m).$$

Substitution of $w_* = w + c_m$ in (C.30) gives

$$t_1 = \frac{V - \eta w}{L} + \frac{\varsigma - 1}{2\zeta}, \quad t_2 = \frac{U + w}{\varsigma} - \frac{(\varsigma - 1)L}{2\eta\zeta}, \quad t_3 = w.$$

The first four rows represent the same trilinear equations. Uniqueness of their coefficients in (UV, U, V, Q) gives the matrix identity.

Let $A(w)$ be the polynomial matrix specified in Appendix C.24. With the row factors

$$\begin{aligned} d_1^{\text{row}} &= \frac{421654016}{\varsigma\zeta^2L}, & d_2^{\text{row}} = d_3^{\text{row}} = d_4^{\text{row}} &= \frac{6588344}{\varsigma\zeta^2L}, \\ d_5^{\text{row}} &= \frac{1}{2\eta\varsigma\zeta^2(\eta + 1)(\eta - \zeta)(\zeta - 1)L}, \end{aligned}$$

we have

$$\begin{aligned} M_*(w) &= \text{diag}(d_1^{\text{row}}, \dots, d_5^{\text{row}})A(w), \\ M(w) &= \text{diag}(d_1^{\text{row}}, \dots, d_4^{\text{row}})A(w + c_m)_{1:4, 1:4}. \end{aligned}$$

These factors are units in the localization and are independent of w . The five maximal minors of M_* use its fifth row as well as the four rows of M . The translation $w \mapsto w + c_m$ extends to an automorphism of $\mathbb{P}^1$ fixing infinity. Thus it preserves ranks, multiplicities, and degree drops.

The coordinates $t_i = w_i/m_i$ describe the pair directions. The six equations consist of four trilinears and two mixed differences f_0, f_1 . The fifth row of M_* is their combination $(1 + \eta\varsigma - \zeta)f_0 + (\varsigma - 1)f_1$. The auxiliary vectors are (UV, U, V, Q_*) and (UV, U, V, Q) . If the first vector vanishes, then $U = V = Q_* = 0$. At the second possible root the mixed differences are $-(\varsigma - 1)(L^2 + \eta\varsigma^2 - \zeta)/(\eta\zeta K)$ and $(1 + \eta\varsigma - \zeta)(L^2 + \eta\varsigma^2 - \zeta)/(\eta\zeta K)$. They force $L^2 + \eta\varsigma^2 - \zeta = 0$, so the roots coincide with the distinguished direction. Every other admissible direction therefore has a nonzero auxiliary vector, including when Q_* has a double root.

Let $F_1^{\text{mom}}, \dots, F_4^{\text{mom}}$ be the cubic direction equations in centered coordinates, with the row normalizations of Appendix C.18, and put $F_5^{\text{mom}} = (L + 1)f_0 + (\varsigma - 1)f_1$. For the auxiliary vector $\xi = (UV, U, V, Q_*)$, define

$$\text{coeff}_\xi(F) = \left(\partial_U \partial_V F|_{U=V=0}, \partial_U F|_{U=V=0}, \partial_V F|_{U=V=0}, \frac{F(0, 0, w)}{Q_*(w)} \right). \quad (\text{C.33})$$

The quotient in the last entry is a polynomial in $\mathbb{Q}(\eta, \zeta, \varsigma)[w]$, so it is defined at every root of Q_* . We have

$$F_i^{\text{mom}} = \text{coeff}_\xi(F_i^{\text{mom}})(UV, U, V, Q_*)^\top, \quad (M_*)_{i,:} = \text{coeff}_\xi(F_i^{\text{mom}}), \quad 1 \leq i \leq 5. \quad (\text{C.34})$$

Indeed, the six equations have bidegree at most $(1, 1)$ in (U, V) , and the constant coefficient of each of these five rows is divisible by Q_* . This proves reconstruction and uniqueness of the rows.

C.20 The two eliminants

Any other direction gives a nonzero auxiliary vector in the kernel of M_* . The first condition on (η, ζ, ς) expresses this common kernel. The second imposes the quadratic relation on the kernel of the four rows. Eliminating the direction parameter w produces the polynomials $\mathcal{P}_1$ and $\mathcal{P}_2$ below. On $g_*\mathcal{P}_0 \neq 0$, an admissible full support direction with $m_1 m_2 m_3 \neq 0$, distinct from the distinguished direction, gives $\mathcal{P}_1 = \mathcal{P}_2 = 0$.

For a rational polynomial in w , let pp denote the primitive polynomial numerator after clearing coefficient denominators. Let ng_{g_*} remove only irreducible factors of g_* and normalize the remaining polynomial to leading coefficient one in graded lexicographic order $\eta > \zeta > \varsigma$. Use the matrices with the row normalizations specified in Appendix C.19, and define

$$C_{ij} = [w^{j-1}] \text{pp det}(M_* \text{ with row } i \text{ omitted}), \quad 1 \leq i \leq 5, \quad 1 \leq j \leq 3, \quad (\text{C.35})$$

$$\mathcal{P}_1 = \text{ng}_{g_*} \gcd_{|I|=3} \det C_I, \quad \deg \mathcal{P}_1 = 38, \quad \deg_\varsigma \mathcal{P}_1 = 11. \quad (\text{C.36})$$

The five maximal minors have degree at most two in w . A common root of the maximal minors makes $C(1, w, w^2)^\top = 0$, hence all ten minors of order three vanish. The following residual identity then gives $\mathcal{P}_1 = 0$.

Let Δ_i , $0 \leq i \leq 9$, be the ten minors of order three of the coefficient matrix. Their factorizations and a polynomial identity for the residual factors are

$$\Delta_i = \mathcal{P}_1 \varepsilon_i r_i, \quad d(\eta) \zeta^2 (\zeta - 1)^2 = \sum_i A_i r_i,$$

where ε_i are units in the previous localization, $d \neq 0$, and only the first six polynomial multipliers A_i are nonzero. In the ring localized at the previously inverted factors and $d(\eta)$, these identities give

$$(\Delta_0, \dots, \Delta_9) = (\mathcal{P}_1).$$

Indeed one inclusion follows from the first identities. Multiplying the second identity by $\mathcal{P}_1$ expresses $\mathcal{P}_1$ as a linear combination of the Δ_i , after division by a unit. Let $\mathbf{N} = MP_*$, where the determinant-one column transformation P_* is given by (C.41). Its column degrees are bounded by $(1, 0, 1, 0)$, so columns two and four are constant in w . The six minors of these two columns are unit multiples in the previous localization of the same first six residuals. The same identity therefore proves that those two columns have rank two. Thus $\mathcal{P}_1 = 0$, and the six minors of the constant columns define an open cover.

For the second condition, use the four rows. Where M has rank three and determinant zero, every nonzero adjugate column spans its kernel. Substitution into the quadratic form gives the polynomial f_j below; its resultant with the determinant eliminates w . These columns are local generators of the kernel line, on which the quadratic has the norm constructed below.

Set $D = \text{pp det } M$ and let a_j be the j th column of $\text{adj } M$. Define

$$f_j = \text{pp}(Q(w)(a_j)_2(a_j)_3 - (a_j)_1(a_j)_4), \quad R_j = \text{Res}_w(D, f_j), \quad j = 1, 2, \quad (\text{C.37})$$

$$\mathcal{P}_0 = \eta^3 \zeta \zeta^2 - \eta \zeta^3 - \eta \zeta \zeta^2 - 2\eta^2 \zeta + 2\zeta^2 \zeta + \eta \zeta, \quad (\text{C.38})$$

$$\mathcal{P}_2 = \text{ng}_{g_*}(\text{gcd}(R_1, R_2)/\mathcal{P}_0), \quad \deg \mathcal{P}_2 = 48, \quad \deg_\zeta \mathcal{P}_2 = 14. \quad (\text{C.39})$$

The quotient is polynomial up to a nonzero rational scalar. The zero sets of $\mathcal{P}_1, \mathcal{P}_2$ are unchanged by primitive rescaling of the rows. The resultant identities have the form $R_j = \varepsilon_j \mathcal{P}_0 \mathcal{P}_2 J_j^2$, where ε_j is a unit in the previous localization and $\text{gcd}(J_1, J_2) = 1$. The factor $\mathcal{P}_0$ corresponds to the branch treated by the direction equations in Appendix C.17. The norm argument below gives the necessary condition $\mathcal{P}_0 \mathcal{P}_2 = 0$.

The coefficient identities for the lower rank and identically zero determinant cases introduce a second nonzero polynomial $\delta(\eta)$. Both $d(\eta)$ and $\delta(\eta)$ are units in $\mathbb{C}(\mathcal{B})$ when η is nonconstant. Thus they apply at the generic point of the ratio curve. If η is constant, cyclic relabeling gives a nonconstant ratio coordinate. Characteristic directions, proper supports, and the distinguished direction are treated in Appendices C.14 and C.16.

C.21 Reduction of the determinants

Write

$$Q(w) = a_Q w^2 + b_Q w + c_Q, \quad a_Q = 4\eta \zeta^2 (1 + \eta - \zeta). \quad (\text{C.40})$$

Since a_Q is a unit on the direction chart, the determinant-one column transformation

$$P_* = \begin{pmatrix} 1 & \eta(w + b_Q/a_Q) & -w & \eta/a_Q \\ 0 & 1 & 0 & 0 \\ 0 & -\eta & 1 & 0 \\ 0 & -a_Q w & 0 & 1 \end{pmatrix}. \quad (\text{C.41})$$

is defined there. For $x = P_*y$, multiplication gives

$$\begin{aligned} Qx_2x_3 - x_1x_4 &= w(a_Qy_1y_2 + b_Qy_2y_3 + y_3y_4) + c_Qy_2(y_3 - \eta y_2) - y_1y_4 \\ &\quad - \frac{\eta}{a_Q}y_4(b_Qy_2 + y_4). \end{aligned} \quad (\text{C.42})$$

The coefficient of y_iy_j has degree at most $(1, 0, 1, 0)_i + (1, 0, 1, 0)_j$. These bounds define a quadratic form with the corresponding column weights on the projective line.

Put $\mathbf{N} = MP_*$. Then $\deg_w \mathbf{N}_{ij} \leq (1, 0, 1, 0)_j$ and $P_* \operatorname{adj} \mathbf{N} = \operatorname{adj} M$. For each three-element row set $I \subset \{1, 2, 3, 4\}$ and $j \in \{1, 3\}$, form

$$\mathcal{P}_{I,j}(w) = \operatorname{pp} \det \mathbf{N}_{I,\{2,4,j\}} = a_{I,j} + b_{I,j}w.$$

Division by units of the stated localization gives the eight pencils, which have a common projective root precisely when the 8×2 coefficient matrix $(a_{I,j}, b_{I,j})$ has rank at most one. Let I_{rank} be generated by its twenty-eight 2×2 minors, and let $I_{\det}$ be generated by the three coefficients of $D(w)$, of degree at most two. The coefficient identities are

$$\delta(\eta)\varsigma(\varsigma - 1)\mathcal{P}_0 \in I_{\operatorname{rank}} \cap I_{\det}, \quad \delta \neq 0.$$

Over $\mathbb{C}(\mathcal{B})$ these exceptional loci therefore reduce to $\varsigma = 0$, $\varsigma = 1$, or $\mathcal{P}_0 = 0$.

Lemma C.10 (Reduction to a 2×2 pencil). *Suppose the constant columns $\mathbf{N}_{\{2,4\}}$ have rank two over the base. Their six 2×2 minor charts cover the base. On one chart put its pivot rows and constant columns first:*

$$\tilde{\mathbf{N}}(w) = \begin{pmatrix} A_0 & B_0(w) \\ C_0 & D_0(w) \end{pmatrix}, \quad \Delta = \det A_0.$$

Here A_0, C_0 are constant in w , B_0, D_0 have degree at most one, and Δ is a unit. Define

$$\mathbf{R}_{\operatorname{Schur}}(w) = \Delta D_0 - C_0 \operatorname{adj}(A_0) B_0, \quad \mathbf{L}_{\operatorname{Schur}}(w) = \begin{pmatrix} -\operatorname{adj}(A_0) B_0 \\ \Delta \operatorname{Id}_2 \end{pmatrix}.$$

Then $\mathbf{R}_{\operatorname{Schur}}$ is a 2×2 linear pencil and

$$\det \mathbf{R}_{\operatorname{Schur}} = \Delta \det \tilde{\mathbf{N}}, \quad \tilde{\mathbf{N}} \mathbf{L}_{\operatorname{Schur}} = \begin{pmatrix} 0 \\ \mathbf{R}_{\operatorname{Schur}} \end{pmatrix}, \quad (\text{C.43})$$

$$\mathbf{L}_{\operatorname{Schur}} \operatorname{adj}(\mathbf{R}_{\operatorname{Schur}}) = \Delta \operatorname{adj}(\tilde{\mathbf{N}})_{\{3,4\}}. \quad (\text{C.44})$$

With $E_0 = \operatorname{adj}(A_0)$ and $J_0 = \operatorname{adj}(\mathbf{R}_{\operatorname{Schur}})$, the adjugate satisfies

$$\Delta^2 \operatorname{adj}(\tilde{\mathbf{N}}) = \begin{pmatrix} (\det \mathbf{R}_{\operatorname{Schur}}) E_0 + E_0 B_0 J_0 C_0 E_0 & -\Delta E_0 B_0 J_0 \\ -\Delta J_0 C_0 E_0 & \Delta^2 J_0 \end{pmatrix}. \quad (\text{C.45})$$

On $\det \tilde{\mathbf{N}} = 0$, rank three is equivalent to $\operatorname{rank} \mathbf{R}_{\operatorname{Schur}} = 1$. The two columns in (C.44) generate the kernel locally. Substitution of either column in the transformed quadratic relation gives a polynomial of degree at most four in w .

Proof. Block elimination with the invertible A_0 proves (C.43) and changes rank by two. Block inversion followed by multiplication by the determinant gives (C.44) and (C.45); these identities are polynomial before localization. A 2×2 matrix of rank one has a nonzero adjugate, whose columns generate its kernel wherever an adjugate entry is a unit. This is local on the determinantal scheme and holds at a multiple root.

Put $\mathbf{y}_c = (y_2, y_4)$ and $\mathbf{y}_1 = (y_1, y_3)$. In the column order $(\mathbf{y}_c, \mathbf{y}_1)$, the upper and lower blocks of $\mathbf{L}_{\operatorname{Schur}} \operatorname{adj}(\mathbf{R}_{\operatorname{Schur}})$ have degrees at most two and one. The coefficient degree bounds in (C.42) therefore give degree at most four for the substituted quadratic. $\square$

Remark C.11 (The six minor charts). Over the localization at $d(\eta)$, the six minors are unit multiples of $r_0, r_1, r_3, r_2, r_4, r_5$. The identity

$$d(\eta)\varsigma^2(\varsigma - 1)^2 = \sum_{i=0}^5 A_i r_i$$

makes their ideal the unit ideal. Hence these charts cover the function field of a ratio curve with nonconstant η . A curve with constant η is treated after cyclic relabeling. For permutation matrices U, V with $\tilde{\mathbf{N}} = UMP_*V$,

$$\text{adj}(M) = (\det U)(\det V)P_*V \text{adj}(\tilde{\mathbf{N}})U.$$

The determinant and substituted quadratic form are homogenized with degrees two and four. This includes their roots at infinity when a leading coefficient vanishes. The matrix $P_*(w)$ gives an affine change of presentation; the displayed adjugate identity relates it to the homogenized polynomials defining the resultants.

Lemma C.12 (Norm of the fixed degree resultant). *Let*

$$d(w) = d_2 w^2 + d_1 w + d_0, \quad f(w) = \sum_{i=0}^4 f_i w^i,$$

with resultant defined at the fixed degrees (2, 4). Put

$$\begin{aligned} a_f &= d_2^3 f_1 - d_1 d_2^2 f_2 + d_2(d_1^2 - d_0 d_2) f_3 + (-d_1^3 + 2d_0 d_1 d_2) f_4, \\ b_f &= d_2^3 f_0 - d_0 d_2^2 f_2 + d_0 d_1 d_2 f_3 + (d_0^2 d_2 - d_0 d_1^2) f_4. \end{aligned}$$

Then the universal polynomial identity is

$$\det \begin{pmatrix} b_f & -d_0 a_f \\ a_f & d_2 b_f - d_1 a_f \end{pmatrix} = d_2^3 \text{Res}_{2,4}(d, f). \quad (\text{C.46})$$

The determinant is divisible by d_2^3 in the universal coefficient ring, and its quotient is the fixed degree resultant. In particular, the identity holds at $d_2 = 0$.

Proof. The remainder is $d_2^3 f \equiv a_f w + b_f \pmod{d}$. Where $d_2 \neq 0$, let α, β be the roots with their multiplicities. The norm of $a_f w + b_f$ is $b_f^2 - (d_1/d_2) a_f b_f + (d_0/d_2) a_f^2$. Multiplication by d_2 , together with $\text{Res}_{2,4}(d, f) = d_2^4 f(\alpha) f(\beta)$, proves (C.46) on a dense coefficient open set. It is therefore an identity in the universal polynomial ring. Its quotient by d_2^3 specializes to the fixed degree resultant under every specialization, including $d_2 = 0$ and repeated or infinite roots. $\square$

C.22 The norm on the determinant divisor

The scalar form of (C.46) is

$$d_2 b_f^2 - d_1 a_f b_f + d_0 a_f^2 = d_2^3 \text{Res}_{2,4}(d, f).$$

It lies in the polynomial ring in the eight coefficients of d, f . Thus it applies to nonreduced quadratic algebras as well.

On the remaining regular open set of the parameter space, the homogeneous determinant defines a finite flat divisor of degree two in the relative projective line. The rank three matrix has a kernel line on this divisor, including at a double point. The coefficient degree bounds in (C.42) define a global quadratic form on that line.

Its finite flat norm is a nonzero section. Indeed, if it vanished identically, both adjugate-column resultants would vanish identically, contrary to their nonzero factorizations. Its zero set is therefore a Cartier divisor on the regular parameter space. Each irreducible component of that divisor is contained in both $R_1 = 0$ and $R_2 = 0$ and hence divides their gcd. Its support consequently lies in $\mathcal{P}_0\mathcal{P}_2 = 0$. An admissible direction lies in this support; after the preceding exclusion of $\mathcal{P}_0 = 0$, it follows that $\mathcal{P}_2 = 0$. The norm is constructed on the ambient parameter open set, and its equation is then restricted to $\mathcal{B}$; the curve may be contained in its zero divisor.

C.23 Intersection orders and slope elimination

On $\mathcal{B}$, $\varsigma = d\zeta/d\eta$. Differentiating $\mathcal{P}_1 = \mathcal{P}_2 = 0$ along the curve gives

$$\begin{aligned}(\mathcal{P}_1)_\eta + \varsigma(\mathcal{P}_1)_\zeta + (\mathcal{P}_1)_{\varsigma'} &= 0, \\ (\mathcal{P}_2)_\eta + \varsigma(\mathcal{P}_2)_\zeta + (\mathcal{P}_2)_{\varsigma'} &= 0.\end{aligned}$$

Cross multiplication eliminates ς' and gives $C_* = 0$. Eliminating ς gives

$$\begin{aligned}C_* &= (\mathcal{P}_1)_\varsigma((\mathcal{P}_2)_\eta + \varsigma(\mathcal{P}_2)_\zeta) \\ &\quad - (\mathcal{P}_2)_\varsigma((\mathcal{P}_1)_\eta + \varsigma(\mathcal{P}_1)_\zeta), \\ R &= \text{Res}_\varsigma(\mathcal{P}_1, \mathcal{P}_2), \quad T = \text{Res}_\varsigma(\mathcal{P}_1, C_*).\end{aligned}\tag{C.47}$$

The fixed Sylvester degrees are $(11, 14, 25)$ for $(\mathcal{P}_1, \mathcal{P}_2, C_*)$. Set

$$\begin{aligned}D_0 &= \zeta^{12}(\zeta + 1)^{20}(\zeta + \eta)^{20}(\zeta - \eta)^{25}(\zeta - 1)^{25} \\ &\quad \cdot (\zeta - \eta - 1)^{26}((\eta + 1)\zeta - \eta)^6.\end{aligned}\tag{C.48}$$

Lemma C.13 (Elimination of the slope). *The polynomial D_0 divides both R and T , and*

$$\gcd(R/D_0, T/D_0) = 1 \quad \text{in } \mathbb{Q}(\eta)[\zeta].$$

Proof. Over $\mathbb{Q}(\eta)$, the relevant intersections in (ζ, ς) are the points

$$\begin{aligned}(0, 0), \quad (-1, -2/\eta), \quad (-\eta, 1), \quad (\eta, 1), \\ (1, 0), \quad (\eta + 1, 1), \quad (\eta/(\eta + 1), (\eta + 1)^{-2}),\end{aligned}$$

respectively. The first three order columns refer to the maximal ideal at each point; the last is a common lower bound for the intersection multiplicities of $(\mathcal{P}_1, \mathcal{P}_2)$ and $(\mathcal{P}_1, C_*)$. Table 8 records these orders and the resulting lower bounds for both resultants.

Table 8: Intersection orders on the exceptional lines.

Line in ζ	Order of $\mathcal{P}_1$	Order of $\mathcal{P}_2$	Lower order of C_*	Common resultant order
$\zeta = 0$	4	3	6	12
$\zeta = -1$	4	5	7	20
$\zeta = -\eta$	4	5	7	20
$\zeta = \eta$	5	5	8	25
$\zeta = 1$	5	5	8	25
$\zeta = \eta + 1$	4	5	8	26
$\zeta = \eta/(\eta + 1)$	2	3	4	6

At $(\zeta, \varsigma) = (\eta + 1, 1)$, the two common exceptional tangents are $\alpha = (2\eta + 1)/(2(\eta^2 + \eta + 1))$ and $\beta = (2\eta + 1)/(3\eta(\eta + 1))$. They contribute additional intersection orders at least four

and two. At the first tangent the implicit slope is $-(\eta - 1)(4\eta^2 + 7\eta + 4)/(8(\eta^2 + \eta + 1)^3)$; the cubic initial form of the other equation vanishes on this slope. Hence the common order is at least $4 \cdot 5 + 4 + 2 = 26$. The other entries follow from the displayed orders and the local intersection lower bound. Thus $D_0 \mid R, T$.

To estimate the degree at infinity, put $t = 1/\zeta$ in the fixed 25×25 Sylvester matrix for R . The weight of row r is $13 - r$ for $1 \leq r \leq 14$ and $30 - r$ for $15 \leq r \leq 25$; column j has weight $j - 1$, $1 \leq j \leq 25$. Their sum is 487. The weighted matrix therefore has determinant $t^{487}R(1/t)$ and entries regular at $t = 0$. The first 23 rows and columns form a unit block, leaving a 2×2 Schur matrix $S(t) = t^2S_2 + t^3S_3 + O(t^4)$. The identities $\det S_2 = 0$ and $(S_2)_{11}(S_3)_{22} + (S_3)_{11}(S_2)_{22} - (S_2)_{12}(S_3)_{21} - (S_3)_{12}(S_2)_{21} = 0$ give $\text{ord}_{t=0} \det S(t) \geq 6$. Hence $\deg_\zeta R \leq 487 - 6 = 481$. Under reduction modulo 32003 and specialization at $\eta = 3$, we have

$$\begin{aligned} p_* = 32003, \quad \eta = 3 : \quad \deg_\zeta \bar{R} = 481, \quad \deg_\zeta \bar{D}_0 = 134, \\ \gcd(\bar{R}/\bar{D}_0, \bar{T}/\bar{D}_0) = 1. \end{aligned} \tag{C.49}$$

All coefficient denominators are units at this reduction, and the fixed ς -degrees 11, 14, 25 are preserved. Since D_0 is primitive over $\mathbb{Z}_{(p_*)}[\eta]$, Gauss's lemma gives the quotients in that ring. The first quotient has ζ -degree at most $481 - 134 = 347$ and reduced degree 347; its leading coefficient is therefore a unit at the specialization. A primitive common factor of positive ζ -degree would retain that degree in the reduction and divide both reduced quotients. The displayed gcd, equivalently a Bezout identity equal to one, excludes such a factor. Hence the quotients are coprime in $\mathbb{Q}(\eta)[\zeta]$. $\square$

A nonconstant ratio curve therefore lies in one of the seven factors of D_0 . Four factors are units on the current chart. The lines $\zeta = -1$ and $\zeta = -\eta$ give $\varsigma = 0$ and $L = 0$, respectively. The Möbius curve $(\eta + 1)\zeta = \eta$ gives $\mathcal{P}_0 = 0$. The preceding branch arguments bound their generic state fibres by one and therefore give state rank at most two.

C.24 Coefficients of the linear equations

Supplement S8 gives the twenty entries of $A(w)$ and the row factors relating it to M_* and M . These are the coefficient expansion of (C.33)–(C.34).

D Calculations for common Euclidean isometries

This appendix gives the calculations used in Section 6. We use the metric convention, state, and fixed data introduced there.

D.1 Translation fibres

With the reconstruction (6.10), the inverse planar flux is $G^{-1}\nabla v = (\tau/d_0)(C_0n - G_{nt}t)$. Write

$$D^2v = \begin{pmatrix} a_H & b_H \\ b_H & P - a_H \end{pmatrix}_{n,t}, \quad N = d(E - 2K)\tau^2 + K\tau^6 - 2d^2.$$

With u, w as in (D.2), the inverse flux residual is

$$\text{div}(G^{-1}\nabla v) - Q = ua_H + wb_H + C_0P/d_0 - Q =: \mathcal{E}, \quad w = -\tau G'_{nt}/d_0.$$

The second equation in (6.9), after multiplication by $G_{nt}d_0\tau^4$, is

$$-2G_{nt}d\tau^2a_H + Nb_H + G_{nt}KP\tau^4 + G_{nt}Pd\tau^2 - G_{nt}Qd_0\tau^4 = 0. \tag{D.1}$$

Its direction is $(N, 2G_{nt}d\tau^2)$. At a point align fixed Cartesian axes with (n, t) and set $c_H = v_{xxx}$, $e_H = v_{xyy}$; then $v_{xyy} = -c_H$ and $v_{yyy} = -e_H$. Differentiating the second equation of (6.9) in both directions yields a system for (c_H, e_H) with matrix

$$M = \begin{pmatrix} u & w \\ -w & u \end{pmatrix}, \quad u = -\frac{2d}{d_0\tau^2}, \quad w = \frac{N}{G_{nt}d_0\tau^4}, \quad \det M = \frac{4G_{nt}^2d^2\tau^4 + N^2}{G_{nt}^2d_0^2\tau^8} > 0. \quad (\text{D.2})$$

For $d \neq 0$ the first numerator term is positive; for $d = 0$ the second is $K^2\tau^{12}$. The determinant is therefore positive.

The differential rules are

$$\begin{aligned} n\tau &= a_H, & t\tau &= b_H, & na_H &= c_H + 2b_H^2/\tau, \\ ta_H &= e_H + 2b_H(P - a_H)/\tau, \\ nb_H &= e_H + (P - 2a_H)b_H/\tau, \\ tb_H &= -c_H + (P - 2a_H)(P - a_H)/\tau, \\ V_n &= A'a_H + G'_{nt}b_H + (A - C_0)(P - a_H)/\tau - 2G_{nt}b_H/\tau, \\ V_t &= G'_{nt}a_H + C'_0b_H + (A - C_0)b_H/\tau + 2G_{nt}(P - a_H)/\tau, \\ \text{curl div } G &= nV_t - tV_n + V_nb_H/\tau + V_t(P - a_H)/\tau. \end{aligned}$$

Here $G'_{nt} = (AC_0 - d_0)'/(2G_{nt})$. These rules and the two derivatives of (D.1) determine a quadratic residual after the adjugate solution of (D.2), after multiplying the substituted curl residual by the determinant of that third derivative system. This determinant is strictly positive on the chart. Parametrize the affine line (D.1) by its stated direction and a scalar s_H . The coefficient of s_H^2 is

$$-\frac{G_{nt}\tau^6 H(\tau)}{d_0^2 D(\tau)^3}, \quad D = -\tau^4 G_{nt}^2, \quad \deg H = 24, \quad [\tau^{24}]H = -K^6. \quad (\text{D.3})$$

To obtain this coefficient, take the homogeneous Hessian part of the residual, set $P = Q = 0$, substitute $(a_H, b_H) = (N, 2G_{nt}d\tau^2)$, replace G_{nt}^2 using (6.10), and collect powers of τ . Since $K > 0$, only finitely many τ levels can annul this coefficient. Away from them there are at most two Hessians at fixed τ .

To fix the normalization, write the differentiated inverse flux equations as $M(c_H, e_H)^\top + r = 0$ and denote the planar curl residual by $\mathcal{C}$. The residual used in (D.3) is

$$\hat{\mathcal{C}} = \det M \mathcal{C}|_{c_H=e_H=0} - (\partial_{c_H}\mathcal{C}, \partial_{e_H}\mathcal{C}) \text{adj}(M)r. \quad (\text{D.4})$$

It is $\det M$ times the substituted curl residual. Since $\det M > 0$, its zero set is the zero set of the substituted curl residual.

D.2 Translation and rotation

We use $\mathcal{N}_b = b^\top b$ as in Section 6. The harmonic reduction follows from the identity below. In coordinates (q, v) let

$$\mathcal{E}_{\text{div}} = \text{div}(vG^{-1}a), \quad \mathcal{E}_{\text{tr}} = \text{tr}\{G^{-1}(K + vaa^\top)\} + v - \mathcal{I}_1, \quad \mathcal{E}_{\text{phase}} = (da)_{qv} + 2p/v^2,$$

and $\mathbf{j} = vG^{-1}a$. Put $E_C = d \text{div}_G(K + vaa^\top)$ and $E_L = d \text{div}_G(vK)$. An identity before imposing any equation is

$$\begin{aligned} (E_C - v^{-1}E_L)_{qv} + 4\Delta_G z_f/v &= a_v \partial_q \mathcal{E}_{\text{div}} - a_q \partial_v \mathcal{E}_{\text{div}} + v^{-1} \partial_v \mathcal{E}_{\text{tr}} \\ &\quad + 2\mathcal{E}_{\text{phase}} \mathcal{E}_{\text{div}} + (\partial_q \mathcal{E}_{\text{phase}}) \mathbf{j}^q + (\partial_v \mathcal{E}_{\text{phase}}) \mathbf{j}^v. \end{aligned} \quad (\text{D.5})$$

The proof expands

$$(\operatorname{div}_G T)_b = \sum_{i,j} \partial_i ((G^{-1})^{ij} T_{jb}) + \frac{1}{2} \sum_{i,j} T_{ij} \partial_b (G^{-1})^{ij}.$$

All first and second jets of the three entries of G^{-1} and the two entries of a are independent in this expansion. For the phase function, $\mathcal{E}_{\text{phase}} = 0$, and the constant spectral and vertical equations give $\mathcal{E}_{\text{tr}} = 0$, $\mathcal{E}_{\text{div}} = K_{C_f}$. Equation (D.5) therefore gives $\Delta_G z_f = 0$.

On the remaining chart write

$$a_q = U, \quad a_v = QLU/v, \quad U^2 = \frac{\mathcal{N}_b}{Q(v + L^2)}, \quad P = v^3 - \mathcal{I}_1 v^2 + \mathcal{I}_2 v - j, \quad (\text{D.6})$$

and define the signed reconstruction variables

$$A = \frac{j}{v} - \frac{P}{v\mathcal{N}_b}, \quad R_0 = \mathcal{I}_2 - A - \frac{j(1 + \mathcal{N}_b)}{v}, \quad T^2 = vAR_0 - jv^2. \quad (\text{D.7})$$

Here $\mathcal{N}_b, A, R_0 > 0$ and $T \neq 0$, with signed variables T, U . The spectral identities give equivalently

$$T^2 = -\frac{1}{v\mathcal{N}_b^2} \prod_{i=1}^3 ((v - \lambda_j)(v - \lambda_k) + \lambda_j \lambda_k \mathcal{N}_b), \quad \{i, j, k\} = \{1, 2, 3\}.$$

The metric reconstruction is

$$\begin{aligned} r &= (-K_{22}^{-1} a_v, K_{11}^{-1} a_q), \quad s_0 = \mathcal{I}_2/j - (1 + \mathcal{N}_b)/v, \\ G_{ij} &= \frac{A}{j} K_{ij} + \frac{v}{\mathcal{N}_b} (s_0 - 2A/j) a_i a_j - \frac{T}{j\mathcal{N}_b} (a_i r_j + r_i a_j). \end{aligned} \quad (\text{D.8})$$

Then $G^{-1} = j \operatorname{adj} G$. This gives $\det G^{-1} = j$ and the two trace identities. Substitution in the vertical equations and harmonicity gives the affine system below.

The case $U = 0$. Before using L , suppose $\psi_q = 0$ and write $w = \psi_v$. The inverse vertical equation is

$$(1 + p^2/(2q))w_v = -K_{C_f}^{-1}.$$

For $p \neq 0$, differentiation in q gives $w_v = 0$ and $K_{C_f}^{-1} = 0$. For $p = 0$, write $w_v = \beta = -K_{C_f}^{-1}$. If $\beta \neq 0$, (v, w) are coordinates. Harmonicity and the forward vertical equation give

$$T_q = T/q + \beta A_w, \quad T = -\frac{v}{2w} (K_{C_f}^{-1} A + K_{C_f}), \quad A = \frac{j}{v} - \frac{P}{v^2 w^2}.$$

Substitution into $T^2 = vAR_0 - jv^2$, followed by multiplication by $4v^3 w^6$, gives a polynomial in w with leading term $-4j^2 v^3 w^8$. It cannot vanish on an open (v, w) rectangle. Thus w is constant in both cases. Both metrics then depend only on v , and Lemma 6.6 excludes the branch.

The reconstruction system. Subscripts $v, \mathcal{N}_b$ in what follows hold $\mathcal{I}_1, \mathcal{I}_2, j$ fixed. Put $D = v + L^2$ and

$$\begin{aligned}
A_{\mathcal{N}_b} &= (j/v - A)/\mathcal{N}_b, \\
T_{\mathcal{N}_b} &= \{v(R_0 - A)A_{\mathcal{N}_b} - jA\}/(2T), \\
A_v &= -j/v^2 - (vP_v - P)/(v^2\mathcal{N}_b), \\
T_v &= \{\mathcal{I}_2 A - A^2 - 2jv + v(R_0 - A)A_v\}/(2T), \\
F &= A - TL/v, & G_0 &= AL + T, \\
S &= \frac{Q\{T(v - L^2) - v(R_0 - A)L\}}{v^2 D}, \\
R_1 &= \frac{vR_0 + AL^2 + 2TL}{vD}.
\end{aligned}$$

Let

$$b_q^0 = 2\mathcal{N}_b/Q + 2\mathcal{N}_b/D, \quad b_q = (2\mathcal{N}_b, 0, 2\mathcal{N}_b L/D, 0), \quad b_v = (0, 2\mathcal{N}_b, 0, 2\mathcal{N}_b L/D).$$

For $y = (U_q/U, U_v/U, L_q, L_v)^\top$, the four rows of M are

$$\begin{aligned}
M_1 &= (QL/v, -1, Q/v, 0), \\
M_2 &= (Q, L, 0, 1), \\
M_3 &= Q\{(F, 0, F_L, 0) + F_{\mathcal{N}_b} b_q\} + (0, G_0, 0, (G_0)_L) + (G_0)_{\mathcal{N}_b} b_v, \\
M_4 &= S_{\mathcal{N}_b} b_q + (0, 0, S_L, (R_1)_L) + (R_1)_{\mathcal{N}_b} b_v.
\end{aligned}$$

The equations are

$$My + \mathbf{r}_0 + \frac{K_{C_f}}{U} \mathbf{r}_C + \frac{K_{C_f}^{-1}}{U} \mathbf{r}_{\text{inv}} + \frac{1}{U} \mathbf{r}_{-1} + K_{C_f} \mathbf{r}_{C0} + U \mathbf{r}_U = 0, \quad (\text{D.9})$$

where

$$\begin{aligned}
\mathbf{r}_0 &= (2p^2 L/v^2, 2, 2F + Q(2F_v + F_{\mathcal{N}_b} b_q^0), 2S_v + S_{\mathcal{N}_b} b_q^0)^\top, \\
\mathbf{r}_C &= (0, 0, -1, 0)^\top, & \mathbf{r}_{\text{inv}} &= (0, 1, 0, 0)^\top, \\
\mathbf{r}_{-1} &= (2p/v^2, 0, 0, 0)^\top, & \mathbf{r}_{C0} &= (0, 0, 0, p/v)^\top, \\
\mathbf{r}_U &= (0, 0, 0, -2pQF/v^2)^\top.
\end{aligned}$$

The rows are phase closedness, inverse vertical equilibrium, forward vertical equilibrium, and harmonicity with its forward-equilibrium part subtracted. Its determinant is

$$\det M = -\frac{2\mathcal{N}_b^2 Q^2}{vD} W, \quad W = \alpha L + \beta, \quad (\text{D.10})$$

with

$$\alpha = \frac{2TA_{\mathcal{N}_b} + (R_0 - A)T_{\mathcal{N}_b}}{v\mathcal{N}_b}, \quad \beta = 2A_{\mathcal{N}_b}^2 + 2T_{\mathcal{N}_b}^2/v + j(2j/v - A)/(v\mathcal{N}_b).$$

The matrix follows by substituting the derivatives of (D.6) in the four equations.

The determinant zero set. Set $C_0 = 2TA_{\mathcal{N}_b} + (R_0 - A)T_{\mathcal{N}_b}$ and $D_0 = 4T^2 + v(R_0 - A)^2$. Direct expansion gives

$$\beta = \frac{jA}{v\mathcal{N}_b} + \frac{2j^2 AR_0}{vD_0} + \frac{2C_0(C_0 + 2jT/v)}{D_0}.$$

Thus $\alpha = 0$ implies $\beta > 0$, so every physical point of $W = 0$ has $\alpha \neq 0$ and $L = -\beta/\alpha$. Partition $M = \begin{pmatrix} E & F_0 \\ G_1 & H_0 \end{pmatrix}$ after its first two rows and columns. Its upper-left determinant is $QD/v > 0$. Write

$$N = H_0 - G_1 E^{-1} F_0, \quad \hat{r} = r_{\text{bottom}} - G_1 E^{-1} r_{\text{top}}.$$

On the determinant zero set $W = 0$, $\det N = 0$, and every solution satisfies $N_{11}\hat{r}_2 - N_{21}\hat{r}_1 = 0$. This identity remains necessary when the first column or the matrix N vanishes. Apply this cofactor to the six displayed right-hand columns and multiply by U to obtain $a_0 U + d_1 = 0$. Consequently

$$N_U = d_1^2 - U^2 a_0^2 = 0.$$

This formula, the Schur complement above and (D.9) specify N_U . Substitute $L = -\beta/\alpha$ and the algebraic relations. At $\mathcal{N}_b = \infty$ its coefficients are

$$[\mathcal{N}_b^2]N_U = \frac{j^2 Q K_{C_f}^2}{v^5}, \quad [\mathcal{N}_b^{3/2}]N_U = 0, \quad [\mathcal{N}_b]N_U|_{K_{C_f}=0} = -\frac{36j^2 Q^2}{v^4}. \quad (\text{D.11})$$

They can be obtained by the binomial expansion of T specified below. One of these coefficients is nonzero for every real pressure pair. Moreover

$$T^2 = -\frac{j^2}{v}\mathcal{N}_b + \left(2j\mathcal{I}_2 - j\mathcal{I}_1 v - 3j^2/v\right) + \frac{P(3j/v - \mathcal{I}_2)}{\mathcal{N}_b} - \frac{P^2}{v\mathcal{N}_b^2} \quad (\text{D.12})$$

has an odd-order pole, so it is not a square in $\mathbb{C}(\mathcal{N}_b)$. The T -norm of the nonzero element N_U is a nonzero rational function of $\mathcal{N}_b$. There are finitely many possible $\mathcal{N}_b$ at fixed v ; then T, L, U also have finite fibres. Lemma 6.6 excludes the open part of the determinant zero set.

Regular Frobenius equations. On $W \neq 0$ define y by the adjugate solution of (D.9). The total derivatives on algebraic functions of $(v, \mathcal{N}_b, L, U)$ are specified by

$$\begin{aligned} D_q v &= 2, & D_v v &= 0, \\ D_q U &= U y_1, & D_v U &= U y_2, \\ D_q L &= y_3, & D_v L &= y_4, \\ D_q \mathcal{N}_b &= b_q^0 + 2\mathcal{N}_b y_1 + 2\mathcal{N}_b L y_3 / D, & D_v \mathcal{N}_b &= 2\mathcal{N}_b y_2 + 2\mathcal{N}_b L y_4 / D. \end{aligned}$$

Derivatives of A, T use the displayed $v, \mathcal{N}_b$ formulas. The two necessary compatibilities, with all pressure terms combined, are

$$F_1 = D_q y_2 - D_v y_1 = 0, \quad F_2 = D_q y_4 - D_v y_3 = 0. \quad (\text{D.13})$$

D.2.1 The projective boundary

Fix the principal values and both real pressures first. Put $v = w^2$, $\mathcal{N}_b = \tau^{-2}$, and choose

$$\begin{aligned} T &= \frac{ij}{w\tau} (1 + c_2 \tau^2 + c_4 \tau^4 + c_6 \tau^6)^{1/2}, \\ c_2 &= 2 - v\mathcal{I}_2/j - P/j + v^3/j, \quad c_4 = vP\mathcal{I}_2/j^2 - 3P/j, \quad c_6 = P^2/j^2. \end{aligned}$$

The square root has constant term one. The second T sheet is obtained by conjugation, and U has two signs.

First consider $L \rightarrow 0$, for arbitrary p . Let $r_0 = \sqrt{Q}$ and $U = \epsilon(wr_0\tau)^{-1}(1 + L^2/v)^{-1/2}$, $\epsilon = \pm 1$. For $0 < a = \text{ord } L / \text{ord } \tau < 1$, the six pressure groups $1, K_{C_f}, K_{C_f}^{-1}, K_{C_f}^2, K_{C_f} K_{C_f}^{-1}, K_{C_f}^2$ in F_2 have least weights

$$2, \quad 2 - a, \quad 3, \quad 4 - 2a, \quad 3 - a, \quad 5 - a.$$

The coefficient of the unique lowest K_{C_f} term is

$$-\frac{7v}{j} \left(2p + \epsilon i \frac{v - 2p^2}{r_0} \right).$$

The squared modulus of the parenthesis is $v^2/Q > 0$. Therefore $K_{C_f} = 0$, after which the constant group has unique lowest term with coefficient

$$\frac{6\mathcal{I}_1}{j}(v - 2p^2 - 2\epsilon i p r_0) \neq 0.$$

Its parenthesis has squared modulus v^2 . Each monomial weight is affine in a ; the endpoint inequalities give the stated minimum throughout the interval.

For $L = \tau\ell$, $\ell \neq -iw$, the first nonzero coefficients of F_1 are, successively,

$$-\frac{w^2 K_{C_f}}{j(w - i\ell)^2} \left(\frac{p(3v - 4p^2)}{Q} + \epsilon i \frac{v - 4p^2}{r_0} \right),$$

and, after $K_{C_f} = 0$,

$$-\frac{6w^3\tau}{j(w - i\ell)^2} \left(i(v - 4p^2) + \epsilon \frac{p(3v - 4p^2)}{r_0} \right).$$

The two squared moduli are v^3/Q^2 and v^3/Q , respectively. Near the remaining root use the zero set $W = 0$:

$$L_{\text{wall}} = -\beta/\alpha = -iw\tau + O(\tau^3), \quad x = (L - L_{\text{wall}})/\tau.$$

After extracting powers of τ , the denominators consist of units and x^3 . Hence $N_1 = x^3 F_1$, $N_2 = x^3 F_2/\tau$ are regular at $(x, \tau) = 0$. For $K_{C_f} \neq 0$ their matrix on (x, τ) is

$$\begin{pmatrix} a_0 K_{C_f} & b_0 K_{C_f}^2 \\ -iwa_0 K_{C_f} & iwb_0 K_{C_f}^2 \end{pmatrix}, \quad a_0 = \frac{\epsilon iw^2(r_0 - \epsilon ip)^3}{jr_0^2}, \quad b_0 = \frac{w^8(r_0 - \epsilon ip)^2}{j^2 r_0^2}.$$

Its determinant is a unit. For $K_{C_f} = 0$ both equations are divisible by τ ; their quotients lie in (x, τ^2) and their matrix on those generators is

$$\begin{pmatrix} c_0 & d_0 \\ -iwc_0 & iwd_0 \end{pmatrix}, \quad c_0 = \frac{6iw^3(r_0 - \epsilon ip)^3}{jr_0}, \quad d_0 = \frac{36w^{10}(r_0 - \epsilon ip)^2}{j^2}.$$

Again the determinant is a unit because $|r_0 - \epsilon ip|^2 = v$. The constant and linear coefficients in τ vanish at $x = 0$, giving membership in (x, τ^2) . Thus the saturated ideal of these equations contains τ or τ^2 , respectively. Hence there is no reduced incoming arc.

For $p \neq 0$ and finite $L \neq 0$, $D \neq 0$, write $Y^2 = -v/(QD)$ for the product of the two leading reconstruction roots. Set

$$\begin{aligned} M_0 &= p^2 L^2 + 2vp^2 - v^2, \\ b_E &= -p(p^2 L^2 + 4vp^2 - 3v^2), \\ b_O &= -Q(3p^2 L^2 + 4vp^2 - v^2). \end{aligned}$$

The leading coefficients of $F_1 + LF_2/D$ and of $F_2|_{K_{C_f}=0}$ are

$$\frac{K_{C_f}(b_E + b_O Y)}{jL^2 Q}, \quad -\frac{6\mathcal{I}_1(M_0 + 2pQDY)}{jv}.$$

The polynomial identities

$$QDb_E^2 + vb_O^2 = Q(p^2L^2 + v^2)^3, \quad M_0^2 + 4p^2QvD = (p^2L^2 + v^2)^2$$

leave only $L = \pm iv/p$, $Y = -p/Q$. Near $L = iv/p$ put $\mathcal{N}_b = -j^2/(vz^2)$ and $\delta = L - iv/p$. The regular equation satisfies

$$F_2/z^2 = \kappa\delta^2 + O(\delta^3, z\delta, z^2), \quad \kappa = -\frac{vp^2(6\mathcal{I}_1 + 7iK_{C_f})}{j^3(p^2 - v)} \neq 0.$$

Thus $\delta = O(z)$ for every possible incoming arc. Set

$$\delta = \frac{v(p^2 - v)}{p^2j}(vn - 1)z.$$

After removing nonzero monomials, the coefficients of z^4 in F_2 and z^5 in $F_1 + LF_2/D$ are a quadratic $Q_f(n)$ and a cubic $R_f(n)$. These coefficients define Q_f and R_f . The first is

$$\begin{aligned} 6Q_f &= -j(6\mathcal{I}_1 + 7iK_{C_f})n^2 \\ &+ (2ij\mathcal{I}_1K_{C_f^{-1}} - 2jK_{C_f}K_{C_f^{-1}} - 12j)n \\ &+ 2\mathcal{I}_1^2 + 5i\mathcal{I}_1K_{C_f} - 2K_{C_f}^2 + 2ijK_{C_f^{-1}} + 4\mathcal{I}_2. \end{aligned}$$

Their resultant has degree two in v and leading coefficient $-j^2\overline{\mathcal{F}}\overline{\mathcal{G}}/27$, where the two non-vanishing factors are given below. Hence only finitely many v levels remain exceptional. On $D = 0$, use the double cover $\sigma^2 = D$. The normalized equation F_2/τ^2 is regular and has constant term $6\mathcal{I}_1Q/j > 0$; the odd- U coefficients carry the necessary factor D . At $L = \infty$, put $s = 1/L$. The expression s^2F_2/τ^2 is regular, with constant

$$-\frac{6\mathcal{I}_1p^2}{jv} \pm \frac{7iK_{C_f}p^2}{j\sqrt{vQ}},$$

whose squared modulus is positive.

For $p = 0$, finite $L \neq 0$ gives, with $e_0^2 = v(v + L^2)$,

$$[\tau^2]F_1 = -\frac{6\mathcal{I}_1vL}{jD} + \frac{iw(v + 7L^2)e_0K_{C_f}}{jL^2D}, \quad [\tau^2]F_2 = \frac{6\mathcal{I}_1v}{j} - \frac{7iwe_0K_{C_f}}{jL}.$$

Their combination first forces $K_{C_f} = 0$ and then $\mathcal{I}_1 = 0$, impossible. At $D = 0$ the second equation remains regular and the pressure part is of higher order, so the same contradiction holds.

The expansions at infinity for $p = 0$ are as follows. Write $C_p = K_{C_f}$, $D_p = K_{C_f^{-1}}$ and $L = \ell\tau^{-a}$. For $0 < a < 1$ the lowest term of F_2 is $v(6\mathcal{I}_1 - 7iC_p)\tau^2/j \neq 0$. For $a > 1$ it is $2\ell^2H_2(v)\tau^{4-2a}/j^2$, where

$$\begin{aligned} H_2(v) &= v^2(\mathcal{I}_1^2 - 5i\mathcal{I}_1C_p/2 - C_p^2 - ijD_p + 2\mathcal{I}_2) \\ &+ v(-ij\mathcal{I}_1D_p - jC_pD_p - 6j) + j(-3\mathcal{I}_1 + 7iC_p/2). \end{aligned}$$

Its constant coefficient is nonzero. For $a = 1$, put $\ell = iw\tau$ and take the leading coefficients of F_2 and $F_2 + DF_1/L$. They are a quadratic and a cubic in x whose resultant is divisible by v^4 and has degree-six coefficient $-8j^2\mathcal{F}\mathcal{G}$. The two factors are

$$\begin{aligned} \mathcal{F} &= j\mathcal{I}_1C_p^2D_p^2 + 2j^2C_pD_p^3 + 4ij\mathcal{I}_2C_pD_p^2 + 4\mathcal{I}_1^2C_p^2 + 4i\mathcal{I}_1C_p^3 - C_p^4 \\ &- 24j\mathcal{I}_1C_pD_p + 24ijC_p^2D_p - 36j^2D_p^2 + 48i\mathcal{I}_1\mathcal{I}_2C_p - 24\mathcal{I}_2C_p^2 \\ &+ 432j\mathcal{I}_1 - 144\mathcal{I}_2^2 - 432ijC_p, \\ \mathcal{G} &= j(\mathcal{I}_1 - iC_p)^3D_p^2 + j(-4i\mathcal{I}_1^2 - 17\mathcal{I}_1C_p/2 + 19iC_p^2/4)D_p \\ &+ 4\mathcal{I}_1^4 - 20i\mathcal{I}_1^3C_p - 141\mathcal{I}_1^2C_p^2/4 + 205i\mathcal{I}_1C_p^3/8 + 25C_p^4/4 \\ &+ 8\mathcal{I}_1^2\mathcal{I}_2 - 20i\mathcal{I}_1\mathcal{I}_2C_p - 25\mathcal{I}_2C_p^2/2 + 9j\mathcal{I}_1 - 27ijC_p/2. \end{aligned}$$

For positive, pairwise distinct principal values,

$$\mathcal{I}_1 \mathcal{I}_2 > 9j, \quad \mathcal{I}_2^2 > 3j\mathcal{I}_1, \quad 0 < \mathcal{I}_2/\mathcal{I}_1^2 \leq 1/3, \quad 0 < j/\mathcal{I}_1^3 \leq 1/27.$$

The two factors are nonzero on this domain. Indeed

$$\text{Im } \mathcal{F} = 4C_p \left\{ \mathcal{I}_1(C_p + 3jD_p/\mathcal{I}_1)^2 + j(\mathcal{I}_2 - 9j/\mathcal{I}_1)D_p^2 + 12(\mathcal{I}_1\mathcal{I}_2 - 9j) \right\}.$$

The brace is strictly positive. At $C_p = 0$, $\mathcal{F} = -36\{j^2D_p^2 + 4(\mathcal{I}_2^2 - 3j\mathcal{I}_1)\} < 0$. For $\mathcal{G}$, put $x = \mathcal{I}_2/\mathcal{I}_1^2$, $y = j/\mathcal{I}_1^3$, $z = C_p^2/\mathcal{I}_1^2$. The resultant of its real and imaginary parts in D_p is $64j^2\mathcal{I}_1^{14} \sum_{m=0}^7 q_m(x, y)z^m$, with

$$\begin{aligned} q_0 &= y + 2xy + 9y^2/4, \\ q_1 &= 1 - x + x^2/4 + y(-9 + 179x)/16 + 3141y^2/256, \\ q_2 &= 193/32 - 425x/64 + 29x^2/16 \\ &\quad + y(-8619/512 + 3565x/128) + 3375y^2/128, \\ q_3 &= 60673/4096 - 8661x/512 + 1241x^2/256 \\ &\quad + y(-77465 + 71745x)/2048 + 5625y^2/256, \\ q_4 &= 38519/2048 - 2633x/128 + 725x^2/128 \\ &\quad + y(-128995/4096 + 36125x/2048), \\ q_5 &= 52789/4096 - 6025x/512 + 625x^2/256 - 36125y/4096, \\ q_6 &= 4575/1024 - 625x/256, \quad q_7 = 625/1024. \end{aligned}$$

Since $0 < x \leq 1/3$, $0 < y \leq 1/27$, these are positive: dropping positive terms gives for $q_1, \dots, q_7$ the respective lower bounds

$$\frac{31}{48}, \frac{14719}{4608}, \frac{859649}{110592}, \frac{1192727}{110592}, \frac{477689}{55296}, \frac{11225}{3072}, \frac{625}{1024}.$$

Thus $\mathcal{G} \neq 0$. The inequalities between monomial weights on each side of $a = 1$ establish that there is no other positive breakpoint.

From bounded projection to the second common direction. Lemma 6.7 now gives finitely many $\mathcal{N}_b$ values at each nonexceptional fixed v . Continuity implies $\partial_v \mathcal{N}_b = 0$. For all pitches the latter expression has the form

$$\partial_v \mathcal{N}_b = f_{\text{even}}(L) + U f_{\text{odd}}(L), \quad U^2 = \mathcal{N}_b/[Q(v + L^2)].$$

This quadratic extension of $\mathbb{C}(L)$ is nonsquare because its two poles are simple. Hence an L -free component would require $f_{\text{even}} \equiv 0$. Its numerator is $n_0 + n_1L + n_2L^2 + n_3L^3$ using the explicit matrix. The combinations $n_0 - vn_2$, $n_1 - vn_3$ are pressure independent. Divide them respectively by the factors nonzero on the stated domain A^3QT and A^3QT^2 , and reduce by (D.7). Their $\mathcal{N}_b$ resultant is

$$524288j^6v^{16}P^2(5v^3 - 3\mathcal{I}_1v^2 + \mathcal{I}_2v + j)^3G^{-1}(v), \quad (\text{D.14})$$

where, writing $S_0 = \mathcal{I}_2^2 - 4j\mathcal{I}_1$,

$$[v^8]G^{-1} = 20S_0^2, \quad [v^7]G^{-1} = -132\mathcal{I}_1S_0^2 - 1804j\mathcal{I}_2S_0 - 14112j^3.$$

Either the first or the second coefficient is nonzero, also when $S_0 = 0$. Outside finitely many v levels the U -norm of $\partial_v \mathcal{N}_b$ is therefore a nonzero rational function of L . Hence L , then T, U , have finite fibres. This proves Lemma 6.10. The excluded v levels cannot contain an open solution because $v_q = 2$.

D.3 Compactness transverse to the source axis

We prove Lemma 6.4 using the state and data of Definition 6.2.

Proof. In a largest component chart write $n = (s, t, 1)$ and set

$$\mathbf{S}^{[w_3^Z]} = (w_3^Z \mathbf{p}, (w_3^Z)^2 \boldsymbol{\tau}, (w_3^Z)^3 \mathbf{u}).$$

For an escaping sequence introduce a signed inverse size ε , and put

$$\mathbf{S}^{[\varepsilon w_3^Z]} = (\varepsilon \mathbf{p}^{[w_3^Z]}, \varepsilon^2 \boldsymbol{\tau}^{[w_3^Z]}, \varepsilon^3 \mathbf{u}^{[w_3^Z]}).$$

Write the principal value ratios of the reciprocal metric as $(1, a, b)$; thus the metric called C_f in this section has principal value ratios $(1, a^{-1}, b^{-1})$. The only divisors involving these ratios are $ab(a-1)(b-1)(a-b)$.

The common field equations are

$$H_j = \sum_i n_i \mathcal{Q}_i^j(\mathbf{S}^{[\varepsilon w_3^Z]}) + d_j \sigma \varepsilon \mathbf{S}_j^{[\varepsilon w_3^Z]} = 0, \quad \sigma = 0.$$

The curvature equations and differentiated compatibility equations are independent of σ . Write $\boldsymbol{\tau}_{10}^{[\ell]} = (\tau_1^{[\ell]}, \dots, \tau_5^{[\ell]})$. First the weight-two variables other than $\tau_0^{[\varepsilon w_3^Z]}$ are solved by the rows (1, 3, 6, 7, 8) of the weight-one equations; their axis minor is $(a-b)/(a(b-1))$. The rows (1, 3) of the weight-two equations have identity coefficient block on the two weight-three variables. The weight-four compatibility polynomial $\mathcal{E}_6$ has coefficient 8 on $(\tau_0^{[\varepsilon w_3^Z]})^2$ at the axis. These facts give

$$|\boldsymbol{\tau}^{[w_3^Z]}| \leq C(1 + |\mathbf{p}^{[w_3^Z]}|)^2, \quad |\mathbf{u}^{[w_3^Z]}| \leq C(1 + |\mathbf{p}^{[w_3^Z]}|)^3.$$

The angular and pressure equations and a nonzero quadratic compatibility polynomial rule out escape away from principal axes. On a direction chart with one zero component, $n = (0, v, 1)$ or $(v, 0, 1)$, the respective nonzero leading coefficients are

$$-\frac{vb(a-1)}{a-b}, \quad \frac{vab(b-1)(a-1)}{(a-b)^2},$$

with weight-two minors $2(a-b)v^4/(a(b-1))$ and $-2v^4$. Thus an escaping subsequence approaches a principal axis. At $n = (0, 0, 1)$ write J_{ij} for the standard matrix units, $(J_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$. The limiting homogeneous state is

$$\mathbf{p}^{[\varepsilon w_3^Z]} = pJ_{13} + qJ_{23}, \quad \tau_0^{[\varepsilon w_3^Z]} = pq, \quad \boldsymbol{\tau}_{10}^{[\varepsilon w_3^Z]} = 0, \quad \mathbf{u}^{[\varepsilon w_3^Z]} = 0, \quad (p, q) \neq (0, 0). \quad (\text{D.15})$$

For example, the kernel of the data map initially allows a diagonal term proportional to $(1, a(b-1)/(a-b), b(a-1)/(a-b))$; the weight-one equation forces its square to vanish. The remaining weight-four equation is $8(\tau_0^{[\varepsilon w_3^Z]} - pq)^2 = 0$.

Near (D.15), use normal variables (s, t, h, z) with $h = p_{33}^{[\varepsilon w_3^Z]}$ and $z = \tau_0^{[\varepsilon w_3^Z]} - pq$. Rows of the data equations (1, 2, 3, 6, 7, 8), namely $(A_1, A_2, A_3, B_{*3}, K_1, K_2)$, together with rows (1, 3, 6, 7, 8, 10, 12) of H solve thirteen state variables, with nonzero combined minor $-(b-a)(a-1)/(4a^2(b-1))$. Pull back the eight data equations, seventeen H equations, three cubic, twenty-one quartic and twenty-seven quintic compatibility equations. With $\Lambda = \text{diag}(1, a^{-1}, b^{-1})$, the angular equations are the three components of each of

$$\varepsilon A = ((\mathbf{p}^{[\varepsilon w_3^Z]})^\top - \text{tr}(\mathbf{p}^{[\varepsilon w_3^Z]})I/2)n, \quad \varepsilon B_* = ((\mathbf{p}^{[\varepsilon w_3^Z]})^\top \Lambda - \text{tr}(\Lambda \mathbf{p}^{[\varepsilon w_3^Z]})I/2)n, \quad (\text{D.16})$$

and two independent components of $\varepsilon K = \mathbf{P}(\mathbf{p}^{[\varepsilon w_3^Z]})n$, where $\mathbf{P}$ is the pressure matrix in (E.6). The angular components are bounded, with fixed physical norms and pitches; the pressure data are fixed. The thirteen solved variables are all state components except $p_{13}^{[\varepsilon w_3^Z]} = p, p_{23}^{[\varepsilon w_3^Z]} = q, p_{33}^{[\varepsilon w_3^Z]} = h, \tau_0^{[\varepsilon w_3^Z]} = pq + z$. For the kernel of the data map away from the axes, relabel so $0 < b < a < 1$ and write a nonzero direction as $(s, t, 1)$ with $st \neq 0$. The eight-by-nine matrix of the data map has a nonzero minor on the stated domain

$$\frac{(b-a)s(b-1)}{4a^2b^3} \{b(a-1)s^2t^2 + a(b-1)s^2 - (a-b)t^2\} \neq 0.$$

Normalize its one dimensional homogeneous kernel by $p_{13}^{[\varepsilon w_3^Z]} = 1$. Solve the six weight-two variables from rows $(1, 2, 3, 4, 6, 7)$ of the weight-one common-field equation; their minor is $2(a-b)s^2t/(a(b-1))$. The three remaining quadratic equations on the line forming the kernel of the data map have a simultaneous zero leading coefficient only if

$$(X+Z)G_* = 0, \quad F_* = 0,$$

where

$$X = b(a-1)s^2t^2 < 0, \quad Y = a(b-1)s^2 < 0, \quad Z = (a-b)t^2 > 0,$$

$$F_* = X^2 + 2XY + Y^2 - 2XZ + 2YZ - 3Z^2,$$

$$G_* - F_*^2 = -8(-X + Y - Z)Z(Y - Z)(X + Y + Z).$$

If $X + Z = 0$, then $F_* = Y^2 > 0$. Otherwise $G_* = F_* = 0$ implies $-X + Y - Z = 0$ or $X + Y + Z = 0$; these give respectively $F_* = 4XY > 0$ and $F_* = 4YZ < 0$. Thus at least one quadratic coefficient is nonzero for every admissible triple of principal values and nonzero direction. This conclusion is independent of the angular and pressure data. After the Cartesian Hessian change of variables, it gives the nonzero $(H_{12}^3)^2$ coefficient used below. The cubic and quartic equations follow from the bracket construction, and the precise twenty-seven derivative rows of weight five are indexed in (E.7); these equations are also used in Section 7. These seventy-six equations, in the order just specified, form $\mathcal{G}$. Their linear normal terms are $\varepsilon L(O)$. Its kernel conditions are

$$A_1 = A_2 = B_{*1} = B_{*2} = 0, \quad (\text{D.17})$$

$$2apq(bB_{*3} - A_3) - a(b-1)p^2(K_1 + \sigma) - (a-b)q^2(K_2 + \sigma) = 0. \quad (\text{D.18})$$

Here A, B_* are the two angular vectors and K_1, K_2 the independent pressures in this labeling. The σ column equals the sum of the two pressure columns. Hence the projection annihilating the first order forcing image is independent of σ . To specify the projection, order the forcing columns as the six angular components followed by the two pressure components and the scale. In the nonresonant chart take rows $(4, 5, 10, 12, 28)$ and columns $(1, 2, 3, 4, 5)$ of the forcing matrix L . At the $p = 1, q = 0$ endpoint use rows $(4, 5, 10, 23, 28)$ and columns $(1, 2, 4, 5, 7)$; at $p = 0, q = 1$ use rows $(4, 5, 12, 22, 28)$ and columns $(1, 2, 4, 5, 8)$. For any such invertible block $L_{I,J}$ define $\mathcal{P}\mathcal{G} = \mathcal{G} - L_{:,J}(L_{I,J})^{-1}(\mathcal{G})_I$. At any further rank-chart boundary the same definition with a nonzero maximal minor gives the projection; (D.17) keeps the rank equal to five. Transversality makes $|L(O)|$ bounded away from zero at the limit, since the first two source angular components cannot both vanish.

Let $\mathcal{G}^{(\sigma)}$ denote the seventy-six residuals after solving the range equations for common dilation factor σ , using the same normal coordinates and data. The systems $\sigma = 0$ and $\sigma = 1$ have the same range chart at $\varepsilon = 0$. The projection $\mathcal{P}$ annihilates their first order forcing images. For $N = |(s, t, h, z)|$, the projected residuals therefore satisfy

$$\mathcal{P}(\mathcal{G}^{(0)} - \mathcal{G}^{(1)}) = O(|\varepsilon|(N + |\varepsilon|)). \quad (\text{D.19})$$

Indeed the difference is divisible by ε , and its normal constant term of the quotient vanishes because $\mathcal{P}$ annihilates the forcing image. At $p = 1$ the purely quadratic normal row space is

$$s^2 + \frac{b-a}{a(b-1)}t^2, \quad sh, \quad th, \quad h^2, \quad z\left(s + \frac{(a-b)q}{a(b-1)}t\right), \quad zh, \quad z^2. \quad (\text{D.20})$$

The generic seven-row minor is $-49152(b-a)(b-1)^2a^2q^5((b-a)q^2+a(b-1))/(b^4(a-1)^4)$. At $q = 0$ replace it by $41472(b-a)(b-1)^4a^4/(b^5(a-1)^5)$. On the remaining real zero set $(b-a)q^2+a(b-1) = 0$, put $a = bq^2/(q^2+1-b)$; a replacement minor is $2048q^{10}(7q^2-4b+3)/(b(q^2+1)^3)$. On the further zero set $7q^2-4b+3 = 0$, it is $-49152q^9/((q^2+1)^2(3q^2-1))$, with $0 < q^2 < 1/7$. The $p = 0$ endpoint is covered by interchanging the two transverse labels.

If $N/|\varepsilon| \rightarrow \infty$, division by N^2 in the projected equations and (D.19) gives (D.20). At either extreme principal axis its first form is positive definite, an immediate contradiction. At the middle axis put

$$t = r_0\delta + d, \quad s = \delta, \quad b = a(1+r_0^2)/(a+r_0^2).$$

Then $\varepsilon, d, h, z = o(|\delta|)$. The next coefficient matrices, obtained by the Taylor coefficients of the same seventy-six equations, have nonzero minor on the stated domains

$$\frac{1024r_0q^3p^3(pr_0+q)^3}{1+r_0^2}, \quad -96r_0^4, \quad -96/r_0$$

in the nonresonant region and its two moving endpoints. The columns are (ε, d, h, z) and the respective equation rows are (47, 21, 49, 13), (12, 13, 17, 48) and (10, 13, 17, 47). The endpoint blocks are evaluated in the $p = 1, q = 0$ and $p = 0, q = 1$ charts with their whole moving-parameter neighbourhoods. In each norm contradiction the unprojected term $\varepsilon L(O)$ forces the normalized ε component to zero; the three remaining columns are independent. Thus first $(\varepsilon, d, h, z) = O(\delta^2)$ and then

$$\varepsilon, d, h = o(\delta^2), \quad z = -\frac{2r_0p^2q(1+r_0^2)}{pr_0+q}\delta^2 + o(\delta^2).$$

Subtracting this centre and repeating the argument gives an $O(\delta^3)$ remainder and $\varepsilon = o(\delta^3)$. The fourth order terms now give, respectively, the strictly positive remainders

$$8r_0^2p^4(1+r_0^2)^2, \quad 20r_0^2(1+r_0^2), \quad 20(1+r_0^2).$$

For the first expression the preceding equation gives $q = pr_0$.

When $q + pr_0 \rightarrow 0$, use the derived variables

$$(\varepsilon, d, h, X, Y), \quad X = z^2/\delta, \quad Y = (q + pr_0)z.$$

The five-column minor on rows (47, 21, 13, 22, 35) is $4096r_0^6p^7(r_0^2+a)/a$. The same unprojected forcing removes the first column after each norm normalization. One obtains a centre $(0, 0, 0, 0, -2r_0p^2q(1+r_0^2)\delta^2)$ and then the bound

$$|W| \leq C(|\delta||z| + |\delta|^3)$$

for the remainder W in these five derived variables. Its X component gives $z = O(\delta^2)$, whereas its Y component gives $Y/\delta^2 \rightarrow 2r_0^2p^3(1+r_0^2) \neq 0$, contradicting $q + pr_0 \rightarrow 0$.

If $N = O(|\varepsilon|)$, the unprojected equations read $0 = \varepsilon L(O) + O(\varepsilon^2)$ and again contradict the nonzero forcing. Escape is impossible. Finally $1 \leq \mu/|w_3^Z| \leq \sqrt{3}$ on the largest component chart, so boundedness of $(\mathbf{p}^{[w_3^Z]}, \boldsymbol{\tau}^{[w_3^Z]}, \mathbf{u}^{[w_3^Z]})$ is equivalent to the asserted weighted bound using μ . This proves the lemma. $\square$

D.4 Variations of a cylindrical deformation

Normalize a reference cylinder to

$$\begin{aligned} R &= r, \quad \psi = \omega \log r, \quad z_f = \gamma z, \quad E = \gamma^2, \\ \Delta &= E^2 - E(1 + \omega^2 + k^2) + k^2 \neq 0. \end{aligned}$$

For arbitrary first variations set $x = \delta R/r$, $y = \delta \psi$ and $w = \delta z_f/\gamma$. In cylindrical orthonormal bases, the variation of the gradient is

$$\delta F_f = \begin{pmatrix} x + rx_r & 0 & rx_z \\ \omega x + ry_r & kx & ry_z \\ \gamma w_r & 0 & \gamma w_z \end{pmatrix}.$$

Let $F_{f,0}$ be the gradient of the reference cylinder and put $C_{f,0} = F_{f,0}^\top F_{f,0}$, $\dot{C}_f = F_{f,0}^\top \delta F_f + \delta F_f^\top F_{f,0}$. The three linearized spectral equations are

$$\text{tr } \dot{C}_f = \text{tr}(C_{f,0} \dot{C}_f) = \text{tr}(C_{f,0}^{-1} \dot{C}_f) = 0.$$

Since E is separated from the two planar eigenvalues, they give $w_z = 0$. For $\omega \neq 0$, the remaining equations give

$$\begin{aligned} x &= A(z)/r^2, \quad y = -\gamma_- A(z)/(2r^2) + B(z), \\ w &= w(r), \quad \gamma_\pm = (1 \pm \omega^2 - k^2)/\omega. \end{aligned}$$

The angular curl residuals of $\dot{C}_f$ and $-C_{f,0}^{-1} \dot{C}_f C_{f,0}^{-1}$ are

$$\begin{aligned} R_{Cr} &= 2krB'', & R_{Ir} &= -2rB''/(Ek), \\ R_{Cz} &= -k\gamma_- A'''/2 + kr^2 B''', & R_{Iz} &= \gamma_+ A'''/(2Ek) - r^2 B'''/(Ek). \end{aligned}$$

The two angular curl equations give $B'' = 0$ and $\gamma_- A''' = \gamma_+ A''' = 0$. Since $\gamma_+ - \gamma_- = 2\omega$, $A''' = 0$. The horizontal equations give $(w'' + w'/r)' = 0$. After removing target constants the kernel is

$$A = A_0 + A_1 z + A_2 z^2, \quad B = bz, \quad w = vr^2 + h \log r. \quad (\text{D.21})$$

The pressure variations are $\delta K_{C_f} = -k\gamma_- A_2$ and $\delta K_{C_f^{-1}} = \gamma_+ A_2/(Ek)$, so fixed pressures force $A_2 = 0$. At $\omega = 0$, the distinct principal values give $x = 0$ while $y(r, z)$ is initially arbitrary. Its two angular pressure expressions are

$$k\{r^2(y_{rr} + y_{zz}) + 3ry_r\}, \quad -\{Er^2 y_{rr} + r^2 y_{zz} + 3Ery_r\}/(Ek).$$

Their constancy and $E \neq 1$ make both $r^2 y_{zz}$ and $r^2 y_{rr} + 3ry_r$ constant. Integration, with the development constants removed, gives

$$y = \Gamma \log r + bz + (A_0 + A_1 z + A_2 z^2)/r^2, \quad w = vr^2 + h \log r.$$

The pressure matrix on (Γ, A_2) has determinant $4(E - 1)/E \neq 0$, and fixed pressures again leave exactly five modes.

The reconstruction and radial weights are used in Lemma 6.12.

For the second-order obstruction, use the cylindrical orthonormal frame, for an axisymmetric symmetric tensor Q ,

$$\begin{aligned} d_r &= \partial_r Q_{rr} + \partial_z Q_{rz} + (Q_{rr} - Q_{\theta\theta})/r, \\ d_\theta &= \partial_r Q_{r\theta} + \partial_z Q_{\theta z} + 2Q_{r\theta}/r, \\ d_z &= \partial_r Q_{rz} + \partial_z Q_{zz} + Q_{rz}/r. \end{aligned}$$

Write the three closedness residuals as

$$(R_r(Q), R_z(Q), R_h(Q)) = (\partial_r(rd_\theta), \partial_z(rd_\theta), \partial_z d_r - \partial_r d_z).$$

Closedness is equivalent to their simultaneous vanishing. For the second variation set $R/r = 1 + \epsilon x + \epsilon^2 x_2$, $\psi = \omega \log r + \epsilon y + \epsilon^2 y_2$ and $z_f/\gamma = z + \epsilon w + \epsilon^2 w_2$. Expansion gives $C_f = C_{f,0} + \epsilon \dot{C}_f + \epsilon^2 C_{f,2}$. The inverse coefficient is $C_{f,0}^{-1} \dot{C}_f C_{f,0}^{-1} \dot{C}_f C_{f,0}^{-1} - C_{f,0}^{-1} C_{f,2} C_{f,0}^{-1}$. The second order spectral equations are

$$\text{tr } C_{f,2} = 0, \quad 2 \text{tr}(C_{f,0} C_{f,2}) + \text{tr}((\dot{C}_f)^2) = 0, \quad \text{tr}(C_{f,0}^{-1} C_{f,2}) - \frac{1}{2} \text{tr}\{(C_{f,0}^{-1} \dot{C}_f)^2\} = 0.$$

The second-order axial spectral equation solves

$$(w_2)_z = -\{Q_{zz} + ((\dot{C}_f)_{rz}, (\dot{C}_f)_{\theta z})(EI - C_{f,0,\text{plane}})^{-1}((\dot{C}_f)_{rz}, (\dot{C}_f)_{\theta z})^\top\}/(2E),$$

where Q is the quadratic part from the first variation. The Jacobian and trace then solve

$$r(x_2)_r + 2x_2 = x^2 + rx_z w_r - (w_2)_z,$$

$$\omega r(y_2)_r = -(1 + \omega^2 + k^2)x_2 - r(x_2)_r - E(w_2)_z - \text{tr } Q/2.$$

Integrate these Euler equations to obtain a particular solution; the free part has the same three arbitrary functions as the first-order kernel for constant principal values. Denote the triples obtained by applying (R_r, R_z, R_h) to the particular second-order coefficients of C_f and C_f^{-1} by (R_{Cr}, R_{Cz}, R_{Ch}) and (R_{Ir}, R_{Iz}, R_{Ih}) , respectively. Define

$$B_{\text{var}} = -R_{Cr}/(2kr), \quad A_{\text{var}} = -(R_{Cz} + Ek^2 R_{Iz})/(k\omega), \quad W_{\text{var}} = A_{\text{var}}/(Er) - R_{Ih}.$$

The compatibility conditions are

$$\begin{aligned} &R_{Cr} + Ek^2 R_{Ir}, \quad \partial_r B_{\text{var}}, \quad \partial_r A_{\text{var}}, \\ &-k\gamma_- A_{\text{var}}/2 + kr^2 \partial_z B_{\text{var}} + R_{Cz}, \\ &R_{Ch} + ER_{Ih} - \omega\gamma_- A_{\text{var}}/(2r) + \omega r \partial_z B_{\text{var}}, \quad \partial_z W_{\text{var}}. \end{aligned}$$

The r^{-7} coefficient of $\partial_r B_{\text{var}}$ is

$$-12\{(k^2 + \omega^2 - 1)^2 + 4\omega^2\}A(z)^2/\omega^3.$$

Therefore $A_0 = A_1 = A_2 = 0$. The remaining compatibility is (6.19), where

$$\mathcal{P}(b, v) = (E - 1)Zb^2 + 4E\omega Zbv + 4E\mathcal{T}v^2,$$

$$Z = 3(E - 1)(k^2 - 1) - (E + 1)\omega^2,$$

$$\mathcal{T} = (E - 1)(k^2 - E)(k^2 - 1) + \omega^2\{(5E + 1)k^2 - E(E + 5)\}.$$

The coefficient denominators divide powers of $Ek\omega\Delta$. The identity is

$$(E - 1)\mathcal{T} + \Delta H_* = E\omega^2 Z, \quad H_* = (E - 1)(k^2 - 1) + (E + 1)\omega^2. \quad (\text{D.22})$$

D.5 Connection fibres for two rotations

We use the physical chart (6.22) on the normalized set N introduced in Section 6. Thus $\mu = |Z_s| = 1$ at the base point, and the adapted Cartesian axes give $Z_s = \partial_{x_3}$ and $Z_t = \gamma \partial_{y_3}$. Write $\partial_a = \partial/\partial x_a$. At projective infinity the angular vectors are null and perpendicular to the rotation values, so write

$$\Omega_s^{\text{axial}} = \alpha(1, i, 0), \quad \Omega_t^{\text{axial}} = \beta(1, \delta i, 0), \quad \delta = \pm 1.$$

The opposite sign in the reference configuration is obtained by conjugation. Differentiating equivariance gives

$$H_{a3} = \Omega_t^{\text{axial}} \times F_f \partial_{x_a} - F_f(\Omega_s^{\text{axial}} \times \partial_{x_a}), \quad H_{ab} = \partial_a \partial_b f. \quad (\text{D.23})$$

The nine components of H_{11}, H_{12}, H_{22} satisfy eight linear rows: the Cartesian ∂_1, ∂_2 derivatives of $\text{tr } C_f$, $\text{tr}(C_f^2)/2$, $\log \det F_f$, and the third components of $\text{div } C_f$ and $\text{div } C_f^{-1}$ at homogeneous infinity. Solve the first seven leaving H_{11}^3 and H_{12}^3 . The last coefficient on H_{11}^3 is

$$-\frac{2(t^2 p^2 + s^2 q^2)}{s^2 t^2 \gamma (s^2 - t^2)} \neq 0.$$

Thus the Hessian is a linear function of $(\alpha, \beta, H_{12}^3)$.

The next equivariance derivative is

$$T_{ab3} = \Omega_t^{\text{axial}} \times H_{ab} - H_{\Omega_s^{\text{axial}} \times \partial_{x_a}, b} - H_{a, \Omega_s^{\text{axial}} \times \partial_{x_b}}, \quad T_{abc} = \partial_a \partial_b \partial_c f. \quad (\text{D.24})$$

Order the twelve remaining components by $T_{111}, T_{112}, T_{122}, T_{222}$ and target component. The fifteen equations consist of the (11), (12), (22) second derivatives of the same three spectral functions, followed by the (12), (13), (23) curl-div components of C_f and then of jC_f^{-1} . Their entries follow from

$$\begin{aligned} \partial_a F_f &= (H_{1a}, H_{2a}, H_{3a}), \\ \partial_a \partial_b F_f &= (T_{1ab}, T_{2ab}, T_{3ab}), \\ \partial_a C_f &= (\partial_a F_f)^\top F_f + F_f^\top \partial_a F_f, \\ \partial_a \partial_b C_f &= (\partial_a \partial_b F_f)^\top F_f + F_f^\top \partial_a \partial_b F_f + (\partial_a F_f)^\top \partial_b F_f + (\partial_b F_f)^\top \partial_a F_f, \\ j \partial_a \partial_b C_f^{-1} &= (\partial_a \partial_b C_f) C_f + C_f (\partial_a \partial_b C_f) \\ &\quad + (\partial_a C_f)(\partial_b C_f) + (\partial_b C_f)(\partial_a C_f) - \mathcal{I}_1 \partial_a \partial_b C_f. \end{aligned}$$

The last identity is used modulo the displayed spectral derivative rows. A curl row (i, j) is $\sum_a \{(\partial_i \partial_a C_f)_{aj} - (\partial_j \partial_a C_f)_{ai}\}$. These formulas specify the matrix M and the quadratic inhomogeneous term $b(H, \Omega_s^{\text{axial}}, \Omega_t^{\text{axial}})$.

Its first twelve rows M_0 are invertible throughout the real chart. For an explicit determinant nonzero on the stated domain put $D_0 = s^2 - t^2$,

$$\begin{aligned} L_0 &= t^2(D_0 + p^2) - s^2 q^2 - D_0 \gamma^2, \quad R_0 = L_0 + D_0^2, \\ P_{10} &= p^2 L_0^2 + q^2 R_0^2 + p^2 q^2 \{(s^2 + t^2)^2 (p^2 + q^2) + 2D_0^2 (s^2 + t^2)\} > 0. \end{aligned}$$

Using the second derivatives of $\det F_f$ gives $\det M_0 = 128pqst(s^2 - t^2)\gamma^5 P_{10}$. Using $J^2 d^2 \log J$ multiplies the three determinant rows by $J = st\gamma$. The principal symbol gives the same conclusion: the spectral equations for a transverse cubic displacement imply $\delta f_i = d_i((\nu \times e_i) \cdot x)^3$; the remaining columns are proportional to $\nu - \nu_i e_i$, whose determinant is $2\nu_1 \nu_2 \nu_3$. If M_1 comprises the last three rows, the necessary quadrics are

$$(Q_0, Q_1, Q_2)^\top = (-M_1 M_0^{-1} \mid I_3) b(H, \Omega_s^{\text{axial}}, \Omega_t^{\text{axial}}), \quad (\text{D.25})$$

after the Hessian substitution.

D.6 The resultant for fixed principal values

For $\delta = +1$ set $p = q = e$, $\gamma = 1$, with $1 < s < t$, substitute $H_{12}^3 = e^{-1}x$, and use the invertible row changes

$$f_0 = e^4 Q_0, \quad f_1 = e^7 Q_1, \quad f_2 = e^5 \left(Q_1 + \frac{s^2 - 1}{t^2 - 1} Q_2 \right).$$

Their coefficients are analytic in $h = e^2$. Complete the square in f_1 and remove the square from f_2 . The resulting system is

$$f_0 = Ml + A_1, \quad f_1 = Gl^2 + B_1, \quad f_2 = Nl + C_1. \quad (\text{D.26})$$

M, N are linear and A_1, B_1, C_1 quadratic in (α, β) . At $h = 0$,

$$l = x + \frac{(1-i)(s^2-1)(t^2-1)}{s^2+t^2-2}\alpha + \frac{i(s+it)(s^2-1)(t^2-1)}{s^2+t^2-2}\beta,$$

$$G_0 = -\frac{(s^2-t^2)(s^2-1)(s^2+t^2-2)^5}{8(s^2+t^2)^2((s^2-1)^2+(t^2-1)^2)} \neq 0.$$

The expansions give $A_1, C_1 = O(h)$ and $B_1 = O(h^2)$ throughout a neighbourhood of the principal value ratios. Form

$$MC_1 - NA_1 = hD_1 + O(h^2), \quad GA_1^2 + B_1M^2 = h^2H_2 + O(h^3).$$

These are polynomial consequences of (D.26). This coefficient-extraction rule defines the binary cubic D_1 and quartic H_2 from (D.25). The primitive forms are obtained by clearing coefficient denominators and dividing by their polynomial gcd. The real zero sets of the cleared denominators and removed common factors are contained in the union of the zero sets of the following polynomials:

$$s, t, s^2 - 1, t^2 - 1, s^2 - t^2, s^2 + t^2, s^2 + t^2 - 2, (s^2 - 1)^2 + (t^2 - 1)^2.$$

They are nonzero on both $t > s > 1$ and $t < -s < -1$.

The resultant of the primitive forms equals factors nonzero on the stated domain times a degree-twenty-four complex factor $R(s, t)$. Its real part is symmetric, and

$$\text{Im } R = -2(t-s)(s+t)^3(st-1)S(s, t)$$

with S real symmetric. Put $y = st$, $z = (s-t)^2$. To express the symmetric polynomials in these variables, use

$$U_0 = 2, \quad U_1 = z + 2y, \quad U_m = (z + 2y)U_{m-1} - y^2U_{m-2} \quad (m \geq 2),$$

so that $U_m = s^{2m} + t^{2m}$. This rewrites $\text{Re } R, S$ as $R_0(z, y), S_0(z, y)$ of respective z degrees eight and six. Their resultant is a nonzero constant times $y^6(y+1)^{32}(y-1)^{33}P_{23}(y)$. The integer coefficients of P_{23} , in descending powers, are

$$\begin{aligned} & (649478266, -11416123738, -12417964938, \\ & 187068269041, 3941270024906, -28179287609472, \\ & 48074729011126, 68083917662805, -308529862356652, \\ & 243011868068132, 320451262838796, -720927035777550, \\ & 412914037204036, 186244893175800, -452434890841540, \\ & 333508333383970, -132678789329678, 29745956846182, \\ & -3615699456706, 264797910237, -12968149454, \\ & 436200536, -9000882, 82377). \end{aligned}$$

Over $\mathbb{Q}[y]/(P_{23})$, the monic gcd of R_0, S_0 is linear in z . It determines $z = z(y)$ uniquely. The polynomial P_{23} is irreducible. Sturm isolation gives exactly the following seven real roots with $|y| > 1$, together with rational boxes for $z(y)$. Every interval entry in the table is divided by 10^6 :

	y interval	z interval	exclusion
1	$(-1651383, -1651382)$	$(3036882, 3036883)$	$z < -4y$
2	$(-1453650, -1453649)$	$(5292554, 5292555)$	$z < -4y$
3	$(-1148067, -1148066)$	$(4508485, 4508486)$	$z < -4y$
4	$(1216589, 1216590)$	$(701933, 701934)$	$z > (y - 1)^2$
5	$(3476033, 3476034)$	$(52178050, 52178051)$	$z > (y - 1)^2$
6	$(4762576, 4762577)$	$(-6509531, -6509530)$	$z < 0$
7	$(16921450, 16921451)$	$(-40125866, -40125865)$	$z < 0$

The allowable regions are $y > 1$, $0 < z < (y - 1)^2$, and $y < -1$, $-4y < z < (y - 1)^2$. Each box is disjoint from the corresponding region. In the other target null chart the complete differential ideal transforms as

$$Q_-(s, t, p, q, \gamma; \alpha, \beta, H_{12}^3) = Q_+(s, -t, p, q, \gamma; \alpha, -\beta, H_{12}^3).$$

The sign on β is the axial-vector transformation law under reflection.

If a projective root approached $e = 0$, normalize $[\alpha : \beta : l]$. The initial square forces $l = 0$, so $(\alpha, \beta) \neq (0, 0)$. The two polynomial consequences divided by h, h^2 then give $D_1 = H_2 = 0$, contrary to the nonzero resultant. To keep the principal values fixed, start at $\text{diag}(\sqrt{\lambda_2}, \sqrt{\lambda_3}, \sqrt{\lambda_1})$ and set $p = q = \epsilon$. The spectral Vandermonde is the nonzero Jacobian of the three spectral equations in (s, t, γ) , so their solutions have $O(\epsilon^2)$ changes. Divide all target lengths by $\gamma(\epsilon)$ and put $e = \epsilon/\gamma(\epsilon)$. Since the preceding coefficient vanishings hold throughout a neighbourhood of the principal value ratios, this drift leaves the same leading D_1, H_2 . Thus every fixed triple of principal values has a good direction. Projective projection is closed, so the bad-direction set E is a proper algebraic subset of the direction surface and has real dimension at most one.

D.7 The exceptional direction fibres

We impose the direction equation (6.23). For $I = \nu^\top C_f \nu$ and $II = \nu^\top C_f^2 \nu$, differentiate both the tensor and the unit common field direction:

$$\begin{aligned} \partial_a \nu &= \Omega_s^{\text{axial}} \times \partial_{x_a} - \nu \{ \nu \cdot (\Omega_s^{\text{axial}} \times \partial_{x_a}) \}, \\ \partial_a I &= \nu^\top (\partial_a C_f) \nu + 2(\partial_a \nu)^\top C_f \nu, \\ \partial_a II &= \nu^\top \{ (\partial_a C_f) C_f + C_f (\partial_a C_f) \} \nu + 2(\partial_a \nu)^\top C_f^2 \nu. \end{aligned}$$

These derivatives determine $(dI \wedge dII)(\partial_{x_1}, \partial_{x_2})$. Substitute the Hessian graph into its homogeneous expression D_{phys} and into the equation P_{ax} , the multiple by a factor nonzero on the stated domain of Q_0 . Precisely, with $I = \nu^\top C_f \nu$ and $II = \nu^\top C_f^2 \nu$, use

$$\begin{aligned} D_{\text{phys}} &= (dI \wedge dII)(\partial_{x_1}, \partial_{x_2}), \\ P_{\text{ax}} &= \Omega_t^{\text{axial}} \cdot \Delta f + (\det F_f) \Omega_s^{\text{axial}} \cdot \text{div}(C_f^{-1}) \\ &= Q_0 / (4st\gamma). \end{aligned}$$

The derivatives and the Laplacian in this formula are evaluated from the Hessian graph (D.23). Both are linear in H_{12}^3 :

$$D_{\text{phys}} = d_1 H_{12}^3 + d_0, \quad P_{\text{ax}} = p_1 H_{12}^3 + p_0.$$

Their elimination is the binary cubic $\mathcal{P}_{\text{dir}} = d_1 p_0 - p_1 d_0$. Only its two extreme coefficients are needed. Write $S = s^2$, $T = t^2$, $P = p^2$, $Q = q^2$, $C_0 = \gamma^2$, $\Delta_0 = S - T$. Their numerators, over respectively $pqst(t^2 p^2 + s^2 q^2)$ and $pq(t^2 p^2 + s^2 q^2)$, are

$$4i\gamma^5(pf_0 - iqf_1), \quad -4i\gamma^2(ptf_2 - iqs f_3).$$

The displayed Hessian matrix gives

$$\begin{aligned} f_2 - f_0 &= \Delta_0\{T\Delta_0 P + TP^2 + 5S\Delta_0 Q \\ &\quad + 5(S+T)PQ - 3SQ^2 - \Delta_0(P+5Q)C_0\}, \\ f_1 - f_3 &= \Delta_0\{-5T\Delta_0 P - 3TP^2 - S\Delta_0 Q \\ &\quad + 5(S+T)PQ + SQ^2 + \Delta_0(5P+Q)C_0\}. \end{aligned}$$

Hence

$$\begin{aligned} (P+5Q)(f_1 - f_3) + (5P+Q)(f_2 - f_0) &= 2\Delta_0\Psi, \\ \Psi &= TP^3 + (15S+8T)P^2Q + (8S+15T)PQ^2 + SQ^3 + 12(S-T)^2PQ > 0. \end{aligned}$$

If $\mathcal{P}_{\text{dir}}$ were identically zero, realness and $pqst \neq 0$ would imply $f_0 = f_1 = f_2 = f_3 = 0$, a contradiction. Reflection covers the second null chart. Thus there are at most three projective ratios $[\alpha : \beta]$. For each ratio, a nonzero d_1 or p_1 determines H_{12}^3 . If both vanish, the three Q_j cannot all have zero $(H_{12}^3)^2$ coefficient: this is the same calculation for the kernel of the data map underlying the estimate away from the axes in Lemma 6.4, performed directly in (D.23)–(D.25). Therefore H_{12}^3 has at most two values. The pure H_{12}^3 point $\alpha = \beta = 0$ is excluded by the same coefficient. The strengthened projective set at infinity is finite. Each strengthened affine fibre has dimension at most one, since a component of dimension at least two would meet the hyperplane at infinity in positive dimension.

D.8 Two screws

We give the linearized pitch equations and the analytic normalization used in Lemma 6.17. The matched reference cylinder has $R = r$, $\psi = \omega \log r$, $z_f = \gamma z$, with $\gamma = q/p$. At a normalized base point the common generators are

$$Z_{s,e} = \partial_\theta + e\partial_z, \quad Z_{t,e} = k\partial_\Theta + \gamma e\partial_Z.$$

The reference cylinder intertwines these fields for every e . Differentiate equivariance with respect to a variation parameter at $e = 0$:

$$\partial_\theta \delta f - kJ_{\text{axis}} \delta f = \delta e(\gamma e_z - Df_c e_z) = 0.$$

After subtracting the derivative of that reference curve, the first variation satisfies the (R, R) system. The reference pressure potentials have zero axial derivative, so $\delta K_i = \partial_\theta \delta \phi_i$. The five modes already derived in (D.21) remain, and the independent pitch mode is added.

At $\omega = 0$ the potentially extra $\Gamma \log r$ and $A_2 z^2/r^2$ terms are removed by the two pressure equations

$$\delta K_{C_f} = 2k(\Gamma + A_2), \quad \delta K_{C_f^{-1}} = -2(\Gamma + A_2/E)/k,$$

whose coefficient determinant is $4(E-1)/E \neq 0$.

The pitch variation is consequently an independent eigenvector of weight -1 , and its angular pressure variation is zero.

For the stable normal form, the three weight -1 coordinates are denoted by x and the weight -2 coordinate by a , as in (6.29), with Q_2 a homogeneous quadratic form. A finite quadratic coordinate change removes every nonresonant quadratic monomial; its divisors are

nonzero integers. Only the x^2 terms in the a equation resonate. Write the transformed field as $N + E_3$, $E_3 = O(|(x, a)|^3)$, where N is the right side of (6.29). Its flow is

$$\Psi_t(x, a) = (e^{-t}x, e^{-2t}(a + tQ_2(x))).$$

If Φ_t is the flow of $N + E_3$, for any small $\delta < 1/3$ it satisfies $|\Phi_t(y)| \leq Ce^{-(1-\delta)t}|y|$ in a fixed small complex neighbourhood. Set $H_t(y) = \Psi_{-t}(\Phi_t(y))$. Differentiation gives

$$\frac{dH_t}{dt} = D\Psi_{-t}(\Phi_t(y))E_3(\Phi_t(y)),$$

bounded by $C(1+t)e^{-(1-3\delta)t}|y|^3$, uniformly in the initial value. Thus H_t converges uniformly holomorphically; its limit is $H(y) = y + O(|y|^3)$ and conjugates the two flows. This proves the analytic normal form.

E Similarity identities and dilation equations

This appendix gives the coefficient calculations used in Sections 5 and 7 and in the compactness estimate of Section 6. The defining equations are reconstructed from flatness of the two metrics and closedness of the pressure forms.

E.1 Similarity identities before the restriction to constant principal values

Use the tensors and Jacobian of Section 5. In the following identities, quantities involving $F_{f^{-1}}$ are evaluated at $f(x)$. With $\ell = \nabla_x \log |J|$,

$$\operatorname{div}_x C_f = F_f^\top \Delta_x f + \frac{1}{2} \nabla_x \operatorname{tr} C_f, \quad \operatorname{div}_x C_f^{-1} = (\Delta_y f^{-1}) \circ f - C_f^{-1} \ell. \quad (\text{E.1})$$

These identities use only invertibility and equality of mixed second derivatives. For the first, expand $\partial_j((F_f)_{ai}(F_f)_{aj})$ and replace $\partial_j(F_f)_{ai}$ by $\partial_i(F_f)_{aj}$. For the second, write $(C_f^{-1})_{ij} = (F_{f^{-1}})_{ia}(F_{f^{-1}})_{ja}$ and use $\partial_{x_j} = (F_f)_{bj}\partial_{y_b}$. The derivative of the first factor gives $\Delta_y(f^{-1})_i$; that of the second gives $(F_{f^{-1}})_{ia}\partial_{y_a} \log |\det F_{f^{-1}}| = -(C_f^{-1}\ell)_i$. Exchanging f and f^{-1} gives the other two formulas in the main text.

Differentiating the inverse and the two metric products gives

$$\begin{aligned} D_j F_{f^{-1}} &= -F_{f^{-1}}(D_j F_f)F_{f^{-1}}, \\ D_j C_f &= (D_j F_f)^\top F_f + F_f^\top D_j F_f, \\ D_j C_f^{-1} &= (D_j F_{f^{-1}})F_{f^{-1}}^\top + F_{f^{-1}}^\top (D_j F_{f^{-1}})^\top, \end{aligned}$$

The first identity follows from differentiating $F_f F_{f^{-1}} = I$; the other two follow from the product rule.

Here λ_i denote the eigenvalues of $C_f = F_f^\top F_f$. For constant positive, pairwise distinct principal values, J , $\operatorname{tr} C_f$ and $\operatorname{tr} C_f^{-1}$ are constant. Thus the two gradient pressures are exchanged by composition with f^{-1} . The labelled pressures are recovered using

$$\det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^{-1} & \lambda_2^{-1} & \lambda_3^{-1} \end{pmatrix} = -\frac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}{\lambda_1 \lambda_2 \lambda_3}. \quad (\text{E.2})$$

If $\bar{\alpha}_i$ is the transported unit principal form, then $f^* \bar{\alpha}_i = \sqrt{\lambda_i} \alpha_i$, and consequently $\bar{h}_i \circ f = (\lambda_i/J)h_i$.

The constant-factor Killing equation implies $D^2 Z = 0$ by its three cyclic derivatives. A skew matrix in three dimensions has eigenvalues $0, i\omega, -i\omega$, hence $\det(\sigma I + \Omega) = \sigma(\sigma^2 + \omega^2)$.

For $\sigma \neq 0$ there is exactly one centre; for $\sigma = 0$ the translation splits uniquely into axial and perpendicular parts, and only the latter can be removed by shifting the origin. Equality of the two scales follows by differentiating the constant positive trace of the relative metric. Differentiating the orbit formula gives

$$Df(\Phi_s^t x_0) = e^{t\Omega_t} [Z_t(y_0), y_{0,u}, y_{0,v}] [Z_s(x_0), x_{0,u}, x_{0,v}]^{-1} e^{-t\Omega_s}.$$

The invariant pressure forms have coefficients independent of the orbit parameter. Their closedness is exactly $\partial_u a_t = \partial_v a_t = 0$ and $\partial_u a_v - \partial_v a_u = 0$; thus their orbit coefficients are arbitrary real constants.

E.2 State equations and pure dilation

For the coefficient calculations, the fixed ordering of the state is

$$\Lambda = \text{diag}(1, a^{-1}, b^{-1}),$$

$$\mathbf{S} = (p_{11}, p_{12}, \dots, p_{33}, \tau_0, \dots, \tau_5, u_0, u_1).$$

These are the coordinates of (2.21). The nine entries of $\mathbf{p}$ are ordered by rows and have weight one; the entries of $\boldsymbol{\tau}$ have weight two and those of $\mathbf{u}$ have weight three. The coordinate weights d_j are those of Section 2. The state convention is

$$p_{ij} = e_j \cdot \text{curl } e_i, \quad [e_i, e_j] = C_{ij}^k e_k, \quad C_{ij}^k = -\epsilon_{ij\ell} p_{k\ell}, \quad D_i \mathbf{S} = \mathcal{Q}_i(\mathbf{S}). \quad (\text{E.3})$$

Write $\mathbf{w}^Z = (w_1^Z, w_2^Z, w_3^Z)$, where $w_i^Z = \alpha_i(Z_s)$. On a chart where $w_3^Z \neq 0$, put

$$n = (s, t, 1) = \mathbf{w}^Z / w_3^Z, \quad \mathbf{S}^{[w_3^Z]} = (w_3^Z \mathbf{p}, (w_3^Z)^2 \boldsymbol{\tau}, (w_3^Z)^3 \mathbf{u}).$$

The coordinate $t = n_2$ is distinct from the state block $\boldsymbol{\tau}$. Weighted homogeneity and $\mathcal{L}_{Z_s} \alpha_i = \alpha_i$ give

$$H_j := \sum_{i=1}^3 n_i \mathcal{Q}_i^j(\mathbf{S}^{[w_3^Z]}) + d_j \mathbf{S}_j^{[w_3^Z]} = 0. \quad (\text{E.4})$$

The angular vectors and projector pressure constants are

$$\begin{aligned} A &= ((\mathbf{p}^{[w_3^Z]})^\top - \tfrac{1}{2} \text{tr } \mathbf{p}^{[w_3^Z]} I) n, \\ B_* &= ((\mathbf{p}^{[w_3^Z]})^\top \Lambda - \tfrac{1}{2} \text{tr}(\Lambda \mathbf{p}^{[w_3^Z]}) I) n, \\ K &= \mathbf{P}(\mathbf{p}^{[w_3^Z]}) n, \quad \sum_i K_i = 0, \end{aligned} \quad (\text{E.5})$$

where

$$\mathbf{P}(\mathbf{p}^{[w_3^Z]}) = \begin{pmatrix} p_{23}^{[w_3^Z]} - p_{32}^{[w_3^Z]} & p_{13}^{[w_3^Z]} & -p_{12}^{[w_3^Z]} \\ -p_{23}^{[w_3^Z]} & p_{31}^{[w_3^Z]} - p_{13}^{[w_3^Z]} & p_{21}^{[w_3^Z]} \\ p_{32}^{[w_3^Z]} & -p_{31}^{[w_3^Z]} & p_{12}^{[w_3^Z]} - p_{21}^{[w_3^Z]} \end{pmatrix}. \quad (\text{E.6})$$

The vector A gives the source angular vector in the principal frame; the target angular vector has components $\sqrt{\Lambda} B_* / \det \sqrt{\Lambda}$. Their norms are controlled by the fixed angular speeds. The components K_i are constant.

Use the cubic compatibilities C_1, C_2, C_3 and the ordered quartic compatibilities $\mathcal{E}_0, \dots, \mathcal{E}_{20}$ of Appendix A.2. Define $G_0^{\text{quint}}, \dots, G_{26}^{\text{quint}}$ as follows. In the ordered list

$$(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_3, \mathcal{E}_5, \mathcal{E}_6, \mathcal{E}_{10}, \mathcal{E}_{11}, \mathcal{E}_{12}, \mathcal{E}_{13}, \mathcal{E}_{15}, \mathcal{E}_{18}, \mathcal{E}_{19})$$

take the three derivatives $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ of each entry, in this order, and retain the positions

$$1, 2, \dots, 19, 21, 22, 23, 26, 28, 31, 32, 35. \quad (\text{E.7})$$

This finite list supplies necessary differential consequences. Reconstruction and continuation use their full differential closure.

The six-column coefficient block for $\boldsymbol{\tau}$ in rows (1, 2, 3, 4, 6, 7) of the first homothety equations has determinant $2(a-b)s^2t/[a(b-1)]$.

E.2.1 Pure dilation

We prove the algebraic step in Lemma 7.2 on the dense direction chart established there. Relabel and normalize the principal values so that $0 < b < a < 1$, and put

$$\kappa = \frac{a(b-1)}{a-b} < -1, \quad \ell = \kappa + 1 = \frac{b(a-1)}{a-b} < 0.$$

Both angular vectors in (E.5) are zero. Their six equations solve the six off-diagonal entries of $\mathbf{p}^{[w_3^Z]}$ using only s, t and spectral differences. The first nine equations in (E.4) solve the six $\boldsymbol{\tau}^{[w_3^Z]}$ variables with the row indices 1, 2, 3, 4, 6, 7 and determinant

$$\frac{2(a-b)s^2t}{a(b-1)}.$$

After substituting these six coordinates and the zero angular vectors, the remaining rows H_5, H_8, H_9 satisfy

$$H_5 - \frac{\kappa}{\ell} H_9 = H_{\text{dil}} := \kappa p_{11}^{[w_3^Z]} + p_{22}^{[w_3^Z]} - \frac{\kappa}{\ell} p_{33}^{[w_3^Z]}. \quad (\text{E.8})$$

Here $p_{ii}^{[w_3^Z]} = w_3^Z h_i$.

The identity $H_{\text{dil}} = 0$ holds on a solution open set, and its directional derivatives vanish there. With $D_i^{[w_3^Z]} = w_3^Z D_i$, the scaled directional derivatives are

$$\begin{aligned} D_i^{[w_3^Z]} \mathbf{p}^{[w_3^Z]} &= (\mathcal{Q}_i^{p_{jk}}(\mathbf{S}^{[w_3^Z]}))_{j,k} + \left(\delta_{i3} + \sum_k C_{ki}^3(\mathbf{p}^{[w_3^Z]}) n_k \right) \mathbf{p}^{[w_3^Z]}, \\ D_i^{[w_3^Z]} n_j &= \delta_{ij} + \sum_k C_{ki}^j(\mathbf{p}^{[w_3^Z]}) n_k - n_j \left(\delta_{i3} + \sum_k C_{ki}^3(\mathbf{p}^{[w_3^Z]}) n_k \right). \end{aligned}$$

Writing $H_u = p_{11}^{[w_3^Z]}$, $Z_u = p_{33}^{[w_3^Z]}$ and eliminating $p_{22}^{[w_3^Z]}$ by (E.8), the three derivatives are

$$\begin{aligned} D_1^{[w_3^Z]} H_{\text{dil}} &= \frac{\kappa H_u}{\ell t} [\ell^2 t^2 H_u + (\kappa - \ell t^2) Z_u], \\ D_2^{[w_3^Z]} H_{\text{dil}} &= -\frac{p_{22}^{[w_3^Z]}}{\ell s} [Z_u - \ell^2 s^2 H_u], \\ D_3^{[w_3^Z]} H_{\text{dil}} &= \frac{\kappa Z_u}{\ell^2 s t} [t^2 Z_u - \ell(\kappa s^2 + t^2) H_u]. \end{aligned} \quad (\text{E.9})$$

All the prefactors being nonzero, these imply

$$Z_u = \ell^2 s^2 H_u, \quad t^2 = \frac{\kappa s^2}{\ell s^2 - 1}, \quad p_{22}^{[w_3^Z]} = \kappa(\ell s^2 - 1) H_u. \quad (\text{E.10})$$

Set $z = \ell s^2 < 0$. Then $z^2 - z + 1 > 0$. The remaining equations are $\mathcal{E}_j = H_u(A_j + B_j H_u)$, $j = 0, 1$, where

$$\begin{aligned} A_0 &= 2\kappa \ell s^2, & B_0 &= (2\kappa t/s)(z^2 - 1)(z^2 - z + 1), \\ A_1 &= \ell t, & B_1 &= \kappa \ell s(2z - 1)(z^2 - z + 1)/(z - 1). \end{aligned}$$

Their necessary determinant is

$$A_0 B_1 - A_1 B_0 = \frac{2\kappa^2 \ell s(z^2 - z + 1)^2}{z - 1} \neq 0. \quad (\text{E.11})$$

The determinant is nonzero for every $z < 0$, so the two equations are inconsistent when $H_u \neq 0$.

E.3 Normal estimates at principal directions

Lemma E.1 (Normal estimate at a principal direction). *For each fixed triple of positive, pairwise distinct principal values and bounded angular and pressure data, an unbounded normalized state can approach only a principal direction. Its homogeneous limit belongs to the plane (E.14). Let ε be the signed inverse size of $\mathbf{p}^{[w_3^Z]}$ and let (s, t, h, z) be the normal coordinates defined below. Then $N = (s^2 + t^2 + h^2 + z^2)^{1/2} = O(|\varepsilon|)$.*

Proof. We prove the lemma from the homothety equations and the angular and pressure equations. The estimate includes every principal label and every limiting point of the projective cylindrical circle.

Nonaxial directions and the higher jets. Use a maximum component chart $n = (s, t, 1)$, $|s|, |t| \leq 1$, and the weighted state $\mathbf{S}^{[w_3^Z]}$. For $st \neq 0$, the eight component functions in (E.5) define a linear map of rank eight on the nine entries of $\mathbf{p}^{[w_3^Z]}$. In the ordering $0 < b < a < 1$, a minor is

$$\frac{(b-a)s(b-1)}{4a^2b^3} \{b(a-1)s^2t^2 + a(b-1)s^2 - (a-b)t^2\}. \quad (\text{E.12})$$

The expression in braces is strictly negative. The remaining affine line is controlled by the quadratic residuals of the first nine homothety equations. For their quadratic leading coefficients, put

$$X = b(a-1)s^2t^2 < 0, \quad Y = a(b-1)s^2 < 0, \quad Z = (a-b)t^2 > 0,$$

and

$$F_* = (X + Y)^2 - 2XZ + 2YZ - 3Z^2, \\ G_* - F_*^2 = -8(-X + Y - Z)Z(Y - Z)(X + Y + Z).$$

Their simultaneous vanishing would require $F_* = 0$ and $(X + Z)G_* = 0$. If $X + Z = 0$, then $F_* = Y^2 > 0$. Otherwise the factored identity forces $-X + Y - Z = 0$ or $X + Y + Z = 0$, which respectively give $F_* = 4XY > 0$ and $F_* = 4YZ < 0$. Both are impossible. Thus the state is bounded on compact sets of directions with all components nonzero.

At $n = (0, v, 1)$, $v \neq 0$, the minor of the data matrix and the six- $\boldsymbol{\tau}^{[w_3^Z]}$ minor can be chosen as

$$\frac{(a-b)^3v^4}{4a^3b^3}, \quad \frac{2(a-b)v^4}{a(b-1)};$$

a nonzero quadratic coefficient on the kernel of the data map is $-vb(a-1)/(a-b)$. At $n = (v, 0, 1)$ the corresponding three quantities are

$$\frac{(b-a)(b-1)^2v^4}{4ab^3}, \quad -2v^4, \quad \frac{vab(b-1)(a-1)}{(a-b)^2}.$$

Permutation of the principal labels gives the remaining direction charts with one zero component. Only principal axis directions can therefore support unbounded $\mathbf{p}^{[w_3^Z]}$.

Write $\boldsymbol{\tau}_{\perp 0}^{[\ell]} = (\tau_1^{[\ell]}, \dots, \tau_5^{[\ell]})$ for the last five entries of the weight-two block. Near $n = (0, 0, 1)$, the first homothety block solves $\tau_1^{[w_3^Z]}, \dots, \tau_5^{[w_3^Z]}$ with minor $(a-b)/(a(b-1))$. The second block solves $u_0^{[w_3^Z]}, u_1^{[w_3^Z]}$ with an identity coefficient block. With $R_0 = 1 + |\mathbf{p}^{[w_3^Z]}|$ these equations give

$$\boldsymbol{\tau}_{\perp 0}^{[w_3^Z]} = \mathbf{a}_\tau(n)\boldsymbol{\tau}_0^{[w_3^Z]} + \mathbf{b}_\tau(n, \mathbf{p}^{[w_3^Z]}), \\ \mathbf{a}_\tau(0, 0, 1) = 0, \quad |\mathbf{b}_\tau| \leq CR_0^2, \quad |\mathbf{u}^{[w_3^Z]}| \leq C(R_0|\boldsymbol{\tau}^{[w_3^Z]}| + R_0^3).$$

The quartic compatibility polynomial $\mathcal{E}_6$ has coefficient $8(\tau_0^{[w_3^Z]})^2$ when $\mathbf{p}^{[w_3^Z]} = \mathbf{u}^{[w_3^Z]} = \boldsymbol{\tau}_{\perp 0}^{[w_3^Z]} = 0$; every $\mathbf{u}^{[w_3^Z]}$ term has a factor $\mathbf{p}^{[w_3^Z]}$. On a smaller axis neighbourhood its equation consequently gives

$$c|\tau_0^{[w_3^Z]}|^2 \leq C(R_0^2|\tau_0^{[w_3^Z]}| + R_0^4).$$

It follows that

$$|\boldsymbol{\tau}^{[w_3^Z]}| \leq C(1 + |\mathbf{p}^{[w_3^Z]}|)^2, \quad |\mathbf{u}^{[w_3^Z]}| \leq C(1 + |\mathbf{p}^{[w_3^Z]}|)^3. \quad (\text{E.13})$$

This prevents a higher jet from escaping faster than the weighted compactification of $\mathbf{p}^{[w_3^Z]}$.

The limiting plane at a principal axis. Take a sequence with $|\mathbf{p}^{[w_3^Z]}| \rightarrow \infty$ and $n \rightarrow (0, 0, 1)$. Choose a signed inverse size ε and form $\mathbf{S}^{[\varepsilon w_3^Z]} = (\varepsilon \mathbf{p}^{[w_3^Z]}, \varepsilon^2 \boldsymbol{\tau}^{[w_3^Z]}, \varepsilon^3 \mathbf{u}^{[w_3^Z]})$. In the limiting state the kernel of the data map is

$$\text{diag} \left(p_{11}^{[\varepsilon w_3^Z]}, \frac{a(b-1)}{a-b} p_{11}^{[\varepsilon w_3^Z]}, \frac{b(a-1)}{a-b} p_{11}^{[\varepsilon w_3^Z]} \right) + pJ_{13} + qJ_{23}.$$

The leading first homothety block gives $\tau_1^{[\varepsilon w_3^Z]} = \dots = \tau_5^{[\varepsilon w_3^Z]} = 0$ and $a(b-1)(p_{11}^{[\varepsilon w_3^Z]})^2/(a-b) = 0$. The next block gives $\mathbf{u}^{[\varepsilon w_3^Z]} = 0$, and $\mathcal{E}_6$ then gives $8(\tau_0^{[\varepsilon w_3^Z]} - pq)^2 = 0$. Thus every leading state has the form

$$\begin{aligned} \mathbf{p}^{[\varepsilon w_3^Z]} &= pJ_{13} + qJ_{23}, & \tau_0^{[\varepsilon w_3^Z]} &= pq, \\ \tau_1^{[\varepsilon w_3^Z]} &= \dots = \tau_5^{[\varepsilon w_3^Z]} = \mathbf{u}^{[\varepsilon w_3^Z]} = 0, & (p, q) &\neq (0, 0). \end{aligned} \quad (\text{E.14})$$

An analytic range system and its normal equations. Use this compactified state. In the chart $p_{13}^{[\varepsilon w_3^Z]} = 1$, write $p_{23}^{[\varepsilon w_3^Z]} = q$, $p_{33}^{[\varepsilon w_3^Z]} = h$, $\tau_0^{[\varepsilon w_3^Z]} = q + z$, $n = (s, t, 1)$. The weighted homothety equations now read

$$\hat{H}_j = \sum_i n_i \mathcal{Q}_i^j(\mathbf{S}^{[\varepsilon w_3^Z]}) + d_j \varepsilon \mathbf{S}_j^{[\varepsilon w_3^Z]} = 0. \quad (\text{E.15})$$

If $\mathcal{O}(\mathbf{p}^{[\varepsilon w_3^Z]}, n)$ denotes the six angular expressions and the first two pressure expressions in (E.5), the eight rescaled data equations are

$$\mathcal{O}(\mathbf{p}^{[\varepsilon w_3^Z]}, n) - \varepsilon O = 0.$$

Thus all angular and pressure data carry the factor ε . The rows of the data equations (1, 2, 3, 6, 7, 8), namely $(A_1, A_2, A_3, B_{*3}, K_1, K_2)$, together with the rows 1, 3, 6, 7, 8, 10, 12 of (E.15), solve the other thirteen state coordinates by the analytic implicit-function theorem. The product of the three successive pivots is

$$\frac{(a-1)(a-b)}{4a^2(b-1)}. \quad (\text{E.16})$$

The determinant is nonzero. The chart with $p_{23}^{[\varepsilon w_3^Z]} = 1$ covers the other projective endpoint. For general p, q , the identities are written in homogeneous coordinates on the plane (E.14); either $p = 1$ or $q = 1$ gives the corresponding analytic range chart.

To obtain the normal equations, pull back, in order, all eight data equations, the seventeen equations (E.15), the three C 's, the twenty-one quartic compatibilities, and the twenty-seven derivatives (E.7); this gives a vector $\mathcal{G}$ of seventy-six analytic equations, in the order

$$\begin{aligned} \mathcal{G} &= (\mathcal{O} - \varepsilon O, \hat{H}_1, \dots, \hat{H}_{17}, C_1, C_2, C_3, \\ &\quad \mathcal{E}_0, \dots, \mathcal{E}_{20}, G_0^{\text{quint}}, \dots, G_{26}^{\text{quint}}). \end{aligned} \quad (\text{E.17})$$

Its coefficient of ε is an affine map $L(O)$ of the eight data

$$O = (A_1, A_2, A_3, B_{*1}, B_{*2}, B_{*3}, K_1, K_2).$$

Its image has rank five. For general (p, q) its zero set is

$$\begin{aligned} A_1 = A_2 = B_{*1} = B_{*2} = 0, \\ 2apq(bB_{*3} - A_3) - a(b-1)p^2(K_1+1) - (a-b)q^2(K_2+1) = 0. \end{aligned} \quad (\text{E.18})$$

Let M_0 be the Jacobian of the thirteen selected range equations in their thirteen dependent state entries. Given their Taylor coefficients of degrees less than m , the coefficient of degree m is $-M_0^{-1}$ times the degree- m coefficient of those equations after substitution of the lower terms. This recursion determines every coefficient used below. For the forcing projection, regard L as a 76×9 matrix in the eight data and the constant 1. On the $pq \neq 0$ chart take its rows (4, 5, 10, 12, 28) and columns (1, 2, 3, 4, 5), write the resulting invertible block as L_0 , and set

$$\mathcal{P} = I - L_{:, \{1,2,3,4,5\}} L_0^{-1} I_{(4,5,10,12,28), :}$$

Here I is the 76×76 identity matrix. At a projective endpoint the same construction uses an invertible five-row block. For $(p, q) = (1, v)$ use rows (4, 5, 10, 23, 28) and columns (1, 2, 4, 5, 7); for $(p, q) = (v, 1)$ use rows (4, 5, 12, 22, 28) and columns (1, 2, 4, 5, 8). Their endpoint determinants are specified below. This defines the projection onto the quotient by $\text{im } L$. The quadratic coefficients in (s, t, h, z) then span

$$s^2 + \frac{b-a}{a(b-1)}t^2, \quad sh, \quad th, \quad h^2, \quad z \left(s + \frac{(a-b)q}{a(b-1)}t \right), \quad zh, \quad z^2. \quad (\text{E.19})$$

The quadratic terms follow by expanding the range equations using (E.16) and reducing their coefficient matrix in the columns $(s^2, sh, th, h^2, sz, hz, z^2)$. The pivots are

$$\begin{array}{l|l} q \neq 0, T_* \neq 0 & -\frac{49152(b-a)(b-1)^2 a^2 q^5 T_*}{b^4(a-1)^4} \\ q = 0 & \frac{41472(b-a)(b-1)^4 a^4}{b^5(a-1)^5} \\ T_* = 0 & \frac{2048q^{10}(7q^2-4b+3)}{b(q^2+1)^3} \\ T_* = 0, 7q^2-4b+3=0 & -\frac{49152q^9}{(q^2+1)^2(3q^2-1)} \end{array} \quad (\text{E.20})$$

In the order of the four rows of (E.20), the seven equation rows are, respectively,

$$\begin{aligned} (13, 17, 19, 21, 35, 47, 74), & \quad (12, 13, 19, 22, 35, 48, 76), \\ (13, 17, 19, 22, 35, 47, 74), & \quad (13, 17, 19, 23, 35, 47, 74). \end{aligned}$$

More explicitly, let M_{quad} consist of these rows in the coefficient matrix of the projected quadratic terms, and let D be the coefficient matrix of the seven displayed generators in (E.19). If I selects the seven monomials above, the row identity being used is $D_{[:,I]}(M_{\text{quad}})_{[:,I]}^{-1} M_{\text{quad}} = D$. Here $T_* = (b-a)q^2 + a(b-1)$. Relabel so that $a < 1$. The real zero set $T_* = 0$ is parametrized by $a = bq^2/(q^2+1-b)$, with $a < b < 1$. On the further zero set $7q^2-4b+3=0$, $b = (7q^2+3)/4$, $a = q^2(7q^2+3)/(1-3q^2)$ and $0 < q^2 < 1/7$, so the replacement pivot in the last row is nonzero. The $p=0$ chart has pivot $4608(a-b)^5 q^{10}/[b^4(a-1)^4(b-1)a]$.

At either extreme principal axis, the first quadratic form in (E.19) is positive definite. If $N = (s^2 + t^2 + h^2 + z^2)^{1/2}$, analytic remainders give

$$N^2 \leq C\{N^3 + |\varepsilon|N + \varepsilon^2\},$$

and hence $N = O(|\varepsilon|)$. The middle axis requires a further argument because that quadratic form is indefinite.

The middle principal axis. Suppose $N/|\varepsilon|$ were unbounded. By (E.19), after passing to a subsequence,

$$\delta = s, \quad t = r_0\delta + d, \quad b = \frac{a(1+r_0^2)}{a+r_0^2}, \quad r_0 \neq 0, \quad \varepsilon, d, h, z = o(|\delta|).$$

The signed variable δ may have either sign. In what follows p, q and all moving data are allowed to vary along the sequence. The Taylor estimates are uniform on compact parameter neighbourhoods.

First suppose $pq(pr_0 + q) \neq 0$ at the limiting projective point. Put $u = (\varepsilon, d, h, z)$. Four rows of the projected residual vector have the expansion

$$(\mathcal{P}\mathcal{G})_{(47,21,49,13)} = \delta M_4(O, p, q)u + \delta^3 \mathbf{c}_4(p, q) + O(\delta^4 + \delta^2|u| + |u|^2). \quad (\text{E.21})$$

On the compatibility set (E.18),

$$\det M_4 = \frac{1024r_0q^3p^3(pr_0 + q)^3}{1 + r_0^2},$$

and the last three columns of M_4 are independent of the data. The unique solution of $M_4u_2 + \mathbf{c}_4 = 0$ is

$$u_2 = \left(0, 0, 0, -\frac{2r_0p^2q(1+r_0^2)}{pr_0 + q}\right). \quad (\text{E.22})$$

Substituting this vector cancels the cubic coefficient of the residual vector $\mathcal{G}$ before projection. Eliminating the four next coefficients from the same rows leaves, in residual rows 17 and 35, respectively,

$$\frac{8pq(pr_0 - q)(1 + r_0^2)^2}{pr_0 + q} = 0, \quad 8r_0^2p^4(1 + r_0^2)^2 = 0 \quad \text{when } q = pr_0. \quad (\text{E.23})$$

The latter is impossible.

To estimate the remainders, suppose $m = |u| \gg \delta^2$. The equations give

$$\mathcal{G} = \varepsilon L(O) + O(|\delta|m + m^2 + |\delta|^3).$$

If ε/m has a nonzero limit, division by m forces (E.18). If that ratio tends to zero, the first column of M_4 disappears and its other three columns may be evaluated at any compatible set of data. Division of (E.21) by δm contradicts the invertibility of M_4 in either case. Thus $u = O(\delta^2)$, and its limit is (E.22). Set $w = u - \delta^2u_2$ using the *moving* p, q . The cancellation of the cubic coefficient of $\mathcal{G}$ gives

$$\mathcal{G} = \varepsilon L(O) + O(|\delta||w| + |w|^2 + \delta^4).$$

The same division, now by $|w|$ and $\delta|w|$, proves $w = O(|\delta|^3)$. The quartic limit therefore gives (E.23).

At the projective endpoints use $(p, q) = (1, v)$ or $(v, 1)$, with v variable. The forcing projection is regular there: its pivots at $v = 0$ are $(r_0^2 + a)^2/r_0^4$ and $(r_0^2 + a)^2/(a^2r_0^2)$, respectively. The four-column determinants use rows (12, 13, 17, 48) and (10, 13, 17, 47) and are $-96r_0^4$ and $-96/r_0$. Formula (E.22) is analytic near either endpoint, since its denominator tends to r_0 or 1. The preceding two estimates therefore apply in these endpoint neighbourhoods. The remaining quartic equations at the endpoints are respectively

$$20r_0^2(1 + r_0^2) = 0, \quad 20(1 + r_0^2) = 0,$$

again impossible. These are residual rows 22 and 23: if M is the coefficient matrix with four columns, I is the selected set of four rows and b_4 is the vector of quartic constant terms, the displayed residual is $(b_4)_j - M_{j,:}M_{I,:}^{-1}(b_4)_I$ at the endpoint.

When $\Delta_{\text{res}} = q + pr_0 \rightarrow 0$ and p, q are nonzero, arrange the equations in the derived variables

$$V = (\varepsilon, d, h, X, Y), \quad X = z^2/\delta, \quad Y = \Delta_{\text{res}}z.$$

Five rows of the projected residual vector have the form

$$(\mathcal{P}\mathcal{G})_{(47,21,13,22,35)} = \delta M_5 V + \delta^3 \mathbf{c}_5 + \mathbf{r}_{\text{rem}}, \quad \det M_5|_{\Delta_{\text{res}}=0} = \frac{4096r_0^6 p^7 (r_0^2 + a)}{a}. \quad (\text{E.24})$$

The corresponding unprojected equations have the additional term $\varepsilon L(O)$. The last four columns and the constant vectors are independent of the data. With $C_* = -2r_0 p^2 q(1 + r_0^2)$, the projected system satisfies

$$M_5(0, 0, 0, 0, C_*)^\top + \mathbf{c}_5 = 0 \quad (\text{E.25})$$

for the moving parameter q . The unprojected system has the same centre, since the additional forcing affects only the first column.

Writing $v = (\varepsilon, d, h)$, the unprojected and projected remainders obey

$$|\mathbf{r}_{\text{rem}}| \leq C\{\delta^2|v| + |v|^2 + |v||z| + |z|^3 + |\delta|z^2 + \delta^2|z| + \delta^4\}. \quad (\text{E.26})$$

The quadratic z^2 term is included in X , the $\delta\Delta_{\text{res}}z$ term in Y , and all δv terms, including the $\delta\varepsilon$ term that depends on the data, are included in M_5 . These assignments give (E.26) directly by Taylor's formula.

If $|V| \gg \delta^2$, the bound $z^2 \leq |\delta||V|$, together with $v, z = o(|\delta|)$, makes the remainder $o(|\delta||V| + |\delta|^3)$. The same compatibility dichotomy for the unprojected equations and the determinant in (E.24) force a contradiction. Hence $V = O(\delta^2)$ and, by (E.25),

$$W := V - (0, 0, 0, 0, C_*\delta^2) = o(\delta^2).$$

Now $|v|, |X| \leq |W|$, so division of (E.26) by $|\delta|$ gives

$$\mathbf{r}_{\text{rem}}/\delta = O(|\delta||z| + |\delta|^3) + o(|W|).$$

Repeating the invertible-matrix estimate yields

$$|W| \leq C(|\delta||z| + |\delta|^3), \quad z^2 \leq C\delta^2|z| + C\delta^4.$$

Thus $z = O(\delta^2)$. But then $\Delta_{\text{res}}z/\delta^2 \rightarrow 0$, whereas (E.25) requires it to tend to $2r_0^2 p^3(1 + r_0^2) \neq 0$. Hence

$$N = O(|\varepsilon|). \quad (\text{E.27})$$

The unprojected equations have the expansion

$$\mathcal{G} = \varepsilon L(O) + O(N^2 + |\varepsilon|N + \varepsilon^2). \quad (\text{E.28})$$

Equation (E.27) and the nonzero forcing pivots therefore give $A_1, A_2 = O(|\varepsilon|)$ in each principal axis chart. The direction is also within $O(|\varepsilon|)$ of that axis, so

$$|\mathbf{p}^{[w_3^Z]}| |A \times n/n| \leq C.$$

The other projective chart and a finite cover of the projective circle make the constant uniform. This is the forcing estimate used in the proof of Lemma 7.3. $\square$

E.4 The cylindrical Jacobian and its invariant subspaces

All state entries in this subsection are coordinates of $\mathbf{S}^{[r]}$; the induced principal operators are $D_i^{[r]} = rD_i$. Their action on the normalized cylindrical state is

$$D_i^{[r]}(\mathbf{S}^{[r]})_j = \mathcal{Q}_i^j(\mathbf{S}^{[r]}) + d_j v_i(\mathbf{S}^{[r]})_j, \quad D_i^{[r]}a_0 = -\Gamma_i(\mathbf{S}^{[r]})a_0, \quad D_i^{[r]}v = w_i w - \Gamma_i(\mathbf{S}^{[r]})v. \quad (\text{E.29})$$

With the axial principal label 3, the reference circle is

$$\begin{aligned} \mathbf{p}^{[r]} &= p\mathbf{J}_{13} + q\mathbf{J}_{23}, \quad \tau_0^{[r]} = pq, \quad \tau_1^{[r]} = \dots = \tau_5^{[r]} = u_0^{[r]} = u_1^{[r]} = 0, \\ v &= (q, -p, 0), \quad a_0 = (0, 0, 1), \quad p^2 + q^2 = 1, \quad \epsilon = 0. \end{aligned} \quad (\text{E.30})$$

We use the fields (E.29) and the reference circle (E.30), with its axial principal label 3. Substitution in (E.29) makes every $D_i^{[r]}$ zero. Let $L_i = DD_i^{[r]}$ at this point, and let J_0 be the Jacobian of the ordered list

$$(C_1, C_2, C_3, \mathcal{E}_0, \dots, \mathcal{E}_{20}, G_0^{\text{quint}}, \dots, G_{26}^{\text{quint}}, |v|^2 - 1, |a_0|^2 - 1, v \cdot a_0).$$

The precise 23×23 formula for L_i is obtained by differentiating (E.29); for its first seventeen rows,

$$(L_i)_{jk} = \partial_k \mathcal{Q}_i^j + d_j v_i \delta_{jk}, \quad 1 \leq k \leq 17, \quad (L_i)_{j,17+k} = d_j(\mathbf{S}^{[r]})_j \delta_{ik}. \quad (\text{E.31})$$

Their axial-frame columns are zero. The last six rows are the derivatives of $w_i w - \Gamma_i(\mathbf{S}^{[r]})v$ and $-\Gamma_i(\mathbf{S}^{[r]})a_0$, with $w = a_0 \times v$. These derivatives determine the matrices from the state fields and cross products.

Since all three fields vanish at the reference point, the Jacobian of every differentiated compatibility constraint $\mathcal{E}$ obeys

$$D(D_i^{[r]}\mathcal{E}) = D\mathcal{E} L_i. \quad (\text{E.32})$$

The Jacobian of (7.9) is $[D\mathcal{T}, 0]$, because $\mathcal{R} = \mathcal{A} = 0$ on the circle. Form the matrix

$$J_* = \begin{bmatrix} D\mathcal{T} \\ J_0 \end{bmatrix}.$$

It has 77 rows and 23 columns. The following table gives the selected rows and columns and the corresponding determinants:

$$\begin{aligned} I_{\text{gen}} &= (1, 2, 3, 4, 6, 7, 8, 10, 12, 18, 26, 75, 76, 77), \\ J_{\text{gen}} &= (1, 2, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 20, 23), \\ I_{p=0} &= (1, 3, 4, 6, 7, 8, 10, 12, 14, 19, 26, 75, 76, 77), \\ J_{p=0} &= (1, 4, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 20, 23), \\ I_{q=0} &= (1, 2, 3, 6, 7, 8, 10, 12, 15, 18, 26, 75, 76, 77), \\ J_{q=0} &= (1, 2, 7, 8, 11, 12, 13, 14, 15, 16, 17, 19, 20, 23). \end{aligned} \quad (\text{E.33})$$

In order, the determinants are

$$\frac{2p^4 q^2 (a-1)^2}{a(b-1)^2}, \quad \frac{2(a-1)^2}{a^2(b-1)^2}, \quad \frac{2(b-a)(a-1)^2}{a(b-1)^3}. \quad (\text{E.34})$$

The endpoint entries use $(p, q) = (0, 1)$ and $(1, 0)$; the opposite signs are obtained by orientation preserving principal-frame sign changes.

The kernel has the following description. For $pq \neq 0$ choose the nine free increments

$$(x_1, \dots, x_9) = (\delta p_{13}^{[r]}, \delta p_{21}^{[r]}, \delta p_{22}^{[r]}, \delta p_{23}^{[r]}, \delta p_{33}^{[r]}, \delta \tau_0^{[r]}, \delta v_2, \delta(a_0)_1, \delta(a_0)_2).$$

Put $g = b(a-1)/[a(b-1)]$, $d = (a-b)/[a(b-1)]$. The dependent increments are

$$\begin{aligned} \delta p_{11}^{[r]} &= -x_3 + x_5 + 2px_8 + 2qx_9, \\ \delta p_{12}^{[r]} &= d\frac{q^2}{p^2}x_2 - g\frac{q}{p}x_3 + \frac{q}{p}x_5 \\ &\quad + \frac{q(1-gq^2)}{p^2}x_8 + \frac{1+gq^2}{p}x_9, \\ \delta \tau_1^{[r]} &= pqx_8 - p^2x_9, \quad \delta \tau_3^{[r]} = q^2x_8 - pqx_9, \\ \delta u_0^{[r]} &= 2pq^2x_8 - 2p^2qx_9, \\ \delta v_1 &= (p/q)x_7, \quad \delta v_3 = -qx_8 + px_9. \end{aligned} \tag{E.35}$$

The remaining dependent entries $\delta p_{31}^{[r]}, \delta p_{32}^{[r]}, \delta \tau_2^{[r]}, \delta \tau_4^{[r]}, \delta \tau_5^{[r]}, \delta u_1^{[r]}, \delta(a_0)_3$ are zero. At $p = 0, q = 1$, take instead the free entries $(\delta p_{12}^{[r]}, \delta p_{13}^{[r]}, \delta p_{22}^{[r]}, \delta p_{23}^{[r]}, \delta p_{33}^{[r]}, \delta \tau_0^{[r]}, \delta v_2, \delta(a_0)_1, \delta(a_0)_2)$; the nonzero dependent entries are

$$\delta p_{11}^{[r]} = -x_3 + x_5 + 2x_9, \quad \delta p_{21}^{[r]} = x_8, \quad \delta \tau_3^{[r]} = x_8, \quad \delta v_3 = -x_8.$$

At $p = 1, q = 0$, use the same free list as in (E.35), with δv_1 replacing δv_2 ; the nonzero dependent entries are

$$\delta p_{11}^{[r]} = -x_3 + x_5 + 2x_8, \quad \delta p_{12}^{[r]} = x_9, \quad \delta \tau_1^{[r]} = -x_9, \quad \delta v_3 = x_9.$$

These formulas define a 23×9 matrix N_{ker} with identity in its free rows. Substitution in the explicitly defined matrices yields

$$J_* N_{\text{ker}} = 0, \quad J_* L_i N_{\text{ker}} = 0 \quad (i = 1, 2, 3), \quad (DT) N_{\text{ker}} = 0. \tag{E.36}$$

Together with (E.34), they prove that the kernel has dimension nine and is invariant under the linearized derivations. By (E.32), the same kernel is obtained after adjoining all differential consequences.

Let $\mathcal{P}_f$ select the nine free rows of N_{ker} and define the restricted matrices by

$$R_0 = \mathcal{P}_f \left(\sum_i v_i L_i \right) N_{\text{ker}}, \quad A_0 = \mathcal{P}_f \left(\sum_i w_i L_i \right) N_{\text{ker}}. \tag{E.37}$$

Equations (E.31) and (E.35) give the following finite polynomial identities:

$$\begin{aligned} \det(xI - R_0) &= x(x+2)(x-1)^2(x+1)^5, \\ R_0(R_0 - I)(R_0 + 2I)(R_0 + I)^2 &= 0, \\ [R_0, A_0] &= 0, \\ A_0 \mathcal{P}_{\text{Hermite}}(R_0) &= 0, \quad \mathcal{P}_{\text{Hermite}}(R_0)^2 = \mathcal{P}_{\text{Hermite}}(R_0), \\ \text{tr } \mathcal{P}_{\text{Hermite}}(R_0) &= 3, \end{aligned} \tag{E.38}$$

where the single scalar polynomial is

$$\mathcal{P}_{\text{Hermite}}(x) = 1 + \frac{5}{3}x - \frac{x^2}{12} - \frac{7}{6}x^3 - \frac{5}{12}x^4. \tag{E.39}$$

The polynomial satisfies $\mathcal{P}_{\text{Hermite}}(0) = \mathcal{P}_{\text{Hermite}}(1) = 1$, $\mathcal{P}_{\text{Hermite}}(-2) = \mathcal{P}_{\text{Hermite}}(-1) = \mathcal{P}'_{\text{Hermite}}(-1) = 0$. In particular $R_0(R_0 - I)\mathcal{P}_{\text{Hermite}}(R_0) = 0$; the positive eigenspace is semisimple, even when the negative -1 block is not. The unique centre is the tangent of the explicit equilibrium circle (E.30). Adding ϵ gives one further radial eigenvalue 1, with zero angular linearization. The characteristic polynomial on the augmented tangent space is therefore

$$x(x+2)(x-1)^3(x+1)^5. \tag{E.40}$$

E.5 Pressure forms

With $\phi_1 + \phi_2 + \phi_3 = 0$, substitution of the three rows of $\mathbf{P}(\mathbf{p})$ gives the structure equation (2.16), including the off-diagonal cancellations. Exterior differentiation and Cartan's formula give (7.23) and (7.24) for the coefficients κ_i and $\ell_i = \vartheta_i(Z)$. The strict pressure interval $-1 < K_i < 2$ is a geometric conclusion from the spherical flux identity and the nonexistence of a nowhere-zero tangent field on the sphere. Together with the similarity weights, it gives the principal field flow contradiction in Section 7.

Supplementary calculations

Relation to the proof

Supplementary outputs	Results in the proof
Section S1: derivative matrices, compatibility polynomials and reconstruction	Proposition 2.4; Proposition 2.5
Section S2: ratio equations, normalization factors and similarity orbits	Lemmas 3.1 and 3.2; Theorem 4.3
Section S3: connection fibres and cylindrical linearization	Theorem 6.1; Lemma 6.4; Lemma 6.12
Section S4: homothety equations, principal-direction normal equations and carrier matrices	Theorem 7.1; Lemma 7.3; Lemma 7.6
Section S5: local ideals, normal determinants and ordinary boundary obstructions	Lemma 8.6; Propositions 8.9 and 8.10
Section S6: characteristic pencils, direction factors and amplitude conics	Proposition 8.13
Sections S7 and S8: moment matrix, quadratic norm and slope resultants	Propositions 8.12 and 8.13
Section S9: deformation gradients and pressures	Proposition 9.7

S1 The finite frame system

We use the notation and derivative ordering of Section 2. In particular $h_i = p_{ii}$, $\tau = (\tau_0, \dots, \tau_5)^\top$ and $u = (u_0, u_1)^\top$. Coefficients lie in $\mathbb{Q}(a, b)$, where

$$a, b > 0, \quad ab(a-1)(b-1)(a-b) \neq 0.$$

The raw state has coordinate weights $(1^9, 2^6, 3^2)$. All row and column indices below start at one.

S1.1 The three derivative matrices

The curvature and pressure expressions define $M^{(0)}$; commutators of the frame coefficients define $M^{(1)}$; commutators of the retained first derivatives define $M^{(2)}$. Their dimensions and unknown columns are

$$\begin{aligned} M^{(0)} &\in \text{Mat}_{24 \times 27}(\mathbb{Q}(a, b)), & (D_i p_{jk})_{i,j,k}, \\ M^{(1)} &\in \text{Mat}_{27 \times 18}(\mathbb{Q}(a, b)), & (D_i \tau_r)_{i=1,2,3; r=0,\dots,5}, \\ M^{(2)} &\in \text{Mat}_{18 \times 6}(\mathbb{Q}(a, b)), & (D_i u_r)_{i=1,2,3; r=0,1}. \end{aligned}$$

The indices j, k are lexicographic. The commutator pairs are $(2, 3), (3, 1), (1, 2)$, followed by the coordinate index. The exact coefficient definitions are those of Appendix A.2. Tables S1–S3 give the resulting matrices.

Table S1: The first derivative matrix $M^{(0)}$. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row	Nonzero column entries
1	$(16, -1) \quad (22, 1)$
2	$(7, 1) \quad (19, -1 \cdot 2^{-1}) \quad (23, 2^{-1}) \quad (27, 2^{-1})$
3	$(4, -1) \quad (10, 2^{-1}) \quad (14, -1 \cdot 2^{-1}) \quad (18, -1 \cdot 2^{-1})$
4	$(17, -1) \quad (19, -1 \cdot 2^{-1}) \quad (23, 2^{-1}) \quad (27, -1 \cdot 2^{-1})$
5	$(8, 1) \quad (20, -1)$
6	$(1, 2^{-1}) \quad (5, -1 \cdot 2^{-1}) \quad (9, 2^{-1}) \quad (11, 1)$
7	$(10, 2^{-1}) \quad (14, 2^{-1}) \quad (18, -1 \cdot 2^{-1}) \quad (24, 1)$
8	$(1, -1 \cdot 2^{-1}) \quad (5, -1 \cdot 2^{-1}) \quad (9, 2^{-1}) \quad (21, -1)$
9	$(6, -1) \quad (12, 1)$
10	$(16, (-1) \cdot b^{-1}) \quad (22, a^{-1})$
11	$(7, b^{-1}) \quad (19, -1 \cdot 2^{-1}) \quad (23, (\frac{1}{2}) \cdot a^{-1}) \quad (27, (\frac{1}{2}) \cdot b^{-1})$

Row Nonzero column entries

12	$(4, (-1) \cdot a^{-1})$	$(10, 2^{-1})$	$(14, (-\frac{1}{2}) \cdot a^{-1})$	$(18, (-\frac{1}{2}) \cdot b^{-1})$
13	$(17, (-1) \cdot b^{-1})$	$(19, -1 \cdot 2^{-1})$	$(23, (\frac{1}{2}) \cdot a^{-1})$	$(27, (-\frac{1}{2}) \cdot b^{-1})$
14	$(8, b^{-1})$	$(20, -1)$		
15	$(1, 2^{-1})$	$(5, (-\frac{1}{2}) \cdot a^{-1})$	$(9, (\frac{1}{2}) \cdot b^{-1})$	$(11, 1)$
16	$(10, 2^{-1})$	$(14, (\frac{1}{2}) \cdot a^{-1})$	$(18, (-\frac{1}{2}) \cdot b^{-1})$	$(24, a^{-1})$
17	$(1, -1 \cdot 2^{-1})$	$(5, (-\frac{1}{2}) \cdot a^{-1})$	$(9, (\frac{1}{2}) \cdot b^{-1})$	$(21, -1)$
18	$(6, (-1) \cdot a^{-1})$	$(12, 1)$		
19	$(11, (b-1))$	$(13, (a-b))$	$(21, (a-1))$	$(25, (-1) \cdot (a-b))$
20	$(2, (-1) \cdot (b-1))$	$(4, (-1) \cdot (a-b))$	$(24, (-1) \cdot (a-1))$	$(26, (b-1))$
21	$(3, (-1) \cdot (a-1))$	$(7, (a-b))$	$(15, (a-1))$	$(17, (-1) \cdot (b-1))$
22	$(11, (-1) \cdot b^{-1} \cdot (b-1))$	$(13, (-1) \cdot b^{-1} \cdot a^{-1} \cdot (a-b))$	$(21, (-1) \cdot a^{-1} \cdot (a-1))$	$(25, b^{-1} \cdot a^{-1} \cdot (a-b))$
23	$(2, b^{-1} \cdot (b-1))$	$(4, b^{-1} \cdot a^{-1} \cdot (a-b))$	$(24, a^{-1} \cdot (a-1))$	$(26, (-1) \cdot b^{-1} \cdot (b-1))$
24	$(3, a^{-1} \cdot (a-1))$	$(7, (-1) \cdot b^{-1} \cdot a^{-1} \cdot (a-b))$	$(15, (-1) \cdot a^{-1} \cdot (a-1))$	$(17, b^{-1} \cdot (b-1))$

Table S2: The first prolongation matrix $M^{(1)}$. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row Nonzero column entries

1	$(9, (-1) \cdot (b-1)^{-1} \cdot a^{-1} \cdot (a-b))$	$(16, (-2) \cdot (b-1)^{-1} \cdot a^{-1} \cdot (a-b))$
2	$(14, 1)$	
3	$(8, 1)$	
4	$(14, 1)$	$(17, -1)$
5	$(9, 1)$	
6	$(10, 1)$	$(13, -1)$
7	$(11, 1)$	
8	$(12, 1)$	$(15, (-\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$
9	$(16, (-2) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$	
10	$(3, (b-1)^{-1} \cdot a^{-1} \cdot (a-b))$	
11	$(16, -1)$	$(18, 1)$
12	$(2, -1)$	$(13, 1)$
13	$(16, -1)$	$(15, (-\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$
14	$(3, -1)$	$(14, (2) \cdot (a-b)^{-1} \cdot (b-1) \cdot a)$
15	$(4, -1)$	
16	$(5, -1)$	$(15, (-\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$
17	$(6, -1)$	
18	$(14, (2) \cdot (a-b)^{-1} \cdot b \cdot (a-1))$	
19	$(4, (2) \cdot (b-1)^{-1} \cdot a^{-1} \cdot (a-b))$	
20	$(2, -1)$	$(10, 1)$
21	$(7, -1)$	$(9, (\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$
22	$(2, -1)$	$(5, 1)$
23	$(8, (-2) \cdot (a-b)^{-1} \cdot (b-1) \cdot a)$	
24	$(1, 1)$	
25	$(9, (\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$	
26	$(3, (\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$	
27	$(4, (2) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$	$(8, (-2) \cdot (a-b)^{-1} \cdot b \cdot (a-1))$

Table S3: The second prolongation matrix $M^{(2)}$. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row	Nonzero column entries
1	$(3, 1)$
3	$(4, 1)$
4	$(5, -1)$
6	$(6, (-\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$
7	$(1, -1)$
8	$(5, 1) \quad (6, (-\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$
9	$(2, -1)$
11	$(6, (-\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$
14	$(3, -1) \quad (4, (\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$
16	$(1, 1)$
17	$(4, (\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$
18	$(2, (\frac{1}{2}) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1))$

For the row and column selections in Appendix A.2, the selected determinants are

$$\det M_{I_0, J_0}^{(0)} = -\frac{(a-1)^4(a-b)^5(b-1)^6}{32a^6b^8}, \quad M^{(0)}Q - r^{(0)} = 0. \quad (\text{S1.1})$$

$$\det M_{I_1, J_1}^{(1)} = -\frac{2(a-b)^2}{a^2(b-1)^2}. \quad (\text{S1.2})$$

$$\det M_{I_2, \{1, \dots, 6\}}^{(2)} = \frac{b(a-1)}{2a(b-1)}. \quad (\text{S1.3})$$

They determine the three raw state fields Q_i . Their weights satisfy

$$\sum_{\ell=1}^{17} d_\ell S_\ell \frac{\partial Q_i^j}{\partial S_\ell} = (d_j + 1) Q_i^j. \quad (\text{S1.4})$$

S1.2 Cubic and quartic compatibility

The first prolongation has three monic cubic residuals, in the order used to define the quartic equations.

$$\begin{aligned}
C_1 = & p_{11}^2 p_{23} - p_{11}^2 p_{32} + \frac{p_{21} \tau_3 (a-b)}{ab-a} + \frac{p_{22} \tau_1 (a-b)}{ab-a} \\
& + \left(\frac{p_{21}^2 p_{23} (-a+b)}{ab-a} + \frac{p_{22} \tau_4 (-a+b)}{ab-a} \right) \\
& + \left(\frac{p_{21}^2 p_{32} (-a+b)}{ab-a} + \frac{p_{31} \tau_2 (-a+b)}{2ab-2a} \right) \\
& + \left(\frac{p_{11} p_{32} p_{33} (a-b)}{ab-b} + \frac{p_{33} \tau_4 (a-b)}{ab-b} \right) \\
& + \left(\frac{p_{21} p_{22} p_{31} (2a-2b)}{ab-a} + \frac{p_{21} p_{31} p_{33} (-a+b)}{ab-b} \right)
\end{aligned} \quad (\text{S1.5})$$

$$\begin{aligned}
C_2 = & p_{11}p_{12}p_{32} - p_{11}\tau_3 + p_{11}\tau_5 - p_{12}^2p_{13} - p_{12}^2p_{31} - p_{12}\tau_1 \\
& + \left(\frac{p_{13}p_{22}^2(a-b)}{ab-a} + \frac{p_{22}^2p_{31}(-a+b)}{ab-a} \right) \\
& + \left(\frac{p_{32}\tau_2(-a+b)}{2ab-2a} + \frac{p_{33}\tau_5(-a+b)}{ab-b} \right) \\
& + \left(\frac{p_{12}p_{32}p_{33}(-a+b)}{ab-b} + \frac{p_{21}p_{22}p_{32}(a-b)}{ab-a} \right) \\
& + \left(\frac{p_{22}p_{31}p_{33}(a-b)}{ab-b} \right)
\end{aligned} \tag{S1.6}$$

$$\begin{aligned}
C_3 = & p_{11}p_{12}p_{33} - p_{11}p_{13}p_{32} - p_{11}p_{23}p_{31} - p_{11}p_{31}p_{32} - p_{11}\tau_0 - p_{12}p_{13}^2 - p_{13}\tau_1 \\
& + (2p_{11}p_{13}p_{23} + \frac{p_{11}\tau_2(ab-b)}{2ab-2a} - p_{12}p_{13}p_{31}) \\
& + \left(\frac{p_{12}p_{33}^2(-a+b)}{ab-b} + \frac{p_{21}p_{23}^2(a-b)}{ab-a} \right) \\
& + \left(\frac{p_{21}p_{33}^2(a-b)}{ab-b} + \frac{p_{22}\tau_0(-a+b)}{ab-a} \right) \\
& + \left(\frac{p_{11}p_{21}p_{22}(-ab+b)}{ab-a} + \frac{p_{23}\tau_3(-a+b)}{ab-a} \right) \\
& + \left(\frac{p_{21}p_{22}p_{33}(-a+b)}{ab-a} + \frac{p_{21}p_{23}p_{32}(a-b)}{ab-a} \right)
\end{aligned} \tag{S1.7}$$

The twenty-one quartics consist of the twelve nonzero residuals of the second prolongation, followed by $\mathcal{Q}_1C_1, \mathcal{Q}_2C_1, \mathcal{Q}_3C_1$, and the analogous triples for C_2, C_3 . Write them as $M\mathbf{u} - \mathbf{r}$. Table S4 gives all coefficients of the 21×2 matrix M .

Table S4: The coefficient matrix of the retained second derivatives. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row	Nonzero column entries
1	$(1, (\frac{1}{ab-a}) \cdot ((5ab-5a)h_1 + (-a+b)h_2)) \quad (2, (\frac{1}{2ab-2a}) \cdot ((-5ab+5b)h_1 + (-a+b)h_3))$
2	$(1, (\frac{-1}{a^2b-ab^2-a^2+ab}) \cdot ((2a^2b-2ab^2-2a^2+2ab)h_1 + (-a^2b+2a^2-2ab+2b^2-b)h_2 + (a^2b-a^2-ab+a)h_3)) \quad (2, (\frac{-a+1}{2a^2b-2ab^2-2a^2+2ab}) \cdot ((7ab-7b^2)h_1 + (-b^2+b)h_2 + (b^2-a)h_3))$
4	$(1, (\frac{-1}{a-b}) \cdot ((ab-a)h_1 + (-5a+5b)h_2)) \quad (2, (\frac{-1}{2a-2b}) \cdot ((-ab+b)h_1 + (-a+b)h_3))$
6	$(1, (\frac{1}{ab-b^2-a+b}) \cdot ((a^2b-2ab^2+2ab-2a+b)h_1 + (-2ab+2b^2+2a-2b)h_2 + (-a^2+ab+a-b)h_3)) \quad (2, (\frac{a-1}{2a^2b^2-2ab^3-4a^2b+4ab^2+2a^2-2ab}) \cdot ((-a^2b^2+2ab^3-a^2b-b^3+2ab-b^2)h_1 + (9ab^2-9b^3-9ab+9b^2)h_2 + (a^2b-2ab^2+b^3+a^2-2ab+b^2)h_3))$
7	$(1, (\frac{-1}{a^3b-a^2b^2-a^3+ab^2+a^2-ab}) \cdot ((2a^3b^2-a^2b^3-3a^3b+a^2b^2-ab^3+a^3-a^2b+3ab^2+a^2-2ab)h_1 + (-a^3b+ab^3-a^3+7a^2b-7ab^2+b^3-a^2+b^2)h_2 + (-9a^3b+9a^2b^2+9a^3-9ab^2-9a^2+9ab)h_3)) \quad (2, (\frac{b}{2a^2b-2ab^2-2a^2+2ab}) \cdot ((2a^2b-ab^2-2ab-a+2b)h_1 + (-ab+b^2+a-b)h_2 + (-4a^2+4ab+4a-4b)h_3))$
9	$(1, (\frac{-6}{a-b}) \cdot ((ab-a)h_1 + (-a+b)h_2)) \quad (2, (\frac{-1}{a-b}) \cdot ((-3ab+3b)h_1 + (-a+b)h_3))$
11	$(2, (\frac{-ab+b}{ab-a}) \cdot p_{31})$
12	$(2, (\frac{ab-b}{ab-a}) \cdot p_{32})$
13	$(1, (\frac{a-b}{ab-a}) \cdot h_2) \quad (2, (\frac{-a+b}{2ab-2a}) \cdot h_3)$
14	$(1, (\frac{a-b}{ab-a}) \cdot p_{21})$
15	$(2, (\frac{-a+b}{2ab-2a}) \cdot p_{31})$
16	$(1, (-1) \cdot p_{12}) \quad (2, (\frac{ab-b}{2ab-2a}) \cdot p_{12})$
17	$(1, (-1) \cdot h_1) \quad (2, (\frac{1}{2ab-2a}) \cdot ((ab-b)h_1 + (-a+b)h_3))$
18	$(2, (\frac{-a+b}{2ab-2a}) \cdot p_{32})$
19	$(1, (-1) \cdot p_{13}) \quad (2, (\frac{ab-b}{2ab-2a}) \cdot p_{13})$
20	$(1, (\frac{-a+b}{ab-a}) \cdot p_{23})$

Row	Nonzero column entries
21	$(1, \left(\frac{1}{ab-a}\right) \cdot ((-ab+a)h_1 + (-a+b)h_2)) \quad (2, \left(\frac{ab-b}{2ab-2a}\right) \cdot h_1)$

Put $\Delta_a = a(b-1)$, $\Delta_b = b(a-1)$ and $\Delta_{ab} = a-b$. Let S_0, S_1 select rows (1, 4) and (13, 17) of M , and put $M_0 = S_0 M$, $M_1 = S_1 M$. With

$$X_0 = \Delta_b h_1 h_2, \quad Y_0 = \Delta_a h_1 h_3, \quad Z_0 = \Delta_{ab} h_2 h_3,$$

the determinant outputs are

$$\det M_0 = \frac{2(6X_0 + Y_0 + Z_0)}{\Delta_a}, \quad \det M_1 = \frac{\Delta_{ab}(X_0 - Y_0 - Z_0)}{2\Delta_a^2}.$$

The following combined adjugate gives a left inverse without choosing either determinant as a denominator.

$$N = \frac{\Delta_a}{2} \text{adj}(M_0)S_0 + \frac{2\Delta_a^2}{\Delta_{ab}} \text{adj}(M_1)S_1 \tag{S1.8}$$

$$NM = 7\Delta_b h_1 h_2 \text{Id}_2, \quad \mathbf{u} = \frac{N\mathbf{r}}{7\Delta_b h_1 h_2}.$$

These are the outputs used in Proposition 2.4 and in the normalization of Section 3.

S2 Normalization and the lower ranks

S2.1 The normalization factors and constant ratios

The factors μ_i and the retained state $\mathbf{X}$ are those of (3.1) and (3.2). The cubic reconstruction, together with the left inverse above, removes $\tau_4, \tau_5, \tau_0, u_0, u_1$ on $h_1 h_2 h_3 \mu_1 \mu_2 \mu_3 \neq 0$. For constant abnormality ratios, use $\alpha_{ij} = 1 - \lambda_j / \lambda_i$ and the three constants R_i of Lemma 3.1. The matrix of the squared off-diagonal coefficients is

$$\begin{pmatrix} R_1(R_1 + 1) & -\frac{\alpha_{31}}{\alpha_{21}}R_2 & -\frac{\alpha_{13}}{\alpha_{23}}R_1R_3 \\ -\frac{\alpha_{21}}{\alpha_{31}}R_1R_2 & R_2(R_2 + 1) & -\frac{\alpha_{23}}{\alpha_{12}}R_3 \\ -\frac{\alpha_{23}}{\alpha_{13}}R_1 & -\frac{\alpha_{32}}{\alpha_{12}}R_2R_3 & R_3(R_3 + 1) \end{pmatrix}.$$

Its determinant is 4, using $R_1 R_2 R_3 = 1$ and $\alpha_{12} \alpha_{23} \alpha_{31} / (\alpha_{21} \alpha_{32} \alpha_{13}) = -1$.

S2.2 The open normalization divisor

On $\mu_1 = 0$, use the variables $q, d, g, z, X, Y, U, V, W$ of Appendix B.3. The opposite-pair equations are

$$(X^2 + d)Y + 2z(1 + z/d)XW = 0, \quad (\text{S2.1})$$

$$(dq - Y^2)X + 2z(1 - z/d)YW = 0, \quad (\text{S2.2})$$

$$2X^2Y^2 + (d - z)Y^2 - q(d + z)X^2 = 0, \quad (\text{S2.3})$$

$$-(1 + aq)(z^2 + d^2) + 2(1 + qg)z - 4XU - 4gYV = 0. \quad (\text{S2.4})$$

$$UX = -\frac{gW^2}{ad} - \frac{(z - d)^2}{4d}, \quad VY = \frac{W^2}{ad} + \frac{q(z + d)^2}{4d}. \quad (\text{S2.5})$$

Set $R = X^2$ and $e = ad - a - 1$. The two necessary cubics in R have the short combination

$$F_1 + F_2 = 4adz(z + d)H, \quad H = 4R(R + d)^2 + eK, \quad (\text{S2.6})$$

$$K = 2(d^2 + 3dz - z^2)R^2 + (2d^3 + 4d^2z - 9dz^2 + z^3)R + 3dz^2(z - d).$$

The Laurent roots at infinity and the first cubic evaluated on those roots are

$$\begin{aligned} R_0 &= -3d + O(z^{-1}), \\ R_1 &= \frac{1}{2}z + \frac{1}{2e} + O(z^{-1}), \\ R_2 &= \frac{1}{2}ez^2 + O(z). \end{aligned} \quad (\text{S2.7})$$

$$\begin{aligned} F_1(R_0) &= 12ade z^6 + O(z^5), \\ F_1(R_1) &= a(d - 1)(ad - 1)z^6 + O(z^5), \\ F_1(R_2) &= ae^3 z^9 + O(z^8). \end{aligned} \quad (\text{S2.8})$$

$$\text{Res}_R(F_1, H) = -768a^3 d(d - 1)(ad - 1)e^4 z^{21} + O(z^{20}). \quad (\text{S2.9})$$

The eight cubic coefficients are given in Appendix B.7. On the exceptional value $e = 0$, $d = 1 + a^{-1}$ and $H = 4R(R + d)^2$. These outputs are used in Lemma 3.2.

S2.3 Two common similarity generators

The orbit matrices and their constant eigenvalue equations are

$$\det\{\Gamma - \lambda_i \text{diag}(r^2, 1)\} = 0. \quad (\text{S2.10})$$

$$(a_0 - \lambda_i)(n_{22} - \lambda_i) = 0, \quad n_{11}(n_{22} - \lambda_i) - n_{12}^2 = 0. \quad (\text{S2.11})$$

$$(\lambda_j - \lambda_k)^2(U_j^2 + V_j^2) = (\lambda_j - m)(m - \lambda_k). \quad (\text{S2.12})$$

For the nonabelian similarity pair, the harmonic height functions and constant planar determinant satisfy

$$\begin{aligned} |F_\perp|^2 + s &= E, & |F_\perp|^2/J_\perp^2 + t/\kappa^2 + 1/\kappa^2 &= E', \\ t &= c_0 s + d_0, & c_0 &= \kappa^2/J_\perp^2 > 0. \end{aligned} \quad (\text{S2.13})$$

$$b_0 = J_\perp c_0 + J_\perp d_0/s, \quad (\text{S2.14})$$

$$a_0^2 = C_0 - c_0 s + C_1/s - J_\perp^2 d_0^2/s^2, \quad (\text{S2.15})$$

For the commuting dilation and rotation, the two norm equations reduce to the quadratic alternative

$$\mathcal{N} = u + b_n x^2 > 0, \quad ub_n \neq 0, \quad y = \frac{Jx}{ku\sqrt{\mathcal{N}}}, \quad \tilde{\mathcal{N}} = \frac{1}{u} - \frac{ub_n k^2}{J^2} y^2. \quad (\text{S2.16})$$

The cylindrical phase uses $t = z/r$, $Q = a_n + mt^2$ and ψ_c from Appendix B.10. The physical gradient and the coefficient that removes the nonconstant phase are

$$F_f = \begin{pmatrix} a_n/\sqrt{Q} & 0 & mt/\sqrt{Q} \\ \sqrt{Q}(\omega - t\psi'_c) & k\sqrt{Q} & \sqrt{Q}\psi'_c \\ 0 & 0 & \nu \end{pmatrix}, \quad \det F_f = a_n k \nu,$$

$$[t^4]P = k^2 m^2 + (K_0 - \omega m)^2 + c_1^2.$$

The remaining constant norm identity is

$$(E/a_n - 1)X_0 W_0 = \omega^2 X_0 W_0^2 + \ell^2 X_0^2 + k^2 W_0^2 + h^2. \quad (\text{S2.17})$$

S3 Common Euclidean isometries

S3.1 Translation fibres

For two translations, use the metric variables of Appendix D.1. The Hessian line, third derivative system and nonzero quadratic coefficient are

$$-2G_{nt}d\tau^2a_H + Nb_H + G_{nt}KP\tau^4 + G_{nt}Pd\tau^2 - G_{nt}Qd_0\tau^4 = 0. \quad (\text{S3.1})$$

$$M = \begin{pmatrix} u & w \\ -w & u \end{pmatrix}, \quad u = -\frac{2d}{d_0\tau^2}, \quad w = \frac{N}{G_{nt}d_0\tau^4}, \quad \det M = \frac{4G_{nt}^2d^2\tau^4 + N^2}{G_{nt}^2d_0^2\tau^8} > 0. \quad (\text{S3.2})$$

$$-\frac{G_{nt}\tau^6 H(\tau)}{d_0^2 D(\tau)^3}, \quad D = -\tau^4 G_{nt}^2, \quad \deg H = 24, \quad [\tau^{24}]H = -K^6. \quad (\text{S3.3})$$

$$\hat{\mathcal{C}} = \det M \mathcal{C}|_{c_H=e_H=0} - (\partial_{c_H}\mathcal{C}, \partial_{e_H}\mathcal{C}) \text{adj}(M)r. \quad (\text{S3.4})$$

For a translation paired with a rotation or a screw motion, the variables $v, \mathcal{N}_b, L, U, A, T, Q, D$ are those of Appendix D.2. The metric reconstruction and the determinant outputs are

$$a_q = U, \quad a_v = QLU/v, \quad U^2 = \frac{\mathcal{N}_b}{Q(v+L^2)}, \quad P = v^3 - \mathcal{I}_1 v^2 + \mathcal{I}_2 v - j, \quad (\text{S3.5})$$

$$A = \frac{j}{v} - \frac{P}{v\mathcal{N}_b}, \quad R_0 = \mathcal{I}_2 - A - \frac{j(1+\mathcal{N}_b)}{v}, \quad T^2 = vAR_0 - jv^2. \quad (\text{S3.6})$$

$$G_{ij} = \frac{A}{j}K_{ij} + \frac{v}{\mathcal{N}_b}(s_0 - 2A/j)a_i a_j - \frac{T}{j\mathcal{N}_b}(a_i r_j + r_i a_j). \quad (\text{S3.7})$$

Here $Q = 2q + p^2$. The four coefficient rows, in the order phase closedness, inverse vertical equilibrium, forward vertical equilibrium and harmonicity, are as follows.

Subscripts $v, \mathcal{N}_b$ in what follows hold $\mathcal{I}_1, \mathcal{I}_2, j$ fixed. Put $D = v + L^2$ and

$$\begin{aligned} A_{\mathcal{N}_b} &= (j/v - A)/\mathcal{N}_b, \\ T_{\mathcal{N}_b} &= \{v(R_0 - A)A_{\mathcal{N}_b} - jA\}/(2T), \\ A_v &= -j/v^2 - (vP_v - P)/(v^2\mathcal{N}_b), \\ T_v &= \{\mathcal{I}_2 A - A^2 - 2jv + v(R_0 - A)A_v\}/(2T), \\ F &= A - TL/v, \\ S &= \frac{Q\{T(v - L^2) - v(R_0 - A)L\}}{v^2 D}, \\ R_1 &= \frac{vR_0 + AL^2 + 2TL}{vD}. \end{aligned} \quad G_0 = AL + T,$$

Let

$$b_q^0 = 2\mathcal{N}_b/Q + 2\mathcal{N}_b/D, \quad b_q = (2\mathcal{N}_b, 0, 2\mathcal{N}_b L/D, 0), \quad b_v = (0, 2\mathcal{N}_b, 0, 2\mathcal{N}_b L/D).$$

For $y = (U_q/U, U_v/U, L_q, L_v)^\top$, the four rows of M are

$$\begin{aligned} M_1 &= (QL/v, -1, Q/v, 0), \\ M_2 &= (Q, L, 0, 1), \\ M_3 &= Q\{(F, 0, F_L, 0) + F_{\mathcal{N}_b}b_q\} + (0, G_0, 0, (G_0)_L) + (G_0)_{\mathcal{N}_b}b_v, \\ M_4 &= S_{\mathcal{N}_b}b_q + (0, 0, S_L, (R_1)_L) + (R_1)_{\mathcal{N}_b}b_v. \end{aligned}$$

The equations are

$$My + \mathbf{r}_0 + \frac{K_{C_f}}{U} \mathbf{r}_C + \frac{K_{C_f}^{-1}}{U} \mathbf{r}_{\text{inv}} + \frac{1}{U} \mathbf{r}_{-1} + K_{C_f} \mathbf{r}_{C0} + U \mathbf{r}_U = 0, \quad (\text{S3.8})$$

where

$$\begin{aligned} \mathbf{r}_0 &= (2p^2 L/v^2, 2, 2F + Q(2F_v + F_{\mathcal{N}_b} b_q^0), 2S_v + S_{\mathcal{N}_b} b_q^0)^\top, \\ \mathbf{r}_C &= (0, 0, -1, 0)^\top, & \mathbf{r}_{\text{inv}} &= (0, 1, 0, 0)^\top, \\ \mathbf{r}_{-1} &= (2p/v^2, 0, 0, 0)^\top, & \mathbf{r}_{C0} &= (0, 0, 0, p/v)^\top, \\ \mathbf{r}_U &= (0, 0, 0, -2pQF/v^2)^\top. \end{aligned}$$

$$\det M = -\frac{2\mathcal{N}_b^2 Q^2}{vD} W, \quad W = \alpha L + \beta, \quad (\text{S3.9})$$

$$[\mathcal{N}_b^2]N_U = \frac{j^2 Q K_{C_f}^2}{v^5}, \quad [\mathcal{N}_b^{3/2}]N_U = 0, \quad [\mathcal{N}_b]N_U|_{K_{C_f}=0} = -\frac{36j^2 Q^2}{v^4}. \quad (\text{S3.10})$$

$$524288j^6 v^{16} P^2(5v^3 - 3\mathcal{I}_1 v^2 + \mathcal{I}_2 v + j)^3 G^{-1}(v), \quad (\text{S3.11})$$

The two remaining compatibilities are (D.13). The asymptotic coefficients in (S3.10) give the finite fibre condition on its determinant zero set.

S3.2 Cylindrical linearization and two rotations

At a cylinder with fixed pressure data, the five radial weights are

$$(-2, -1, -1, 1, 1).$$

The quadratic obstruction uses $E, k, \omega, \Delta, \mathcal{T}, H_*, Z$ defined in Appendix D.4. Its coefficient identity is

$$(E - 1)\mathcal{T} + \Delta H_* = E\omega^2 Z, \quad H_* = (E - 1)(k^2 - 1) + (E + 1)\omega^2. \quad (\text{S3.12})$$

The physical Hessian and third derivative systems for two rotations are

$$H_{a3} = \Omega_t^{\text{axial}} \times F_f \partial_{x_a} - F_f (\Omega_s^{\text{axial}} \times \partial_{x_a}), \quad H_{ab} = \partial_a \partial_b f. \quad (\text{S3.13})$$

$$T_{ab3} = \Omega_t^{\text{axial}} \times H_{ab} - H_{\Omega_s^{\text{axial}} \times \partial_{x_a}, b} - H_{a, \Omega_s^{\text{axial}} \times \partial_{x_b}}, \quad T_{abc} = \partial_a \partial_b \partial_c f. \quad (\text{S3.14})$$

$$(Q_0, Q_1, Q_2)^\top = (-M_1 M_0^{-1} \mid I_3) b(H, \Omega_s^{\text{axial}}, \Omega_t^{\text{axial}}), \quad (\text{S3.15})$$

For the first twelve rows, with the determinant differentiated instead of its logarithm, put

$$D_0 = s^2 - t^2, \quad L_0 = t^2(D_0 + p^2) - s^2 q^2 - D_0 \gamma^2, \quad R_0 = L_0 + D_0^2,$$

$$P_{10} = p^2 L_0^2 + q^2 R_0^2 + p^2 q^2 \{(s^2 + t^2)^2 (p^2 + q^2) + 2D_0^2 (s^2 + t^2)\}.$$

The selected determinant is

$$\det M_0 = 128pqst(s^2 - t^2)\gamma^5 P_{10}.$$

Here $P_{10} > 0$ on the real chart $pqst\gamma(s^2 - t^2) \neq 0$. The three determinant rows are multiplied by $st\gamma$ when one uses $(st\gamma)^2 d^2 \log \det F_f$. The three homogeneous compatibility conics are those in (S3.15); their source rows and material resultant are specified in Appendices D.5 and D.6. The remaining coefficient on H_{11}^3 , after solving seven Hessian rows, is

$$-\frac{2(t^2 p^2 + s^2 q^2)}{s^2 t^2 \gamma(s^2 - t^2)}.$$

For the transverse compactness argument, the homogeneous escape plane is

$$\mathbf{p}^{[\varepsilon w_3^Z]} = p\mathbf{J}_{13} + q\mathbf{J}_{23}, \quad \tau_0^{[\varepsilon w_3^Z]} = pq, \quad \boldsymbol{\tau}_{\perp 0}^{[\varepsilon w_3^Z]} = 0, \quad \mathbf{u}^{[\varepsilon w_3^Z]} = 0, \quad (p, q) \neq (0, 0). \quad (\text{S3.16})$$

$$s^2 + \frac{b-a}{a(b-1)}t^2, \quad sh, \quad th, \quad h^2, \quad z \left(s + \frac{(a-b)q}{a(b-1)}t \right), \quad zh, \quad z^2. \quad (\text{S3.17})$$

The source forcing and its comparison with the physical field length are

$$A_1 = A_2 = B_{*1} = B_{*2} = 0, \quad (\text{S3.18})$$

$$2apq(bB_{*3} - A_3) - a(b-1)p^2(K_1 + \sigma) - (a-b)q^2(K_2 + \sigma) = 0. \quad (\text{S3.19})$$

$$\mathcal{P}(\mathcal{G}^{(0)} - \mathcal{G}^{(1)}) = O(|\varepsilon|(N + |\varepsilon|)). \quad (\text{S3.20})$$

For two screws, the matched cylindrical carrier has radial weights $(-2, -1, -1, -1, 1, 1)$. The pitch relation fixes the axial stretch as $\gamma = q/p$; the fixed-pressure equations are given in Appendix D.8.

S4 Common dilations and rotations

S4.1 Homothety and pressure matrices

The seventeen homothety equations and the eight angular and pressure observations are those of Appendix E.2. The projector pressure matrix and pure dilation determinant are

$$\mathbf{P}(\mathbf{p}^{[w_3^Z]}) = \begin{pmatrix} p_{23}^{[w_3^Z]} - p_{32}^{[w_3^Z]} & p_{13}^{[w_3^Z]} & -p_{12}^{[w_3^Z]} \\ -p_{23}^{[w_3^Z]} & p_{31}^{[w_3^Z]} - p_{13}^{[w_3^Z]} & p_{21}^{[w_3^Z]} \\ p_{32}^{[w_3^Z]} & -p_{31}^{[w_3^Z]} & p_{12}^{[w_3^Z]} - p_{21}^{[w_3^Z]} \end{pmatrix}. \quad (\text{S4.1})$$

$$A_0 B_1 - A_1 B_0 = \frac{2\kappa^2 \ell s(z^2 - z + 1)^2}{z - 1} \neq 0. \quad (\text{S4.2})$$

The eight-column angular minor away from principal directions is

$$\frac{(b-a)s(b-1)}{4a^2b^3} \{b(a-1)s^2t^2 + a(b-1)s^2 - (a-b)t^2\}. \quad (\text{S4.3})$$

The normal equations at a principal direction have the following coefficient outputs. The coordinates and row selections are those of Appendix E.3.

$$\begin{aligned} \mathbf{p}^{[\varepsilon w_3^Z]} &= p\mathbf{J}_{13} + q\mathbf{J}_{23}, & \tau_0^{[\varepsilon w_3^Z]} &= pq, \\ \tau_1^{[\varepsilon w_3^Z]} &= \dots = \tau_5^{[\varepsilon w_3^Z]} = \mathbf{u}^{[\varepsilon w_3^Z]} = 0, & (p, q) &\neq (0, 0). \end{aligned} \quad (\text{S4.4})$$

$$s^2 + \frac{b-a}{a(b-1)}t^2, \quad sh, \quad th, \quad h^2, \quad z \left(s + \frac{(a-b)q}{a(b-1)}t \right), \quad zh, \quad z^2. \quad (\text{S4.5})$$

$$\begin{array}{l|l} q \neq 0, T_* \neq 0 & -\frac{49152(b-a)(b-1)^2a^2q^5T_*}{b^4(a-1)^4} \\ q = 0 & \frac{41472(b-a)(b-1)^4a^4}{b^5(a-1)^5} \\ T_* = 0 & \frac{2048q^{10}(7q^2-4b+3)}{b(q^2+1)^3} \\ T_* = 0, 7q^2-4b+3=0 & -\frac{49152q^9}{(q^2+1)^2(3q^2-1)} \end{array} \quad (\text{S4.6})$$

$$(\mathcal{P}\mathcal{G})_{(47,21,49,13)} = \delta M_4(O, p, q)u + \delta^3 \mathbf{c}_4(p, q) + O(\delta^4 + \delta^2|u| + |u|^2). \quad (\text{S4.7})$$

$$u_2 = \left(0, 0, 0, -\frac{2r_0 p^2 q(1+r_0^2)}{pr_0 + q} \right). \quad (\text{S4.8})$$

$$\frac{8pq(pr_0 - q)(1 + r_0^2)^2}{pr_0 + q} = 0, \quad 8r_0^2 p^4(1 + r_0^2)^2 = 0 \quad \text{when } q = pr_0. \quad (\text{S4.9})$$

$$(\mathcal{P}\mathcal{G})_{(47,21,13,22,35)} = \delta M_5 V + \delta^3 \mathbf{c}_5 + \mathbf{r}_{\text{rem}}, \quad \det M_5|_{\Delta_{\text{res}}=0} = \frac{4096r_0^6 p^7 (r_0^2 + a)}{a}. \quad (\text{S4.10})$$

$$M_5(0, 0, 0, 0, C_*)^\top + \mathbf{c}_5 = 0 \quad (\text{S4.11})$$

$$N = O(|\varepsilon|). \quad (\text{S4.12})$$

S4.2 The cylindrical carrier matrices

Use the reference circle and the ordered twenty-three coordinates of Appendix E.4. The three ambient linearizations L_i are defined by differentiating (E.29); their first seventeen rows are

$$(L_i)_{jk} = \partial_k \mathcal{Q}_i^j + d_j v_i \delta_{jk}, \quad 1 \leq k \leq 17, \quad (L_i)_{j,17+k} = d_j (S^{[r]})_j \delta_{ik}. \quad (\text{S4.13})$$

The last six rows are the derivatives of $w_i w - \Gamma_i v$ and $-\Gamma_i a_0$, where $w = a_0 \times v$. For the nine free rows in (E.35), the kernel matrix N_{ker} satisfies

$$J_* N_{\text{ker}} = 0, \quad J_* L_i N_{\text{ker}} = 0 \quad (i = 1, 2, 3), \quad (DT) N_{\text{ker}} = 0. \quad (\text{S4.14})$$

$$R_0 = \mathcal{P}_f \left(\sum_i v_i L_i \right) N_{\text{ker}}, \quad A_0 = \mathcal{P}_f \left(\sum_i w_i L_i \right) N_{\text{ker}}. \quad (\text{S4.15})$$

Tables S5 and S6 give the full restricted matrices in this free basis. We parametrize the reference circle by

$$p = \frac{2t}{1+t^2}, \quad q = \frac{1-t^2}{1+t^2}.$$

Here t is only the rational parameter of this circle. The endpoint bases are those specified after (E.35).

Table S5: The radial matrix R_0 on $pq \neq 0$. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row	Nonzero column entries
1	$(1, -1) \quad (4, (4) \cdot (t-1) \cdot t \cdot (t+1) \cdot (t^2+1)^{-2}) \quad (6, (-1) \cdot (t-1) \cdot (t+1) \cdot (t^2+1)^{-1})$
2	$(2, (\frac{1}{4}) \cdot t^{-2} \cdot (b-1)^{-1} \cdot a^{-1} \cdot (t-1)^2 \cdot (t+1)^2 \cdot (t^2+1)^{-2} \cdot (-at^4 + bt^4 + 4abt^2 - 2at^2 - 2bt^2 - a + b))$ $(3, (-\frac{1}{2}) \cdot t^{-1} \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1) \cdot (t-1)^3 \cdot (t+1)^3 \cdot (t^2+1)^{-2}) \quad (5, (\frac{1}{2}) \cdot t^{-1} \cdot (t-1) \cdot (t+1))$ $(8, (\frac{1}{4}) \cdot t^{-2} \cdot (b-1)^{-1} \cdot a^{-1} \cdot (t-1) \cdot (t+1) \cdot (t^2+1)^{-3} \cdot (-at^8 + bt^8 + 8abt^6 - 4at^6 - 4bt^6 - 6at^4 + 6bt^4 + 8abt^2 - 4at^2 - 4bt^2 - a + b))$ $(9, (-\frac{1}{2}) \cdot t^{-1} \cdot (b-1)^{-1} \cdot a^{-1} \cdot (t^2+1)^{-3} \cdot (2abt^8 - at^8 - bt^8 - 4at^6 + 4bt^6 + 12abt^4 - 6at^4 - 6bt^4 - 4at^2 + 4bt^2 + 2ab - a - b))$
3	$(2, (-1) \cdot t^{-1} \cdot (t-1)^3 \cdot (t+1)^3 \cdot (t^2+1)^{-2}) \quad (3, (-1) \cdot (t^2+1)^{-2} \cdot (3t^4 - 2t^2 + 3))$ $(8, (-1) \cdot t^{-1} \cdot (t-1)^4 \cdot (t+1)^4 \cdot (t^2+1)^{-3}) \quad (9, (-2) \cdot (t-1)^3 \cdot (t+1)^3 \cdot (t^2+1)^{-3})$
4	$(4, (-1) \cdot (t^2+1)^{-2} \cdot (t^2 - 2t - 1) \cdot (t^2 + 2t - 1)) \quad (6, (-2) \cdot t \cdot (t^2+1)^{-1})$
5	$(2, (-1) \cdot t^{-1} \cdot (b-1)^{-1} \cdot a^{-1} \cdot (t-1) \cdot (t+1) \cdot b \cdot (a-1)) \quad (3, (-2) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1)) \quad (5, 1)$ $(8, (-1) \cdot t^{-1} \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1) \cdot (t-1)^2 \cdot (t+1)^2 \cdot (t^2+1)^{-1})$ $(9, (-2) \cdot (b-1)^{-1} \cdot a^{-1} \cdot (t-1) \cdot (t+1) \cdot b \cdot (a-1) \cdot (t^2+1)^{-1})$
6	$(1, (-1) \cdot (t-1) \cdot (t+1) \cdot (t^2+1)^{-1}) \quad (4, (-2) \cdot t \cdot (t^2+1)^{-3} \cdot (t^2 - 4t + 1) \cdot (t^2 + 4t + 1))$ $(6, (-1) \cdot (t^2+1)^{-2} \cdot (t^4 + 10t^2 + 1))$
7	$(1, (-1) \cdot (t-1)^2 \cdot (t+1)^2 \cdot (t^2+1)^{-2}) \quad (4, (-2) \cdot (t-1) \cdot t \cdot (t+1) \cdot (t^2+1)^{-2}) \quad (7, -1)$
8	$(2, (-\frac{1}{4}) \cdot t^{-2} \cdot (b-1)^{-1} \cdot a^{-1} \cdot (-a+b) \cdot (t-1)^3 \cdot (t+1)^3 \cdot (t^2+1)^{-1})$ $(3, (\frac{1}{2}) \cdot t^{-1} \cdot (b-1)^{-1} \cdot a^{-1} \cdot (t^2+1)^{-1} \cdot (abt^4 - bt^4 + 2abt^2 - 4at^2 + 2bt^2 + ab - b))$ $(5, (-\frac{1}{2}) \cdot t^{-1} \cdot (t^2+1))$ $(8, (-\frac{1}{4}) \cdot t^{-2} \cdot (b-1)^{-1} \cdot a^{-1} \cdot (t^2+1)^{-2} \cdot (-at^8 + bt^8 + 4abt^6 - 4bt^6 + 8abt^4 - 14at^4 + 6bt^4 + 4abt^2 - 4bt^2 - a + b))$ $(9, (\frac{1}{2}) \cdot t^{-1} \cdot (b-1)^{-1} \cdot a^{-1} \cdot (t-1) \cdot (t+1) \cdot (t^2+1)^{-2} \cdot (2abt^4 - at^4 - bt^4 + 4abt^2 - 6at^2 + 2bt^2 + 2ab - a - b))$
9	$(2, (-2) \cdot t \cdot (t^2+1)^{-1}) \quad (3, (t-1) \cdot (t+1) \cdot (t^2+1)^{-1}) \quad (8, (-2) \cdot (t-1) \cdot t \cdot (t+1) \cdot (t^2+1)^{-2})$ $(9, (t-1)^2 \cdot (t+1)^2 \cdot (t^2+1)^{-2})$

Table S6: The angular matrix A_0 on $pq \neq 0$. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row	Nonzero column entries
1	$(4, (-8) \cdot t^2 \cdot (t^2 + 1)^{-2})$ $(6, (2) \cdot t \cdot (t^2 + 1)^{-1})$ $(7, (-2) \cdot (t - 1)^{-1} \cdot (t + 1)^{-1} \cdot t)$
2	$(2, (-\frac{1}{2}) \cdot t^{-1} \cdot (b - 1)^{-1} \cdot a^{-1} \cdot (t - 1) \cdot (t + 1) \cdot (t^2 + 1)^{-2} \cdot (-at^4 + bt^4 + 4abt^2 - 2at^2 - 2bt^2 - a + b))$ $(3, (-1) \cdot (b - 1)^{-1} \cdot a^{-1} \cdot (t^2 + 1)^{-2} \cdot (-at^4 + bt^4 + 4abt^2 - 2at^2 - 2bt^2 - a + b))$ $(8, (-\frac{1}{2}) \cdot t^{-1} \cdot (b - 1)^{-1} \cdot a^{-1} \cdot (t - 1)^2 \cdot (t + 1)^2 \cdot (t^2 + 1)^{-3} \cdot (-at^4 + bt^4 + 4abt^2 - 2at^2 - 2bt^2 - a + b))$ $(9, (-1) \cdot (b - 1)^{-1} \cdot a^{-1} \cdot (t - 1) \cdot (t + 1) \cdot (t^2 + 1)^{-3} \cdot (-at^4 + bt^4 + 4abt^2 - 2at^2 - 2bt^2 - a + b))$
3	$(2, (2) \cdot (t - 1)^2 \cdot (t + 1)^2 \cdot (t^2 + 1)^{-2})$ $(3, (4) \cdot (t - 1) \cdot t \cdot (t + 1) \cdot (t^2 + 1)^{-2})$ $(8, (2) \cdot (t - 1)^3 \cdot (t + 1)^3 \cdot (t^2 + 1)^{-3})$ $(9, (4) \cdot t \cdot (t - 1)^2 \cdot (t + 1)^2 \cdot (t^2 + 1)^{-3})$
4	$(4, (4) \cdot (t - 1) \cdot t \cdot (t + 1) \cdot (t^2 + 1)^{-2})$ $(6, (-1) \cdot (t - 1) \cdot (t + 1) \cdot (t^2 + 1)^{-1})$ $(7, 1)$
6	$(4, (16) \cdot (t - 1) \cdot (t + 1) \cdot t^2 \cdot (t^2 + 1)^{-3})$ $(6, (-4) \cdot (t - 1) \cdot t \cdot (t + 1) \cdot (t^2 + 1)^{-2})$ $(7, (4) \cdot t \cdot (t^2 + 1)^{-1})$
7	$(1, (2) \cdot (t - 1) \cdot t \cdot (t + 1) \cdot (t^2 + 1)^{-2})$ $(4, (-1) \cdot (t - 1)^2 \cdot (t + 1)^2 \cdot (t^2 + 1)^{-2})$
8	$(2, (\frac{1}{2}) \cdot t^{-1} \cdot (b - 1)^{-1} \cdot a^{-1} \cdot (-a + b) \cdot (t - 1)^2 \cdot (t + 1)^2 \cdot (t^2 + 1)^{-1})$ $(3, (b - 1)^{-1} \cdot a^{-1} \cdot (-a + b) \cdot (t - 1) \cdot (t + 1) \cdot (t^2 + 1)^{-1})$ $(8, (\frac{1}{2}) \cdot t^{-1} \cdot (b - 1)^{-1} \cdot a^{-1} \cdot (-a + b) \cdot (t - 1)^3 \cdot (t + 1)^3 \cdot (t^2 + 1)^{-2})$ $(9, (b - 1)^{-1} \cdot a^{-1} \cdot (-a + b) \cdot (t - 1)^2 \cdot (t + 1)^2 \cdot (t^2 + 1)^{-2})$
9	$(2, (-1) \cdot (t - 1) \cdot (t + 1) \cdot (t^2 + 1)^{-1})$ $(3, (-2) \cdot t \cdot (t^2 + 1)^{-1})$ $(8, (-1) \cdot (t - 1)^2 \cdot (t + 1)^2 \cdot (t^2 + 1)^{-2})$ $(9, (-2) \cdot (t - 1) \cdot t \cdot (t + 1) \cdot (t^2 + 1)^{-2})$

At $(p, q) = (0, 1)$, the endpoint matrices are

Table S7: The radial matrix at $(p, q) = (0, 1)$. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row	Nonzero column entries
1	$(8, -1)$
2	$(2, -1)$ $(6, 1)$
3	$(3, (a - b)^{-1} \cdot (2ab - 3a + b))$ $(5, (-2) \cdot (a - b)^{-1} \cdot (b - 1) \cdot a)$ $(9, (-4) \cdot (a - b)^{-1} \cdot (b - 1) \cdot a)$
4	$(4, -1)$
5	$(3, (2) \cdot (a - b)^{-1} \cdot b \cdot (a - 1))$ $(5, (-1) \cdot (a - b)^{-1} \cdot (2ab - a - b))$ $(9, (-4) \cdot (a - b)^{-1} \cdot b \cdot (a - 1))$
6	$(2, 1)$ $(6, -1)$
7	$(2, -1)$ $(7, -1)$
8	$(1, -1)$
9	$(3, -1)$ $(9, 1)$

Table S8: The angular matrix at $(p, q) = (0, 1)$. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row	Nonzero column entries
1	$(3, -1)$ $(5, 1)$ $(9, 2)$
4	$(6, 1)$ $(7, 1)$
7	$(4, -1)$
8	$(3, -1)$ $(5, 1)$ $(9, 2)$

At $(p, q) = (1, 0)$, they are

Table S9: The radial matrix at $(p, q) = (1, 0)$. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row	Nonzero column entries
1	$(1, -1)$
2	$(9, -1)$
3	$(3, -1)$
4	$(4, 1) \quad (6, -1)$
5	$(3, (-2) \cdot (b-1)^{-1} \cdot a^{-1} \cdot b \cdot (a-1)) \quad (5, 1)$
6	$(4, 3) \quad (6, -1 \cdot 3)$
7	$(4, 1) \quad (7, -1)$
8	$(3, 1) \quad (5, -1) \quad (8, -1)$
9	$(2, -1)$

Table S10: The angular matrix at $(p, q) = (1, 0)$. An entry (j, c) in row i means that the (i, j) coefficient is c . Unlisted entries are zero.

Row	Nonzero column entries
1	$(4, -1 \cdot 2) \quad (6, 1) \quad (7, 1)$
2	$(3, -1)$
7	$(1, -1)$
9	$(3, -1)$

For $pq \neq 0$, the kernel basis and the entries of R_0, A_0 do not require division by $b - ap^2 - q^2$. They can therefore be specialized at the material resonance $b = ap^2 + q^2$ in the same nine free coordinates.

The three determinants below correspond to the 14×14 submatrices of the constraint Jacobian J_* selected in (E.33): the generic chart $pq \neq 0$, the endpoint $(p, q) = (0, 1)$, and the endpoint $(p, q) = (1, 0)$, respectively.

$$\frac{2p^4q^2(a-1)^2}{a(b-1)^2}, \quad \frac{2(a-1)^2}{a^2(b-1)^2}, \quad \frac{2(b-a)(a-1)^2}{a(b-1)^3}. \quad (\text{S4.16})$$

Each determinant is nonzero on its chart for positive, pairwise distinct principal values. Together with (S4.14), these minors give $\text{rank } J_* = 14$ and $\dim \ker J_* = 23 - 14 = 9$. The restrictions of the radial and angular linearizations to this kernel satisfy

$$\begin{aligned} \det(xI - R_0) &= x(x+2)(x-1)^2(x+1)^5, \\ R_0(R_0 - I)(R_0 + 2I)(R_0 + I)^2 &= 0, \\ [R_0, A_0] &= 0, \\ A_0 \mathcal{P}_{\text{Hermite}}(R_0) &= 0, \quad \mathcal{P}_{\text{Hermite}}(R_0)^2 = \mathcal{P}_{\text{Hermite}}(R_0), \\ \text{tr } \mathcal{P}_{\text{Hermite}}(R_0) &= 3, \end{aligned} \quad (\text{S4.17})$$

where the Hermite interpolation polynomial is

$$\mathcal{P}_{\text{Hermite}}(x) = 1 + \frac{5}{3}x - \frac{x^2}{12} - \frac{7}{6}x^3 - \frac{5}{12}x^4. \quad (\text{S4.18})$$

The interpolation conditions $\mathcal{P}_{\text{Hermite}}(0) = \mathcal{P}_{\text{Hermite}}(1) = 1$ and $\mathcal{P}_{\text{Hermite}}(-2) = \mathcal{P}_{\text{Hermite}}(-1) = \mathcal{P}'_{\text{Hermite}}(-1) = 0$, together with the annihilating polynomial in (S4.17), give

$$\text{im } \mathcal{P}_{\text{Hermite}}(R_0) = \ker R_0 \oplus \ker(R_0 - I).$$

The image consists of the one dimensional centre, tangent to the equilibrium circle, and the two dimensional eigenspace of eigenvalue 1. The double zero at -1 annihilates its generalized eigenspace. The identity $A_0 \mathcal{P}_{\text{Hermite}}(R_0) = 0$ gives vanishing of the angular linearization on the image of the projection.

The additional coordinate ϵ has radial eigenvalue 1 and zero angular linearization. On the ten dimensional tangent space of the containing manifold in Lemma 7.6, the radial linearization therefore has characteristic polynomial

$$\det(xI - D\mathcal{R}) = x(x+2)(x-1)^3(x+1)^5. \quad (\text{S4.19})$$

Its centre, positive eigenspace and negative generalized eigenspace have dimensions one, three and six, respectively, as used in Lemma 7.7.

S5 The relative projective boundary

S5.1 Ratio coordinates and the valuation regions

Use κ_0, q, η, ζ from (8.1). In this section $k = a\kappa_0$ and $m = 1+k$ are the spectral constants of Appendix C.1. The affine retained coordinates R_i are specified by

$$\begin{aligned}
\hat{\tau}_1 &= -\hat{p}_{12}(\hat{p}_{13} + \hat{p}_{31}) + \left(1 - \frac{\eta\zeta}{\eta - \zeta}\right)\hat{p}_{23} \\
&\quad - \frac{\eta\zeta}{\eta - \zeta}\hat{p}_{32} - \frac{\eta\zeta}{2(\eta - \zeta)}R_1, \\
\hat{\tau}_2 &= -\frac{2\eta}{\kappa_0(1 + \eta)}\hat{p}_{12} + \frac{2\eta^2}{\kappa_0(1 + \eta)}\hat{p}_{21} - \frac{\eta}{\kappa_0(1 + \eta)}R_3, \\
\hat{\tau}_3 &= \hat{p}_{21}(\hat{p}_{23} + \hat{p}_{32}) + \frac{\zeta - \eta + \eta\zeta}{\kappa_0(1 - \zeta)}\hat{p}_{13} \\
&\quad + \frac{\zeta}{\kappa_0(1 - \zeta)}\hat{p}_{31} + \frac{\zeta}{2\kappa_0(1 - \zeta)}R_2.
\end{aligned} \tag{S5.1}$$

For a formal approach, put $n = \text{ord } \tau$ and $s_0 = \min_j \text{ord } v_j$, where $v_j = \tau^2 R_j$. The valuation regions and boundary conclusions are

Table S11: Valuation regions of the weighted boundary.

Region	Valuation condition
Weighted boundary	$s_0 = 0$
Ordinary	$s_0 \geq n$
Subcritical	$n/2 < s_0 < n$
Critical	$s_0 = n/2$
Supercritical	$0 < s_0 < n/2$

Table S12: Boundary conclusions on the primary and critical strata.

Chart	Conclusion	Stratum
Original weighted boundary $\mathbf{v} \neq 0$	$\tau \in I$ locally	All one-, two- and three-coordinate $\mathbf{v}$ supports.
Original weighted boundary nonzero $\bar{p}$ cone	$\tau \in I$ locally	All nonzero amplitudes, including vanishing sums.
Ordinary exceptional family	Boundary contained in the ordinary boundary family	All points on a dominant component.
Critical single R	$\tau \in \sqrt{I}$ locally	Eight roots of the auxiliary equations; all fifteen $\bar{p}$ supports.
Critical two R	$\tau \in \sqrt{I}$ locally	Rank three, rank two and rank one coefficient strata.
Critical three R; at least two amplitude pairs	No invariant boundary component dominating the ratio plane	First integrals, amplitude coefficient equations and their rank strata.
Critical three R; one amplitude pair	$\tau \in \sqrt{I}$ locally	Paired normal equations and the hypersurface kernel quadratic.

Table S13: Subcritical supports with one nonzero component.

Support	Result	Identity or remaining charts
1,4	$\tau^4 \in I$	Proposition C.9.
2,8	$\tau \in I$	Nine generators; selected determinants $-25qk^3/4032$ and $-544qk^3/10935$.
3,6,9,12	$1 \in I$	Units $8x^2y^2$, $16y^2z^2$, $16x^2w^2$, $8z^2w^2$.
5	$\tau \in I$	Eight roots; the unit factors $1 + 2z_0$ and $9 + 20z_0$ are complementary.
7,13	$\tau \in \sqrt{I}$	Both the conic and its complement.
10	$\tau \in \sqrt{I}$	$w^2 \neq \pm k$ and each hypersurface $w^2 = \pm k$.
11,14	$1 \in I$	The odd relation has unit coefficient of xyw or yzw .
15	$\tau \in \sqrt{I}$	The conic and its resonance hypersurfaces.

The transverse coordinates and amplitude symbols in these tables are the local coordinates of Appendix C.8. The quotient determinants on a pure-axis chart, with $\bar{R}_3 = 1$ and $\varrho = \bar{R}_2$, are

$$D_9 = 29\varrho^2 + 49q, \quad D_{15} = q - 19\varrho^2, \quad -\frac{21\varrho q}{2D_9}, \quad \frac{\varrho(\varrho^2 + 16q)}{4D_{15}}.$$

At $D_9 = 0$, the complementary values are $D_{15} = 960q/29$ and $\varrho^2 + 16q = 415q/29$. At the inactive paired resonance, the residual square coefficient is

$$-\frac{16q}{\kappa_0} \left((\kappa_0 + \frac{1}{2})^2 + 3 \right).$$

For the subcritical three-coordinate paired chart, the ideal is

$$J = (sW_\ell, s\kappa, \delta W_\ell, \delta W_\varrho, \delta\kappa, s^2), \quad \det C_J = -\frac{5q\varrho}{1152\kappa_0^5}.$$

The exact coordinate graph and the relations defining this coefficient matrix are in Appendix C.8.

S5.2 Ordinary normal ideals

Use the ordinary family (8.3) and the normal coordinates

$$\bar{p} = (1, n_0, n_1, n_2, n_3, \ell), \quad \mathbf{R} = (\lambda_{\text{in}}(v)\ell + n_\perp, u, v), \quad \mathbf{n} = (n_0, n_1, n_2, n_3, n_\perp, \tau). \quad (\text{S5.2})$$

Put $t = n_3$, $w = v + 2$, and $y = (n_0 + n_3, n_1, n_2, \tau)$. The ordinary ideal and the ideal at $\mu = m\ell^2 - k = 0$ are

$$J_9 = (y, un_\perp, wt, n_\perp^2, n_\perp t, t^2). \quad (\text{S5.3})$$

$$J_{10} = (y, un_\perp, wt, n_\perp^2, n_\perp t, \mu t^2, t^3). \quad (\text{S5.4})$$

At $\nu = m\ell^2 + k = 0$, use $y_0 = n_0 + n_3$, $y_1 = n_1 + mt^2$, $y_2 = n_2 + m\ell t^2$, $y_3 = \tau$. The ideal and the shifted quadratic coefficient are

$$J_{11} = (y_0, y_1, y_2, y_3, un_\perp, wt, n_\perp^2, n_\perp t, \nu t^2, ut^2, t^3). \quad (\text{S5.5})$$

$$g_i(\ell, u) = q_{i,tt} - mM_{i,n_1} - m\ell M_{i,n_2}. \quad (\text{S5.6})$$

The normal coefficient matrices have unit maximal minors in the local coefficient rings specified in Appendix C.11. Their generators include τ . At the material value $k + q = 0$, the replacement base point $(\eta, \zeta) = (1, -2)$ gives coefficient determinants

$$\ell^5 \frac{64512}{25} \frac{q^4}{\kappa_0^2}, \quad -\frac{155648q^2}{135\kappa_0^3}.$$

S5.3 Algebraicity on the ordinary family

The induced differential equations and first integral are

$$D_\zeta v = \frac{(v - 2\eta)(v + 2)}{\Theta}, \quad D_\zeta \ell = 0, \quad D_\zeta(u/\beta) = \frac{v + 1 - \eta}{\Theta}(u/\beta). \quad (\text{S5.7})$$

$$\mathcal{J}_{\text{in}} = \frac{u^2}{\beta^2(v + 2)(v - 2\eta)}, \quad D_\eta \mathcal{J}_{\text{in}} = D_\zeta \mathcal{J}_{\text{in}} = 0.$$

For the normal minors at $u = 0$, the two leading factors have the identity

$$5(4\kappa_0\ell^2 + 5q) - 4(5\kappa_0\ell^2 + 4q) = 9q.$$

At $v = 2\eta$, the highest ζ coefficient of the relevant normal factor is $-16mq(\eta + 1)^5\ell^2$. At the positive amplitude hypersurface, with $N = a\kappa_0^2$,

$$f_1 = 21622N - 989(k + q), \quad f_2 = 14006N + 377(k + q), \quad 377f_1 + 989f_2 = 22003428N.$$

For fixed η and nonconstant v , the incoming connection gives

$$d\left(\frac{\zeta}{v + 2}\right) = -\frac{(\eta + 1)v}{(v - 2\eta)(v + 2)^2} dv. \quad (\text{S5.8})$$

On the branch $v = 2\eta$, where $u \neq 0$,

$$D_\zeta \log(u/\beta) = -\frac{1}{2\eta}. \quad (\text{S5.9})$$

S5.4 The additional ordinary hypersurface

The local coordinates of Appendix C.12 are

$$u = 0, \quad v = -2, \quad \ell_0^2 = d = -\frac{(a - b)\widehat{h}_3^2}{b(a - 1)} \neq 0. \quad (\text{S5.10})$$

$$\begin{aligned} \bar{p}_{13} &= \rho_0 - k_\perp + \kappa\ell\tau, & \bar{p}_{21} &= \rho_1, \\ \bar{p}_{23} &= \rho_2, & \bar{p}_{31} &= k_\perp, \\ n_\perp &= \mathbf{R}_1 - \lambda_{\text{in}}(v)\ell. \end{aligned} \quad (\text{S5.11})$$

Put $w = v + 2$ and $g = \ell^2 - d$. The linear normal coefficients satisfy

$$\begin{aligned} M_{j5} &= uA_j, \\ M_{j4} - M_{j1} &= wB_j, \\ M_{j6} + \kappa\ell M_{j1} &= uC_j + wD_j + gE_j, \end{aligned} \quad (\text{S5.12})$$

$$J = (un_{\perp}, wk_{\perp}, u\tau, w\tau, g\tau). \quad (\text{S5.13})$$

At $u = 0, v = -2$, set $A = M_{(2,18),(2,3)}$, $d_A = \det A$, $c = M_{6,1}$, $\Delta = d_A c$ and $(C_{j1}, C_{j2}) = (M_{j2}, M_{j3}) \text{adj } A$. The cokernel relation is

$$\Delta M_j - cC_{j1}M_2 - cC_{j2}M_{18} - d_A M_{j1}M_6 \equiv 0 \pmod{\ell^2 - d}. \quad (\text{S5.14})$$

$$Q = (n_{\perp}^2, k_{\perp}n_{\perp}, n_{\perp}\tau, P_{\text{hom}}(k_{\perp}, \tau)),$$

$$P_{\text{hom}}(k_{\perp}, \tau) = \tau^2 P\left(\frac{k_{\perp}}{\ell_0 \tau}\right). \quad (\text{S5.15})$$

$$\begin{aligned} J_i + \mathcal{E}_i^{\text{rem}} &= 0 & (1 \leq i \leq 5), \\ Q_j + \mathcal{E}_{5+j}^{\text{rem}} &= 0 & (1 \leq j \leq 4), \\ \mathcal{E}_i^{\text{rem}} &\in (W)(n)^2 + (n)^3. \end{aligned} \quad (\text{S5.16})$$

$$\begin{aligned} k_{\perp} &= \ell_0 y_0 \tau + O(\tau^2), & n_{\perp} &= O(\tau^2), \\ (\ell - \ell_0, u, v + 2) &= O(\tau^2), & P(y_0) &= 0. \end{aligned} \quad (\text{S5.17})$$

$$O_{jc} = R_{8, \text{L}_2} R_{j,c} - R_{j, \text{L}_2} R_{8,c}. \quad (\text{S5.18})$$

$$\mathcal{N} = f_0^2 - P_1 f_0 f_1 + P_0 f_1^2. \quad (\text{S5.19})$$

$$\mathcal{N}(T, \pm T) = N_{\pm}(a, b) \epsilon^{-30} + O(\epsilon^{-29}). \quad (\text{S5.20})$$

Here P is the monic quadratic of (C.20); the normal cubic coefficient matrix O has the row order specified in (C.23). Its quadratic norm is

$$\text{Nm}_P(f_0 + f_1 y) = f_0^2 - P_1 f_0 f_1 + P_0 f_1^2, \quad P(y) = y^2 + P_1 y + P_0.$$

The two amplitude hypersurfaces have the determinant outputs

$$\begin{aligned} N_- &= \frac{9216}{49} \frac{a^2(b-1)^4(b-a)^{14}}{(a-1)^{18}b^{17}} C_- \\ &\quad \times \frac{(7ab - a - 6b)^2(11ab - 3a - 8b)^4}{(5ab - a - 4b)^6(2ab - a - b)^2}, \end{aligned} \quad (\text{S5.21})$$

$$\begin{aligned} N_+ &= \frac{9216}{49} \frac{a^2(b-1)^4(b-a)^{12}}{(a-1)^{18}b^{17}} C_+ \\ &\quad \times \frac{(5ab + a - 6b)^2(5ab + 3a - 8b)^4}{(3ab + a - 4b)^6}, \end{aligned} \quad (\text{S5.22})$$

S6 Fibres over ratio curves

S6.1 The characteristic affine systems

The curve slope is $\varsigma = d\zeta/d\eta$ and $L = \eta\varsigma - \zeta$. The characteristic factors are

$$\varsigma - 1, \quad L + 1, \quad L + \varsigma, \quad \varsigma, \quad L,$$

giving respectively $\zeta = \eta + c$, $\zeta = 1 + c\eta$, $\zeta = c(\eta + 1)$, $\zeta = c$, $\zeta = c\eta$. For the first pencil, put $T = \zeta - \eta$ and $s = \hat{p}_{23}$; thus $\hat{h}_3 = q\hat{h}_2 + qT/\kappa_0$. On $T(T - 1)s \neq 0$, the three cubic relations and $E_2L_p = E_3L_p = E_1B_p = 0$ have six-column determinant

$$\frac{4q^3T^2(1 + \kappa_0\hat{h}_2)(1 - \kappa_0\hat{h}_3/q)(1 - T)^2s}{\kappa_0^2\hat{h}_2}.$$

The two equations $E_2^2L_p = E_3^2L_p = 0$ have u_0, u_1 coefficient determinant $-2q^2(1 - T)^2$. Here $L_p = p_{33} - qp_{22} - (qT/\kappa_0)p_{11}$ and $B_p = p_{33}p_{23} + qp_{22}p_{32}$. The affine 2×2 matrix and its Cramer numerators, including its rank strata, are those of Appendix C.14. On the common line $\zeta = \eta + 1$, the reconstructed fibre has two polynomials with coprime quintic leading forms; the two conics in these forms are nonproportional, and one has determinant -1 (Appendix C.15).

S6.2 Directions, amplitudes and exceptional factors

The opposite-pair direction coordinates are

$$\begin{aligned} \hat{p}_{12} &= w_3, & \hat{p}_{13} &= w_2, & \hat{p}_{23} &= w_1, \\ \hat{p}_{21} &= \eta w_3 + \frac{((\eta + 1)\varsigma - \zeta)m_3}{2\varsigma}, \\ \hat{p}_{31} &= -\varsigma w_2 - \frac{(1 + \eta\varsigma - \zeta)m_2}{2\eta}, \\ \hat{p}_{32} &= -\frac{\varsigma w_1}{\eta} - \frac{(1 - \varsigma)m_1}{2\eta}, \\ R_1 &= \frac{(\varsigma - \eta\varsigma)m_1}{\eta\varsigma}, & R_2 &= \frac{\varsigma m_2}{\varsigma}, & R_3 &= -\frac{m_3}{\eta}. \end{aligned} \tag{S6.1}$$

On nonzero pair support, the mixed equations give projective maps between the directions. A non-identity composition has at most two fixed points. For a nonvertical projected factor $f(\eta, \zeta)$, the contact slope is $-f_\eta/f_\zeta$. The exceptional coefficient factors and their contact outputs are

Table S14: Coefficients on the exceptional direction hypersurfaces.

Hypersurface	Nonzero coefficient or consequence
$L = 0$	$(A_{\text{amp}})_{31} = 2(\eta - \zeta)$.
$\varsigma = 0$	$(A_{\text{amp}})_{12} = 2(\zeta - 1)$.
$L - \varsigma + 1 = 0$	The mixed coefficient is $-4\eta^2((\eta - 1)\varsigma + 1)^2/[\varsigma(\varsigma - 1)]$; here $(\eta - 1)\varsigma + 1 = \zeta$.
$L + \varsigma + 1 = 0$	The mixed coefficient is $4\eta(\eta + \zeta)^2/(\zeta - 1)$. Its vanishing gives $\zeta = -\eta$, $\varsigma = -1$, and $(A_{\text{amp}})_{31} = 4\eta$.
$L + \varsigma - 1 = 0$	The mixed coefficient is $-4\eta^4(\zeta + 1)^2/(\eta - \zeta)$. Its vanishing gives $\zeta = -1$, $\varsigma = 0$, and $(A_{\text{amp}})_{12} = -4$.

For two nonzero amplitudes x, y , the quartic combinations give $\pm 4x^2y^2$. With three amplitudes, the exceptional kernel lines give $xy(x + y)^2$ and $xy(x^2 + y^2)$. The distinguished full-support direction is

$$t_1 = \frac{\varsigma}{2\varsigma}, \quad t_2 = t_3 = -\frac{L}{2\eta\varsigma}.$$

Its conics in the squared amplitudes have the coefficient minor

$$-2 \left(\frac{\eta\varsigma}{\varsigma} - 1 \right)^2.$$

Their determinants and the two degenerate values are those of Appendix C.16.

S6.3 The quadratic direction condition

The quadratic factor is

$$\mathcal{P}_0 = \eta\zeta(\eta^2 - 1)\zeta^2 - 2(\eta^2 - \zeta^2)\zeta + \eta\zeta(1 - \zeta^2). \quad (\text{S6.2})$$

For a nonvertical irreducible curve $f(\eta, \zeta) = 0$, contact with $\mathcal{P}_0 = 0$ requires

$$\eta\zeta(\eta^2 - 1)f_\eta^2 - 2(\zeta^2 - \eta^2)f_\eta f_\zeta - \eta\zeta(\zeta^2 - 1)f_\zeta^2 \equiv 0 \pmod{f}.$$

The leading and separation norm polynomials have total degrees 19 and 20, both of bidegree (12, 14); the rank-drop norm polynomial has total degree nine. Their determinant and norm definitions are given in Appendix C.17. For the distinguished direction, $\varsigma' = \varsigma L/(\eta\zeta)$, and its contact condition is

$$\eta\zeta((\mathcal{P}_0)_\eta + \varsigma(\mathcal{P}_0)_\zeta) + \varsigma L(\mathcal{P}_0)_\varsigma = 0.$$

Writing the left side as $F_{\mathcal{P}_0}$, the resultant output is

$$\begin{aligned} \text{Res}_\varsigma(\mathcal{P}_0, F_{\mathcal{P}_0}) = & -12(\zeta - \eta)(\zeta - 1)(\zeta + 1)(\eta - 1)(\eta + 1) \\ & \cdot (\eta + \zeta)\zeta^5\eta^5(\eta^2 + \zeta^2 + 1). \end{aligned}$$

S7 Moment matrices, norms and slope elimination

S7.1 The two matrix presentations

Set $K = 1 + \eta - \zeta$ and $c_m = L/(2\eta\zeta)$. The affine direction change and its quadratic coordinate are

$$t_1 = \frac{V - \eta w_*}{L} + \frac{\zeta}{2\zeta}, \quad t_2 = \frac{U + w_*}{\zeta} - \frac{L}{2\eta\zeta}, \quad t_3 = w_* - \frac{L}{2\eta\zeta}. \quad (\text{S7.1})$$

$$\begin{aligned} Q_*(w_*) &= 2\zeta w_* \{2\eta\zeta K w_* - (L^2 + \eta\zeta^2 - \zeta)\}, \\ Q(w) &= (2\eta\zeta w + L) \{2\zeta K w - (\zeta - 1)(L + \zeta)\}. \end{aligned} \quad (\text{S7.2})$$

The full matrices are determined by the polynomial matrix $\mathbf{A}$ in Section S8 and its five row factors. They satisfy

$$M(w) = M_*(w + c_m)_{1:4,1:4}, \quad Q(w) = Q_*(w + c_m).$$

The coefficient extraction and reconstruction identities are

$$\text{coeff}_\xi(F) = \left(\partial_U \partial_V F|_{U=V=0}, \partial_U F|_{U=V=0}, \partial_V F|_{U=V=0}, \frac{F(0,0,w)}{Q_*(w)} \right). \quad (\text{S7.3})$$

$$F_i^{\text{mom}} = \text{coeff}_\xi(F_i^{\text{mom}}) (UV, U, V, Q_*)^\top, \quad (M_*)_{i,:} = \text{coeff}_\xi(F_i^{\text{mom}}), \quad 1 \leq i \leq 5. \quad (\text{S7.4})$$

S7.2 The determinant and quadratic eliminants

For a rational polynomial, pp denotes its primitive polynomial numerator. Let ng_{g_*} remove only the irreducible factors of

$$g_* = \eta\zeta(\eta - \zeta)(\zeta - 1)(\eta + 1)(1 + \eta - \zeta)(\zeta - 1)(L + 1)(L + \zeta)\zeta L$$

and normalize the result as in Appendix C.20. If $\mathbf{C}$ is the 5×3 coefficient matrix of the five maximal minors of M_* , then

$$\mathcal{P}_1 = \text{ng}_{g_*} \gcd_{|I|=3} \det C_I, \quad (\deg \mathcal{P}_1, \deg_\zeta \mathcal{P}_1) = (38, 11).$$

The residual minors r_i have an identity

$$d(\eta)\zeta^2(\zeta - 1)^2 = \sum_{i=0}^5 A_i r_i, \quad d(\eta) \neq 0.$$

For the second eliminant, put $D = \text{ppdet } M$ and let a_j be column j of $\text{adj } M$. The quadratic kernel equation gives

$$f_j = \text{pp}\{Q(w)(a_j)_2(a_j)_3 - (a_j)_1(a_j)_4\}, \quad R_j = \text{Res}_w(D, f_j), \quad j = 1, 2.$$

The common eliminant is

$$\mathcal{P}_2 = \text{ng}_{g_*} (\gcd(R_1, R_2)/\mathcal{P}_0),$$

with

$$(\deg \mathcal{P}_2, \deg_\zeta \mathcal{P}_2) = (48, 14).$$

The full coefficient representation of these outputs is specified by the matrices in Section S8, the primitive minor definitions and the fixed $(2, 4)$ resultants in Appendix C.20.

S7.3 Schur reduction and the finite flat norm

The determinant-one column transformation and transformed quadric are

$$Q(w) = a_Q w^2 + b_Q w + c_Q, \quad a_Q = 4\eta\zeta^2(1 + \eta - \zeta). \quad (\text{S7.5})$$

$$P_* = \begin{pmatrix} 1 & \eta(w + b_Q/a_Q) & -w & \eta/a_Q \\ 0 & 1 & 0 & 0 \\ 0 & -\eta & 1 & 0 \\ 0 & -a_Q w & 0 & 1 \end{pmatrix}. \quad (\text{S7.6})$$

With $x = P_* y$, the quadratic form becomes

$$Qx_2x_3 - x_1x_4 = w(a_Q y_1y_2 + b_Q y_2y_3 + y_3y_4) + c_Q y_2(y_3 - \eta y_2) - y_1y_4 - \frac{\eta}{a_Q} y_4(b_Q y_2 + y_4). \quad (\text{S7.7})$$

On a constant-column minor chart, write

$$\tilde{\mathbf{N}} = \begin{pmatrix} A_0 & B_0(w) \\ C_0 & D_0(w) \end{pmatrix}, \quad \Delta = \det A_0, \quad \mathbf{R}_{\text{Schur}} = \Delta D_0 - C_0 \text{adj}(A_0) B_0,$$

$$\mathbf{L}_{\text{Schur}} = \begin{pmatrix} -\text{adj}(A_0) B_0 \\ \Delta \text{Id}_2 \end{pmatrix}.$$

Write $E_0 = \text{adj } A_0$ and $J_0 = \text{adj } \mathbf{R}_{\text{Schur}}$. The block identities are

$$\det \mathbf{R}_{\text{Schur}} = \Delta \det \tilde{\mathbf{N}}, \quad \tilde{\mathbf{N}} \mathbf{L}_{\text{Schur}} = \begin{pmatrix} 0 \\ \mathbf{R}_{\text{Schur}} \end{pmatrix}, \quad (\text{S7.8})$$

$$\mathbf{L}_{\text{Schur}} \text{adj}(\mathbf{R}_{\text{Schur}}) = \Delta \text{adj}(\tilde{\mathbf{N}})_{\{3,4\}}. \quad (\text{S7.9})$$

$$\Delta^2 \text{adj}(\tilde{\mathbf{N}}) = \begin{pmatrix} (\det \mathbf{R}_{\text{Schur}}) E_0 + E_0 B_0 J_0 C_0 E_0 & -\Delta E_0 B_0 J_0 \\ -\Delta J_0 C_0 E_0 & \Delta^2 J_0 \end{pmatrix}. \quad (\text{S7.10})$$

For $d(w) = d_2 w^2 + d_1 w + d_0$ and $f(w) = \sum_{i=0}^4 f_i w^i$, put

$$\begin{aligned} a_f &= d_2^3 f_1 - d_1 d_2^2 f_2 + d_2(d_1^2 - d_0 d_2) f_3 + (-d_1^3 + 2d_0 d_1 d_2) f_4, \\ b_f &= d_2^3 f_0 - d_0 d_2^2 f_2 + d_0 d_1 d_2 f_3 + (d_0^2 d_2 - d_0 d_1^2) f_4. \end{aligned}$$

Thus $d_2^3 f \equiv a_f w + b_f \pmod{d}$, and the norm output is

$$\det \begin{pmatrix} b_f & -d_0 a_f \\ a_f & d_2 b_f - d_1 a_f \end{pmatrix} = d_2^3 \text{Res}_{2,4}(d, f). \quad (\text{S7.11})$$

S7.4 Contact orders and elimination of the slope

The differentiated curve conditions are

$$(\mathcal{P}_j)_\eta + \varsigma(\mathcal{P}_j)_\zeta + (\mathcal{P}_j)_{\varsigma'} = 0, \quad j = 1, 2. \quad (\text{S7.12})$$

Eliminating ς' gives

$$C_* = (\mathcal{P}_1)_\varsigma((\mathcal{P}_2)_\eta + \varsigma(\mathcal{P}_2)_\zeta) - (\mathcal{P}_2)_\varsigma((\mathcal{P}_1)_\eta + \varsigma(\mathcal{P}_1)_\zeta).$$

The fixed Sylvester degrees of $(\mathcal{P}_1, \mathcal{P}_2, C_*)$ are $(11, 14, 25)$. Put $R = \text{Res}_\zeta(\mathcal{P}_1, \mathcal{P}_2)$, $T = \text{Res}_\zeta(\mathcal{P}_1, C_*)$. The common divisor is

$$D_0 = \zeta^{12}(\zeta + 1)^{20}(\zeta + \eta)^{20}(\zeta - \eta)^{25}(\zeta - 1)^{25} \\ \cdot (\zeta - \eta - 1)^{26}((\eta + 1)\zeta - \eta)^6.$$

The local orders are

Table S15: Intersection orders on the exceptional lines.

Line in ζ	Order of $\mathcal{P}_1$	Order of $\mathcal{P}_2$	Lower order of C_*	Common resultant order
$\zeta = 0$	4	3	6	12
$\zeta = -1$	4	5	7	20
$\zeta = -\eta$	4	5	7	20
$\zeta = \eta$	5	5	8	25
$\zeta = 1$	5	5	8	25
$\zeta = \eta + 1$	4	5	8	26
$\zeta = \eta/(\eta + 1)$	2	3	4	6

The infinity calculation has coarse weight 487 and a rank 23 unit block. Its 2×2 Schur complement has determinant of order at least six, giving $\deg_\zeta R \leq 481$. At $p_* = 32003$ and $\eta = 3$, the exact finite-field outputs are

$$\deg_\zeta \overline{R} = 481, \quad \deg_\zeta \overline{D}_0 = 134, \quad \deg_\zeta \overline{R/D_0} = 347,$$

$$\gcd(\overline{R/D_0}, \overline{T/D_0}) = 1.$$

The characteristic-zero result is

$$\gcd(R/D_0, T/D_0) = 1 \quad \text{in } \mathbb{Q}(\eta)[\zeta].$$

The seven persistent ratio factors are

$$\zeta = 0, \quad \zeta = -1, \quad \zeta = -\eta, \quad \zeta = \eta, \quad \zeta = 1, \quad \zeta = \eta + 1, \quad (\eta + 1)\zeta = \eta. \quad (\text{S7.13})$$

S8 Polynomial coefficients of the moment matrix

The ratios η, ζ , slope ς and direction parameter w are independent indeterminates in this coefficient list. They are specialized to a solution curve after the matrix has been formed. The definitions of L, c_m are those of Section S7. The five row factors are

$$d_1^{\text{row}} = \frac{421654016}{\varsigma \zeta^2 L}, \quad d_2^{\text{row}} = d_3^{\text{row}} = d_4^{\text{row}} = \frac{6588344}{\varsigma \zeta^2 L},$$

$$d_5^{\text{row}} = \frac{1}{2\eta \varsigma \zeta^2 (\eta + 1)(\eta - \zeta)(\zeta - 1)L}.$$

The coefficient matrices satisfy

$$M_*(w) = \text{diag}(d_1^{\text{row}}, \dots, d_5^{\text{row}}) \mathbf{A}(w),$$

$$M(w) = \text{diag}(d_1^{\text{row}}, \dots, d_4^{\text{row}}) \mathbf{A}(w + c_m)_{1:4, 1:4}.$$

The following expressions define $\mathbf{A} \in \text{Mat}_{5 \times 4}(\mathbb{Q}[\eta, \zeta, \varsigma, w])$. The expressions γ_i are defined successively in terms of $\eta, \zeta, \varsigma, w$. They determine the twenty matrix entries.

S8.1 Auxiliary expressions

$$\gamma_0 = \zeta^3$$

$$\gamma_1 = \eta \gamma_0$$

$$\gamma_2 = 8\gamma_1$$

$$\gamma_3 = -\zeta$$

$$\gamma_4 = \eta + \gamma_3$$

$$\gamma_5 = \gamma_4 + 1$$

$$\gamma_6 = \eta - 1$$

$$\gamma_7 = \gamma_5 \gamma_6$$

$$\gamma_8 = \eta + \zeta$$

$$\gamma_9 = \zeta + 1$$

$$\gamma_{10} = \eta \zeta$$

$$\gamma_{11} = -\eta + \zeta + \gamma_{10}$$

$$\gamma_{12} = \eta^2$$

$$\gamma_{13} = \zeta^2$$

$$\gamma_{14} = \eta \gamma_{13}$$

$$\gamma_{15} = \zeta \gamma_{12}$$

$$\gamma_{16} = -\gamma_{15}$$

$$\gamma_{17} = 2\gamma_{10}$$

$$\gamma_{18} = -\gamma_{12} - \gamma_{13} - \gamma_{14} - \gamma_{16} - \gamma_{17} - \gamma_4$$

$$\gamma_{19} = \gamma_{11} \gamma_{18} \gamma_8 \gamma_9$$

$$\gamma_{20} = \gamma_{19} \gamma_2 \gamma_7$$

$$\gamma_{21} = \eta^3$$

$$\gamma_{22} = -\gamma_{14}$$

$$\gamma_{23} = \gamma_0 \gamma_{12}$$

$$\gamma_{24} = 34\gamma_{23}$$

$$\gamma_{25} = \zeta^6$$

$$\gamma_{26} = \gamma_{12} \gamma_{25}$$

$$\gamma_{27} = \zeta^5$$

$$\gamma_{28} = 2\gamma_{27}$$

$$\gamma_{29} = \varsigma \gamma_{21}$$

$$\begin{aligned}\gamma_{30} &= -\gamma_{28}\gamma_{29} \\ \gamma_{31} &= \zeta^4 \\ \gamma_{32} &= -18\eta\gamma_{31}\end{aligned}$$

$$\begin{aligned}\gamma_{33} &= \eta^5 \\ \gamma_{34} &= \gamma_0\gamma_{33} \\ \gamma_{35} &= \eta^4\end{aligned}$$

$$\begin{aligned}\gamma_{36} &= \gamma_{12}\gamma_{27} \\ \gamma_{37} &= -9\gamma_{36} \\ \gamma_{38} &= \gamma_{31}\gamma_{35}\end{aligned}$$

$$\begin{aligned}\gamma_{39} &= 7\gamma_{38} \\ \gamma_{40} &= \gamma_{21}\gamma_{27} \\ \gamma_{41} &= 3\gamma_{40}\end{aligned}$$

$$\begin{aligned}\gamma_{42} &= \eta^6 \\ \gamma_{43} &= 2\zeta \\ \gamma_{44} &= \gamma_{42}\gamma_{43}\end{aligned}$$

$$\begin{aligned}\gamma_{45} &= \zeta\gamma_{33} \\ \gamma_{46} &= \eta\gamma_{27} \\ \gamma_{47} &= \zeta\gamma_{35}\end{aligned}$$

$$\begin{aligned}\gamma_{48} &= \gamma_0\gamma_{35} \\ \gamma_{49} &= \gamma_0\gamma_{21} \\ \gamma_{50} &= \gamma_{12}\gamma_{13}\end{aligned}$$

$$\begin{aligned}\gamma_{51} &= 21\zeta\gamma_{50} \\ \gamma_{52} &= -\gamma_{51} \\ \gamma_{53} &= \gamma_{14}\gamma_{43}\end{aligned}$$

$$\begin{aligned}\gamma_{54} &= -2\zeta\gamma_{13}\gamma_{42} \\ \gamma_{55} &= 2\gamma_{31} \\ \gamma_{56} &= \gamma_{29}\gamma_{55}\end{aligned}$$

$$\begin{aligned}\gamma_{57} &= \eta\zeta \\ \gamma_{58} &= \gamma_{31}\gamma_{57} \\ \gamma_{59} &= 4\gamma_{58}\end{aligned}$$

$$\begin{aligned}\gamma_{60} &= \zeta\gamma_{42} \\ \gamma_{61} &= 4\zeta\gamma_{60} \\ \gamma_{62} &= \zeta\gamma_{31}\end{aligned}$$

$$\begin{aligned}\gamma_{63} &= \gamma_{35}\gamma_{62} \\ \gamma_{64} &= 8\gamma_0 \\ \gamma_{65} &= \zeta\gamma_{34}\end{aligned}$$

$$\begin{aligned}\gamma_{66} &= \zeta\gamma_{23} \\ \gamma_{67} &= \zeta\gamma_{35} \\ \gamma_{68} &= \gamma_0\gamma_{67}\end{aligned}$$

$$\begin{aligned}\gamma_{69} &= \gamma_0\gamma_{29} \\ \gamma_{70} &= \zeta\gamma_{21} \\ \gamma_{71} &= \zeta\gamma_{70}\end{aligned}$$

$$\begin{aligned}\gamma_{72} &= \zeta\gamma_{45} \\ \gamma_{73} &= \zeta\gamma_{67} \\ \gamma_{74} &= \zeta\gamma_{15}\end{aligned}$$

$$\begin{aligned}\gamma_{75} &= 4\gamma_{74} \\ \gamma_{76} &= -\gamma_{27} \\ \gamma_{77} &= \varsigma\gamma_{12}\end{aligned}$$

$$\begin{aligned}\gamma_{78} &= \gamma_{55}\gamma_{77} \\ \gamma_{79} &= \gamma_{76} + \gamma_{78} \\ \gamma_{80} &= \gamma_0 + \gamma_{75} + \gamma_{79}\end{aligned}$$

$$\begin{aligned}\gamma_{81} &= -\gamma_{31} \\ \gamma_{82} &= 2\gamma_{25} \\ \gamma_{83} &= \eta\gamma_{82}\end{aligned}$$

$$\begin{aligned}\gamma_{84} &= 7\gamma_{21} \\ \gamma_{85} &= \gamma_{31}\gamma_{84} \\ \gamma_{86} &= \gamma_{28}\gamma_{57}\end{aligned}$$

$$\begin{aligned}\gamma_{87} &= \gamma_{81} + \gamma_{83} + \gamma_{85} - \gamma_{86} \\ \gamma_{88} &= \varsigma\gamma_{36} \\ \gamma_{89} &= -4\gamma_{88}\end{aligned}$$

$$\begin{aligned}\gamma_{90} &= 7\gamma_{70} \\ \gamma_{91} &= \gamma_{25} + \gamma_{89} + \gamma_{90} \\ \gamma_{92} &= 4\gamma_0\end{aligned}$$

$$\begin{aligned}\gamma_{93} &= w^2 \\ \gamma_{94} &= \gamma_{13}\gamma_{84} \\ \gamma_{95} &= 2\gamma_{14}\end{aligned}$$

$$\begin{aligned}\gamma_{96} &= -\gamma_{15}\gamma_{43} \\ \gamma_{97} &= \gamma_0\gamma_{84} \\ \gamma_{98} &= \gamma_{57}\gamma_{92}\end{aligned}$$

$$\begin{aligned}\gamma_{99} &= -\gamma_{98} \\ \gamma_{100} &= \varsigma^2 \\ \gamma_{101} &= \gamma_{100}\gamma_{33}\end{aligned}$$

$$\begin{aligned}\gamma_{102} &= 15\gamma_{67} \\ \gamma_{103} &= 23\gamma_{50} \\ \gamma_{104} &= -4\varsigma\gamma_{12}\gamma_{13}\end{aligned}$$

$$\begin{aligned}\gamma_{105} &= \gamma_{100}\gamma_{50} \\ \gamma_{106} &= 2\gamma_{105} \\ \gamma_{107} &= \gamma_{100}\gamma_{70}\end{aligned}$$

$$\begin{aligned}\gamma_{108} &= \gamma_{100}\gamma_{35} \\ \gamma_{109} &= 4\gamma_{108} \\ \gamma_{110} &= 11\gamma_{69}\end{aligned}$$

$$\begin{aligned}\gamma_{111} &= \gamma_{13}\gamma_{29} \\ \gamma_{112} &= 28\gamma_{111} \\ \gamma_{113} &= 2\gamma_{13}\end{aligned}$$

$$\begin{aligned}\gamma_{114} &= \gamma_{100}\gamma_{21} \\ \gamma_{115} &= -\gamma_{113}\gamma_{114} \\ \gamma_{116} &= \gamma_{115} + \gamma_{29}\end{aligned}$$

$$\begin{aligned}\gamma_{117} &= \varsigma\gamma_{14} \\ \gamma_{118} &= \gamma_{27} - \gamma_{78} \\ \gamma_{119} &= \gamma_{117} + \gamma_{118}\end{aligned}$$

$$\begin{aligned}\gamma_{120} &= \eta\gamma_{55} \\ \gamma_{121} &= -\gamma_{120} \\ \gamma_{122} &= \gamma_{33}\gamma_{43}\end{aligned}$$

$$\begin{aligned}\gamma_{123} &= \gamma_{121} + \gamma_{122} \\ \gamma_{124} &= -\gamma_0 \\ \gamma_{125} &= -\gamma_{85}\end{aligned}$$

$$\begin{aligned}\gamma_{126} &= \eta\gamma_{28} \\ \gamma_{127} &= \varsigma\gamma_{33} \\ \gamma_{128} &= \gamma_{113}\gamma_{127} + \gamma_{124} + \gamma_{125} + \gamma_{126} + \gamma_{36}\end{aligned}$$

$$\begin{aligned}\gamma_{129} &= 2\gamma_1 \\ \gamma_{130} &= -\gamma_{129} \\ \gamma_{131} &= \gamma_{100}\gamma_{42}\end{aligned}$$

$$\begin{aligned}\gamma_{132} &= 2\gamma_{131} \\ \gamma_{133} &= \gamma_{130} + \gamma_{132} \\ \gamma_{134} &= 8\gamma_{108}\end{aligned}$$

$$\begin{aligned}\gamma_{135} &= -\gamma_{90} \\ \gamma_{136} &= \gamma_{134} + \gamma_{135} + \gamma_{16} \\ \gamma_{137} &= 8\gamma_{101}\end{aligned}$$

$$\begin{aligned}\gamma_{138} &= \gamma_{13}\gamma_{67} \\ \gamma_{139} &= \gamma_{13}\gamma_{137} + 23\gamma_{138} \\ \gamma_{140} &= \zeta - 1\end{aligned}$$

$$\begin{aligned}\gamma_{141} &= \gamma_3 + \gamma_{57} \\ \gamma_{142} &= \gamma_{113}\gamma_{140}\gamma_{141} \\ \gamma_{143} &= -\gamma_{55}\end{aligned}$$

$$\begin{aligned}\gamma_{144} &= \zeta^7 \\ \gamma_{145} &= \eta^7 \\ \gamma_{146} &= \gamma_{100}\gamma_{145}\end{aligned}$$

$$\begin{aligned}\gamma_{147} &= \gamma_{113}\gamma_{33} \\ \gamma_{148} &= 3\gamma_1 \\ \gamma_{149} &= \eta\gamma_{25}\end{aligned}$$

$$\begin{aligned}\gamma_{150} &= 3\gamma_{149} \\ \gamma_{151} &= \gamma_{21}\gamma_{25} \\ \gamma_{152} &= \zeta\gamma_{146}\end{aligned}$$

$$\begin{aligned}\gamma_{153} &= \gamma_{100}\gamma_{36} \\ \gamma_{154} &= \varsigma\gamma_0 \\ \gamma_{155} &= \gamma_{154}\gamma_{42}\end{aligned}$$

$$\begin{aligned}\gamma_{156} &= 2\gamma_{155} \\ \gamma_{157} &= \gamma_{43}\gamma_{70} \\ \gamma_{158} &= 2\zeta\end{aligned}$$

$$\begin{aligned}\gamma_{159} &= \varsigma\gamma_{158} \\ \gamma_{160} &= \gamma_{159}\gamma_{42} \\ \gamma_{161} &= \gamma_{108}\gamma_{27}\end{aligned}$$

$$\begin{aligned}\gamma_{162} &= \tau\gamma_{108} \\ \gamma_{163} &= \tau\gamma_0 \\ \gamma_{164} &= \gamma_{13}\gamma_{21}\end{aligned}$$

$$\begin{aligned}\gamma_{165} &= \gamma_{100}\gamma_{164} \\ \gamma_{166} &= \zeta\gamma_{131} \\ \gamma_{167} &= \gamma_{12}\gamma_{62}\end{aligned}$$

$$\begin{aligned}\gamma_{168} &= \gamma_{100}\gamma_{34} \\ \gamma_{169} &= 2\gamma_{107} \\ \gamma_{170} &= -\gamma_{169}\end{aligned}$$

$$\begin{aligned}\gamma_{171} &= 2\gamma_0 \\ \gamma_{172} &= \gamma_{170} + \gamma_{171}\gamma_{57} \\ \gamma_{173} &= -\gamma_{131}\end{aligned}$$

$$\begin{aligned}\gamma_{174} &= -\gamma_{100}\gamma_{97} + \gamma_{105} + \gamma_{173} \\ \gamma_{175} &= 12\gamma_{72} \\ \gamma_{176} &= \gamma_{175} + \gamma_{28} + \gamma_{86}\end{aligned}$$

$$\begin{aligned}\gamma_{177} &= \gamma_{13}\gamma_{146} \\ \gamma_{178} &= 3\gamma_{114} \\ \gamma_{179} &= \gamma_{100}\gamma_{45}\end{aligned}$$

$$\begin{aligned}\gamma_{180} &= 21\gamma_{108} \\ \gamma_{181} &= \gamma_0\gamma_{180} + \gamma_{177} + \gamma_{178}\gamma_{27} + 18\gamma_{179} \\ \gamma_{182} &= 4\gamma_{14}\end{aligned}$$

$$\begin{aligned}\gamma_{183} &= \gamma_{182}w \\ \gamma_{184} &= 2\gamma_{23} \\ \gamma_{185} &= -\gamma_{184}\end{aligned}$$

$$\begin{aligned}\gamma_{186} &= \gamma_{108}\gamma_{113} \\ \gamma_{187} &= 8\gamma_{31} \\ \gamma_{188} &= 10\gamma_0\end{aligned}$$

$$\begin{aligned}\gamma_{189} &= 4\gamma_{50} \\ \gamma_{190} &= \gamma_{12}\gamma_{55} \\ \gamma_{191} &= \zeta\gamma_{113}\end{aligned}$$

$$\begin{aligned}\gamma_{192} &= \gamma_{100}\gamma_{12} \\ \gamma_{193} &= \eta\gamma_{31} \\ \gamma_{194} &= \gamma_{100}\gamma_{15}\end{aligned}$$

$$\begin{aligned}\gamma_{195} &= \gamma_0\gamma_{57} \\ \gamma_{196} &= \gamma_{100}\gamma_{17} \\ \gamma_{197} &= \gamma_{158}\gamma_{67}\end{aligned}$$

$$\begin{aligned}\gamma_{198} &= \zeta\gamma_{109} \\ \gamma_{199} &= 4\gamma_{105} \\ \gamma_{200} &= 17\gamma_{117}\end{aligned}$$

$$\begin{aligned}\gamma_{201} &= \gamma_{55}\gamma_{57} \\ \gamma_{202} &= 2\gamma_{77} \\ \gamma_{203} &= \gamma_{201} - \gamma_{202}\end{aligned}$$

$$\begin{aligned}\gamma_{204} &= \gamma_{191}\gamma_4 \\ \gamma_{205} &= -\gamma_1 \\ \gamma_{206} &= -\gamma_{108}\end{aligned}$$

$$\begin{aligned}\gamma_{207} &= 7\gamma_{12} \\ \gamma_{208} &= \gamma_0\gamma_{207} \\ \gamma_{209} &= \gamma_{100}\gamma_{208}\end{aligned}$$

$$\begin{aligned}\gamma_{210} &= 7\gamma_{168} \\ \gamma_{211} &= -\gamma_{210} \\ \gamma_{212} &= -\gamma_{45}\end{aligned}$$

$$\begin{aligned}\gamma_{213} &= \gamma_{31}\gamma_{33} \\ \gamma_{214} &= \gamma_{13}\gamma_{33} \\ \gamma_{215} &= \gamma_{27}\gamma_{84}\end{aligned}$$

$$\begin{aligned}\gamma_{216} &= \gamma_{13}\gamma_{35} \\ \gamma_{217} &= \gamma_{21}\gamma_{31} \\ \gamma_{218} &= \gamma_{12}\gamma_{31}\end{aligned}$$

$$\begin{aligned}\gamma_{219} &= \gamma_1\gamma_{100} \\ \gamma_{220} &= \gamma_{29}\gamma_{31} \\ \gamma_{221} &= \gamma_{108}\gamma_{13}\end{aligned}$$

$$\begin{aligned}\gamma_{222} &= \gamma_{114}\gamma_{31} \\ \gamma_{223} &= \gamma_{100}\gamma_{218} \\ \gamma_{224} &= \gamma_{108}\gamma_{31}\end{aligned}$$

$$\begin{aligned}\gamma_{225} &= 10\gamma_{67} \\ \gamma_{226} &= \gamma_{101}\gamma_{13} \\ \gamma_{227} &= 4\gamma_{25}\end{aligned}$$

$$\begin{aligned}\gamma_{228} &= \varsigma\gamma_{13} \\ \gamma_{229} &= \gamma_{228}\gamma_{42} \\ \gamma_{230} &= \gamma_{13}\gamma_{131}\end{aligned}$$

$$\begin{aligned}\gamma_{231} &= 10\gamma_{13} \\ \gamma_{232} &= \zeta\gamma_{108} \\ \gamma_{233} &= 12\gamma_{67}\end{aligned}$$

$$\begin{aligned}\gamma_{234} &= \gamma_{27}\gamma_{57} \\ \gamma_{235} &= \gamma_{27}\gamma_{29} \\ \gamma_{236} &= \varsigma\gamma_{214}\end{aligned}$$

$$\begin{aligned}\gamma_{237} &= \gamma_0\gamma_{114} \\ \gamma_{238} &= -\gamma_{50} \\ \gamma_{239} &= \gamma_{238} + \gamma_{31}\end{aligned}$$

$$\begin{aligned}\gamma_{240} &= -\gamma_{25} - \gamma_{61} \\ \gamma_{241} &= 4\gamma_{13}w \\ \gamma_{242} &= -\gamma_{112}\end{aligned}$$

$$\begin{aligned}\gamma_{243} &= -4\gamma_{236} \\ \gamma_{244} &= \gamma_{108}\gamma_{158} \\ \gamma_{245} &= \gamma_{17}w\end{aligned}$$

$$\begin{aligned}\gamma_{246} &= \gamma_{245}\gamma_6 \\ \gamma_{247} &= \gamma_{108} + \gamma_{129} \\ \gamma_{248} &= \gamma_{43}\gamma_{45}\end{aligned}$$

$$\begin{aligned}\gamma_{249} &= \gamma_{248} + \gamma_{97} \\ \gamma_{250} &= \gamma_{108}\gamma_{163} \\ \gamma_{251} &= \gamma_{25} + \gamma_{250} + \gamma_{89}\end{aligned}$$

$$\begin{aligned}\gamma_{252} &= -\gamma_{230} + \gamma_{26} + \gamma_{34}\gamma_{43} \\ \gamma_{253} &= -\gamma_{209} + \gamma_{30} \\ \gamma_{254} &= -\gamma_{215} + \gamma_{253}\end{aligned}$$

$$\begin{aligned}\gamma_{255} &= 7\gamma_{13} \\ \gamma_{256} &= -18\zeta\gamma_{12} \\ \gamma_{257} &= -\gamma_{13}\gamma_{21} + \gamma_{21}\end{aligned}$$

$$\begin{aligned}\gamma_{258} &= 18\eta\zeta - 6\eta + 6\zeta - \gamma_{124} - 18\gamma_{14} - \gamma_{189} - \gamma_{207} - \gamma_{23} - \gamma_{255} \\ &\quad - \gamma_{256} - \gamma_{257}\end{aligned}$$

$$\begin{aligned}\gamma_{259} &= \gamma_{258}\gamma_8 \\ \gamma_{260} &= \gamma_{259}\gamma_5 \\ \gamma_{261} &= -\gamma_{29}\gamma_{92}\end{aligned}$$

$$\begin{aligned}\gamma_{262} &= 13\gamma_0 \\ \gamma_{263} &= 6\gamma_{12} \\ \gamma_{264} &= 6\gamma_{13}\end{aligned}$$

$$\begin{aligned}\gamma_{265} &= -4\gamma_{70} \\ \gamma_{266} &= 2\gamma_{47} \\ \gamma_{267} &= 34\gamma_{14}\end{aligned}$$

$$\begin{aligned}\gamma_{268} &= \varsigma\gamma_{10} \\ \gamma_{269} &= \varsigma\gamma_{50} \\ \gamma_{270} &= 56\gamma_{269} + \gamma_{79}\end{aligned}$$

$$\begin{aligned}\gamma_{271} &= -\gamma_{46} \\ \gamma_{272} &= 4\gamma_{164} \\ \gamma_{273} &= \gamma_{271} + \gamma_{272}\end{aligned}$$

$$\begin{aligned}\gamma_{274} &= 6\gamma_{15} \\ \gamma_{275} &= -\gamma_{274} \\ \gamma_{276} &= 3\gamma_{218}\end{aligned}$$

$$\begin{aligned}\gamma_{277} &= \gamma_{275} + \gamma_{276} - 17\gamma_{66} - 47\gamma_{71} \\ \gamma_{278} &= -\gamma_{197} \\ \gamma_{279} &= \varsigma\gamma_{263}\end{aligned}$$

$$\begin{aligned}\gamma_{280} &= 6\gamma_{29} \\ \gamma_{281} &= 3\gamma_{108} \\ \gamma_{282} &= 6\gamma_{10}\end{aligned}$$

$$\begin{aligned}\gamma_{283} &= 18\gamma_{50} \\ \gamma_{284} &= 15\gamma_{71} \\ \gamma_{285} &= 3\gamma_{107}\end{aligned}$$

$$\begin{aligned}\gamma_{286} &= -\gamma_{285} \\ \gamma_{287} &= \gamma_{286} + \gamma_{81} \\ \gamma_{288} &= -3\gamma_{21}\gamma_{31}\end{aligned}$$

$$\begin{aligned}\gamma_{289} &= 12\gamma_0 \\ \gamma_{290} &= 10\gamma_{50} \\ \gamma_{291} &= 3\gamma_{164}\end{aligned}$$

$$\begin{aligned}\gamma_{292} &= 13\gamma_{108} \\ \gamma_{293} &= 6\gamma_{114} \\ \gamma_{294} &= -6\eta\gamma_{13} - 6\zeta\gamma_{100}\gamma_{12} + \gamma_{293}\end{aligned}$$

$$\begin{aligned}\gamma_{295} &= -\gamma_{248} \\ \gamma_{296} &= \gamma_{295} + \gamma_{97} \\ \gamma_{297} &= -\gamma_{56}\end{aligned}$$

$$\begin{aligned}\gamma_{298} &= \gamma_{131} + \gamma_{297} \\ \gamma_{299} &= -39\zeta\gamma_{100}\gamma_{35} - 3\gamma_0\gamma_{100}\gamma_{21} \\ \gamma_{300} &= -\gamma_{82}\end{aligned}$$

$$\begin{aligned}\gamma_{301} &= -\zeta\gamma_{35} \\ \gamma_{302} &= -6\zeta\gamma_0\gamma_{35} + \gamma_{223} + \gamma_{300} + \gamma_{301} + \gamma_{48} + \gamma_{86}\end{aligned}$$

$$\begin{aligned}\gamma_{303} &= \gamma_{114}\gamma_{55} - \gamma_{149} + \gamma_{224} + \gamma_{284} \\ \gamma_{304} &= 12\zeta \\ \gamma_{305} &= 13\gamma_{114}\end{aligned}$$

$$\begin{aligned}\gamma_{306} &= 27\gamma_{10} \\ \gamma_{307} &= \gamma_{130} + \gamma_{154} \\ \gamma_{308} &= \gamma_{169} + \gamma_{195}\end{aligned}$$

$$\begin{aligned}\gamma_{309} &= 3\gamma_{195} \\ \gamma_{310} &= -\gamma_{309} \\ \gamma_{311} &= 3\gamma_{71}\end{aligned}$$

$$\begin{aligned}\gamma_{312} &= \gamma_{114}\gamma_{92} \\ \gamma_{313} &= 6\gamma_0 \\ \gamma_{314} &= 34\gamma_1\end{aligned}$$

$$\begin{aligned}\gamma_{315} &= 4\gamma_{23} \\ \gamma_{316} &= -3\gamma_{36} \\ \gamma_{317} &= 59\gamma_{232} + \gamma_{316}\end{aligned}$$

$$\begin{aligned}\gamma_{318} &= 6\gamma_{50} \\ \gamma_{319} &= \gamma_{100}\gamma_{193} \\ \gamma_{320} &= 4\gamma_{234}\end{aligned}$$

$$\begin{aligned}\gamma_{321} &= \gamma_{149} + \gamma_{192}\gamma_{55} + 12\gamma_{220} + \gamma_{222} + \gamma_{318} + \gamma_{319} - \gamma_{320} \\ \gamma_{322} &= \gamma_{23}\gamma_{43}\end{aligned}$$

$$\begin{aligned}\gamma_{323} &= 28\gamma_{164} \\ \gamma_{324} &= \gamma_{100}\gamma_{323} + \gamma_{162} \\ \gamma_{325} &= -\gamma_{201}\end{aligned}$$

$$\begin{aligned}\gamma_{326} &= \gamma_{101} + \gamma_{108}\gamma_{255} + \gamma_{325} \\ \gamma_{327} &= -6\zeta\gamma_{35} + \gamma_0 \\ \gamma_{328} &= -\gamma_{13}\end{aligned}$$

$$\begin{aligned}\gamma_{329} &= \gamma_{12} + \gamma_{328} \\ \gamma_{330} &= -18\zeta\gamma_{21} + \gamma_{329} + 6\gamma_{35} + \gamma_{84}\end{aligned}$$

$$\begin{aligned}\gamma_{331} &= -\gamma_{182} + \gamma_{23} - \gamma_{256} - \gamma_{283} - \gamma_{327} - \gamma_{330} - \gamma_{94} \\ \gamma_{332} &= \gamma_{331}\gamma_9\end{aligned}$$

$$\begin{aligned}\gamma_{333} &= \gamma_2\gamma_{332}\gamma_5 \\ \gamma_{334} &= -12\zeta\gamma_{35} \\ \gamma_{335} &= \gamma_{264}\gamma_{35}\end{aligned}$$

$$\begin{aligned}\gamma_{336} &= 8\gamma_{117} \\ \gamma_{337} &= \gamma_{10}\gamma_{43} \\ \gamma_{338} &= -\gamma_{66}\end{aligned}$$

$$\begin{aligned}\gamma_{339} &= -\gamma_{195} \\ \gamma_{340} &= 14\gamma_{23} \\ \gamma_{341} &= 4\gamma_{117}\end{aligned}$$

$$\begin{aligned}\gamma_{342} &= 2\gamma_{194} \\ \gamma_{343} &= \gamma_{193} + \gamma_{31} \\ \gamma_{344} &= \gamma_{22} - \gamma_{304}\gamma_{67}\end{aligned}$$

$$\begin{aligned}\gamma_{345} &= \gamma_{10} + \gamma_{205} - \gamma_{77} \\ \gamma_{346} &= -6\gamma_{48} \\ \gamma_{347} &= 13\gamma_{101}\end{aligned}$$

$$\begin{aligned}\gamma_{348} &= 6\gamma_{131} \\ \gamma_{349} &= 3\gamma_{50} \\ \gamma_{350} &= \gamma_1 + \gamma_{201} + \gamma_{271}\end{aligned}$$

$$\begin{aligned}\gamma_{351} &= \gamma_{190} + \gamma_{194} + \gamma_{312} \\ \gamma_{352} &= -\gamma_{114} \\ \gamma_{353} &= \gamma_{14} + \gamma_{352}\end{aligned}$$

$$\begin{aligned}\gamma_{354} &= \gamma_{15} - 12\gamma_{65} \\ \gamma_{355} &= 3\gamma_{105} \\ \gamma_{356} &= -\gamma_{355}\end{aligned}$$

$$\begin{aligned}\gamma_{357} &= -\gamma_{171} - \gamma_{313}\gamma_{57} + \gamma_{356} + \gamma_{53} + \gamma_{55} + \gamma_{96} \\ \gamma_{358} &= 10\gamma_{14} \\ \gamma_{359} &= 2\gamma_{57}\end{aligned}$$

$$\begin{aligned}\gamma_{360} &= 15\gamma_{228} \\ \gamma_{361} &= 25\gamma_{74} \\ \gamma_{362} &= \gamma_{10}\gamma_{100}\end{aligned}$$

$$\begin{aligned}\gamma_{363} &= 8\gamma_{362} \\ \gamma_{364} &= -6\gamma_{71} \\ \gamma_{365} &= 3\gamma_{194}\end{aligned}$$

$$\begin{aligned}\gamma_{366} &= 2\eta \\ \gamma_{367} &= \gamma_{100}\gamma_{366} \\ \gamma_{368} &= 3\gamma_{192} + \gamma_{367}\end{aligned}$$

$$\begin{aligned}\gamma_{369} &= 5\gamma_1 \\ \gamma_{370} &= \gamma_{280} - \gamma_{369} \\ \gamma_{371} &= \gamma_{100}\gamma_{14}\end{aligned}$$

$$\begin{aligned}\gamma_{372} &= 44\gamma_{68} \\ \gamma_{373} &= 22\gamma_{107} \\ \gamma_{374} &= 11\gamma_{71}\end{aligned}$$

$$\begin{aligned}\gamma_{375} &= \gamma_{188}\gamma_{57} \\ \gamma_{376} &= 22\gamma_{58} \\ \gamma_{377} &= 110\gamma_{111}\end{aligned}$$

$$\begin{aligned}\gamma_{378} &= \gamma_2 + \gamma_{81} \\ \gamma_{379} &= -6\zeta\gamma_{100}\gamma_{42} + 12\gamma_{216} \\ \gamma_{380} &= 14\gamma_{50}\end{aligned}$$

$$\begin{aligned}\gamma_{381} &= 14\gamma_{195} \\ \gamma_{382} &= 6\gamma_{179} \\ \gamma_{383} &= 7\gamma_{105}\end{aligned}$$

$$\begin{aligned}\gamma_{384} &= 68\gamma_{111} \\ \gamma_{385} &= \gamma_0 + \gamma_{114} \\ \gamma_{386} &= \eta + 1\end{aligned}$$

$$\begin{aligned}\gamma_{387} &= 17\zeta\gamma_{12} + \gamma_{148} - \gamma_{257} - \gamma_{290} - \gamma_{329} - \gamma_{343} - \gamma_{358} \\ \gamma_{388} &= \gamma_{386}\gamma_{387}\end{aligned}$$

$$\begin{aligned}\gamma_{389} &= \gamma_2\gamma_{388}\gamma_7 \\ \gamma_{390} &= -4\gamma_{167} \\ \gamma_{391} &= \gamma_{81} + \gamma_{95}\end{aligned}$$

$$\begin{aligned}\gamma_{392} &= \gamma_{13} - \gamma_{337} \\ \gamma_{393} &= -\gamma_{160} + \gamma_{214} \\ \gamma_{394} &= -5\gamma_{29}\end{aligned}$$

$$\begin{aligned}\gamma_{395} &= \gamma_{114} + \gamma_{74} \\ \gamma_{396} &= -\gamma_{198} \\ \gamma_{397} &= \gamma_{396} + \gamma_{77}\end{aligned}$$

$$\begin{aligned}\gamma_{398} &= \gamma_{132} - \gamma_{323} \\ \gamma_{399} &= 3\gamma_{222} \\ \gamma_{400} &= \gamma_{100}\gamma_{13}\end{aligned}$$

$$\begin{aligned}\gamma_{401} &= \gamma_{400}\gamma_{84} \\ \gamma_{402} &= 12\gamma_{63} \\ \gamma_{403} &= 46\gamma_{221}\end{aligned}$$

$$\begin{aligned}\gamma_{404} &= \gamma_{194} + \gamma_{353} \\ \gamma_{405} &= -2\gamma_{100}\gamma_{35} + \gamma_{146} + \gamma_{212} + \gamma_{70} + 24\gamma_{73}\end{aligned}$$

$$\begin{aligned}\gamma_{406} &= \gamma_{154} + \gamma_{159} \\ \gamma_{407} &= \gamma_{185} + \gamma_{66} \\ \gamma_{408} &= \gamma_{124} + \gamma_{31}\end{aligned}$$

$$\begin{aligned}\gamma_{409} &= \gamma_{27} - \gamma_{318} \\ \gamma_{410} &= -\gamma_{57} \\ \gamma_{411} &= \gamma_4\gamma_5\end{aligned}$$

$$\begin{aligned}\gamma_{412} &= \varsigma^3 \\ \gamma_{413} &= \gamma_{21}\gamma_{412} \\ \gamma_{414} &= -\gamma_{413}\end{aligned}$$

$$\begin{aligned}\gamma_{415} &= -\gamma_{12}\gamma_{412} \\ \gamma_{416} &= 2\gamma_{15} \\ \gamma_{417} &= \gamma_{35}\gamma_{412}\end{aligned}$$

$$\begin{aligned}\gamma_{418} &= \gamma_{192} + \gamma_{309} + \gamma_{391} + \gamma_{417} \\ \gamma_{419} &= \gamma_{130} + \gamma_{33}\gamma_{412} \\ \gamma_{420} &= \gamma_{412}\gamma_{70}\end{aligned}$$

$$\begin{aligned}\gamma_{421} &= \gamma_{286} + \gamma_{413} + 2\gamma_{420} \\ \gamma_{422} &= \varsigma\gamma_{349} + \gamma_{356} + \gamma_{385}\end{aligned}$$

$$\begin{aligned}\gamma_{423} &= -\gamma_{100}\gamma_{95} + \gamma_{309} + \gamma_{96} \\ \gamma_{424} &= 2\gamma_{50}\end{aligned}$$

The twenty entries are listed by matrix row. Each expression retains the auxiliary quantities defined above.

S8.2 Matrix entries

Row 1

$$A_{1,1} = -\gamma_{20}w$$

$$-\gamma_{92} \begin{pmatrix} 55\varsigma\gamma_{13}\gamma_{21} + 21\varsigma\gamma_{13}\gamma_{33} + 55\varsigma\gamma_{13}\gamma_{35} + 2\varsigma\gamma_{21} \\ + 18\varsigma\gamma_{33} + 18\varsigma\gamma_{35} - 10\gamma_1 + 16\gamma_{12}\gamma_{13} + 9\gamma_{12}\gamma_{31} \\ + 33\gamma_{13}\gamma_{21} + 3\gamma_{13}\gamma_{33} + 15\gamma_{13}\gamma_{35} - \gamma_{16} - \gamma_{21} - \gamma_{22} \\ - \gamma_{24} - \gamma_{26} - \gamma_{30} - \gamma_{32} + \gamma_{33} - \gamma_{34} - \gamma_{37} + \gamma_{39} + \gamma_{41} + \gamma_{44} \\ - 3\gamma_{45} - 7\gamma_{46} - 11\gamma_{47} - 11\gamma_{48} - 21\gamma_{49} - \gamma_{52} + \gamma_{53} - \gamma_{54} \\ - \gamma_{56} - \gamma_{57}\gamma_{64} - \gamma_{59} - \gamma_{61} - 4\gamma_{63} - 8\gamma_{65} - 11\gamma_{66} \\ - 11\gamma_{68} - 18\gamma_{69} - 31\gamma_{71} - 31\gamma_{72} - 66\gamma_{73} - \gamma_{80} - \gamma_{87} \\ - \gamma_{91} \end{pmatrix}$$

$$A_{1,2} = 8\gamma_0\gamma_{11}\gamma_{12}\gamma_{18}\gamma_5\gamma_6\gamma_8\gamma_9\gamma_{93}$$

$$-\gamma_{142} \begin{pmatrix} \eta\varsigma\gamma_{31} + 35\varsigma\gamma_{21} + 4\varsigma\gamma_{33} + 37\varsigma\gamma_{35} + \varsigma\gamma_0\gamma_{35} \\ + \varsigma\gamma_{21}\gamma_{31} + 20\varsigma\gamma_{100}\gamma_{33} + 14\varsigma\gamma_{100}\gamma_{35} + 2\varsigma\gamma_{100}\gamma_{42} \\ - 10\gamma_{101} - \gamma_{102} - \gamma_{103} - \gamma_{104} - \gamma_{106} - 4\gamma_{107} \\ - \gamma_{109}\gamma_{13} - \gamma_{110} - \gamma_{112} - \gamma_{116} - \gamma_{119} + 23\gamma_{12}\gamma_{31} \\ - \gamma_{123} - \gamma_{128} - \gamma_{133} - \gamma_{136} - \gamma_{139} - 8\gamma_{66} - \gamma_{94} - \gamma_{95} \\ - \gamma_{96} - \gamma_{97} - \gamma_{99} \end{pmatrix}$$

$$-\gamma_{183} \begin{pmatrix} 6\eta\varsigma\gamma_{31} + 3\eta\gamma_{144} + 3\eta\gamma_{31} + \varsigma\gamma_{12}\gamma_{27} + 47\varsigma\gamma_{13}\gamma_{21} \\ + 30\varsigma\gamma_{13}\gamma_{33} + 91\varsigma\gamma_{13}\gamma_{35} + 2\varsigma\gamma_{21}\gamma_{25} + 3\varsigma\gamma_{21}\gamma_{27} \\ + 5\varsigma\gamma_{21}\gamma_{31} + 2\varsigma\gamma_{25}\gamma_{35} + 4\varsigma\gamma_{31}\gamma_{33} + 11\varsigma\gamma_{31}\gamma_{35} \\ - \varsigma\gamma_{162} + 3\gamma_0\gamma_{100}\gamma_{12} + \gamma_0\gamma_{100}\gamma_{145} + 7\gamma_{100}\gamma_{12}\gamma_{31} \\ + 34\gamma_{100}\gamma_{13}\gamma_{33} + 9\gamma_{100}\gamma_{13}\gamma_{35} + 16\gamma_{100}\gamma_{13}\gamma_{42} \\ + 11\gamma_{100}\gamma_{21}\gamma_{31} - \gamma_{100}\gamma_{27}\gamma_{33} + 11\gamma_{100}\gamma_{31}\gamma_{33} \\ + 15\gamma_{100}\gamma_{31}\gamma_{35} - \gamma_{101} - \gamma_{104} - \gamma_{108} + 25\gamma_{12}\gamma_{27} \\ + 25\gamma_{12}\gamma_{31} + 10\gamma_{13}\gamma_{21} + 4\gamma_{13}\gamma_{35} - \gamma_{131}\gamma_{163} - \gamma_{143} \\ - \gamma_{144}\gamma_{21} + 2\gamma_{144} + \gamma_{146} + \gamma_{147} - \gamma_{148} - \gamma_{150} \\ - 10\gamma_{151} - \gamma_{152} - \gamma_{153} + \gamma_{156} - \gamma_{157} - \gamma_{160} - 3\gamma_{161} \\ - 9\gamma_{165} - 10\gamma_{166} - 14\gamma_{167} - 33\gamma_{168} - \gamma_{172} - \gamma_{174} \\ - \gamma_{176} - \gamma_{181} + 40\gamma_{21}\gamma_{27} - 25\gamma_{23} - \gamma_{26}\gamma_{43} - 25\gamma_{26} \\ + \gamma_{27}\gamma_{33} + 4\gamma_{27}\gamma_{35} - \gamma_{33}\gamma_{55} - 4\gamma_{38} - \gamma_{45} - 4\gamma_{48} \\ - 40\gamma_{49} - \gamma_{54} - \gamma_{55}\gamma_{60} - \gamma_{57}\gamma_{82} - 22\gamma_{65} - 5\gamma_{66} \\ - 68\gamma_{68} - 55\gamma_{69} + \gamma_{70} - 20\gamma_{73} - \gamma_{82} \end{pmatrix}$$

$$A_{1,3} = -\gamma_{20}\gamma_{93}$$

$$\begin{aligned}
& + \gamma_{204} \left(\begin{aligned} & 17\zeta\gamma_{15} + 14\gamma_1 + 8\gamma_{100}\gamma_{14} - 14\gamma_{107} - 4\gamma_{111} - \gamma_{113} \\ & + 10\gamma_{114} + \gamma_{115} + \gamma_{134} - \gamma_{138} - 20\gamma_{14} + 8\gamma_{15} - 15\gamma_{154} \\ & + \gamma_{167} + \gamma_{17} + \gamma_{185} + \gamma_{186} - \gamma_{187} + \gamma_{188} - \gamma_{189} - \gamma_{190} \\ & + \gamma_{191} + 2\gamma_{192} + 4\gamma_{193} - 20\gamma_{194} - 19\gamma_{195} - \gamma_{196} \\ & + \gamma_{197} + \gamma_{198} + \gamma_{199} + \gamma_{200} + \gamma_{203} - 15\gamma_{29} + \gamma_{62} - 4\gamma_{66} \\ & - \gamma_{67} + 19\gamma_{71} \end{aligned} \right) \\
& + \gamma_{241} \left(\begin{aligned} & 2\eta\gamma_{144} + 12\zeta\gamma_{213} - 8\zeta\gamma_{26} - \zeta\gamma_{225} - 3\gamma_0\gamma_{131} \\ & + \gamma_{100}\gamma_{46} - 7\gamma_{101}\gamma_{31} - \gamma_{101} + \gamma_{105} + \gamma_{107} + \gamma_{12}\gamma_{144} \\ & + \gamma_{131} + 55\gamma_{138} + \gamma_{144} - \gamma_{146}\gamma_{158} + \gamma_{146} + 7\gamma_{149} \\ & - 3\gamma_{151} + 3\gamma_{153} - 4\gamma_{155} + \gamma_{161} - 16\gamma_{165} + 7\gamma_{166} \\ & - 55\gamma_{167} + \gamma_{181} + \gamma_{188}\gamma_{77} + 10\gamma_{193} + \gamma_{205} + \gamma_{206} \\ & + \gamma_{209} + \gamma_{211} + \gamma_{212} + \gamma_{213} + 3\gamma_{214} + \gamma_{215} + 11\gamma_{216} \\ & + 21\gamma_{217} + 34\gamma_{218} - \gamma_{219} - 37\gamma_{220} - 34\gamma_{221} - 15\gamma_{222} \\ & - 11\gamma_{223} - 11\gamma_{224} - 9\gamma_{226} - \gamma_{227}\gamma_{29} - \gamma_{227}\gamma_{57} \\ & + 8\gamma_{229} - 16\gamma_{23} + 9\gamma_{230} + \gamma_{231}\gamma_{29} + 10\gamma_{232} \\ & + \gamma_{233}\gamma_{27} + 22\gamma_{234} + 23\gamma_{235} + 33\gamma_{236} + 33\gamma_{237} \\ & + \gamma_{239} + \gamma_{240} - 9\gamma_{26} - 7\gamma_{27}\gamma_{35} - 3\gamma_{34} + \gamma_{37} + 11\gamma_{38} \\ & - 18\gamma_{46} - 15\gamma_{48} - 33\gamma_{49} - 10\gamma_{58} - 23\gamma_{65} - 37\gamma_{68} + \gamma_{70} \\ & - 22\gamma_{72} + \gamma_{76} + 33\gamma_{88} + \gamma_{94} \end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
A_{1,4} = & 18\gamma_0\gamma_{67} + \gamma_{121} - \gamma_{126} + \gamma_{131}\gamma_{158} - 32\gamma_{138} + 23\gamma_{165} \\
& + 32\gamma_{167} + \gamma_{170} + \gamma_{174} + 2\gamma_{179} + \gamma_{19}\gamma_{246} + \gamma_{210} \\
& + 18\gamma_{220} - 23\gamma_{226} + 23\gamma_{23} + \gamma_{238} + \gamma_{242} + \gamma_{243} - \gamma_{244} \\
& + \gamma_{247} + \gamma_{249} + \gamma_{251} + \gamma_{252} + \gamma_{254} - 23\gamma_{36} + 14\gamma_{58} \\
& - 28\gamma_{66} + 14\gamma_{73} + \gamma_{87} - \gamma_{94}
\end{aligned}$$

Row 2

$$A_{2,1} = \gamma_2\gamma_{260}w$$

$$\begin{aligned}
& + \gamma_{92} \left(\begin{aligned} & -59\gamma_1 + 21\gamma_{111} + 27\gamma_{117} - \gamma_{122} - 8\gamma_{138} + \gamma_{187} \\ & + 26\gamma_{193} - 11\gamma_{195} + \gamma_{201} - \gamma_{216} + \gamma_{225} - 39\gamma_{23} + \gamma_{249} \\ & + \gamma_{261} - \gamma_{262} - \gamma_{263} + \gamma_{264} + \gamma_{265} + \gamma_{266} + \gamma_{267} \\ & - 12\gamma_{268} + \gamma_{270} + \gamma_{273} + \gamma_{277} + 24\gamma_{29} - \gamma_{35} + 48\gamma_{50} \\ & + 4\gamma_{73} - 59\gamma_{74} + 12\gamma_{77} - \gamma_{84} \end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
A_{2,2} = & \gamma_{142} \left(\begin{aligned} & \zeta\gamma_{282} - \zeta\gamma_{283} - \zeta\gamma_{134} + 29\gamma_1 + 2\gamma_{101} + \gamma_{111} + \gamma_{114} \\ & + 21\gamma_{15} + \gamma_{163} - \gamma_{193} + 5\gamma_{194} + \gamma_{195} - \gamma_{200} + \gamma_{208} \\ & - \gamma_{264} - \gamma_{267} + \gamma_{278} - \gamma_{279} - \gamma_{280} + \gamma_{281} + \gamma_{282} + \gamma_{284} \\ & + \gamma_{287} - 28\gamma_{50} + \gamma_{66} + 2\gamma_{67} + 23\gamma_{74} \end{aligned} \right) \\
& + \gamma_{183} \left(\begin{aligned} & 12\eta\zeta\gamma_{13} + 26\eta\zeta\gamma_{31} + 98\eta\gamma_{31} + 5\zeta\zeta\gamma_{35} + 59\zeta\gamma_{12}\gamma_{13} \\ & + 36\zeta\gamma_{12}\gamma_{31} + 32\zeta\gamma_{13}\gamma_{21} + 23\zeta\gamma_{100}\gamma_{21} \\ & + 22\zeta\gamma_{100}\gamma_{33} + 10\gamma_0\gamma_{100}\gamma_{35} - 57\gamma_1 + 68\gamma_{12}\gamma_{31} \\ & - \gamma_{135} - \gamma_{137} - \gamma_{139} - 24\gamma_{165} - 36\gamma_{195} - \gamma_{199} - \gamma_{209} \\ & - \gamma_{216} - 29\gamma_{221} - 72\gamma_{23} - \gamma_{252} - 16\gamma_{27} - \gamma_{275} - \gamma_{288} \\ & - \gamma_{289} + \gamma_{290} - \gamma_{291} - \gamma_{292} - \gamma_{294} - \gamma_{296} - \gamma_{298} - \gamma_{299} \\ & - \gamma_{30} - \gamma_{302} - \gamma_{303} + \gamma_{31}\gamma_{35} + 26\gamma_{31} - 11\gamma_{36} - 48\gamma_{46} \\ & - 71\gamma_{66} - 21\gamma_{69} - 12\gamma_{74} \end{aligned} \right) \\
& - 8\gamma_{23}\gamma_{260}\gamma_{93}
\end{aligned}$$

$$A_{2,3} = 8\eta\gamma_0\gamma_{258}\gamma_5\gamma_8\gamma_{93}$$

$$\begin{aligned} & -\gamma_{204} \begin{pmatrix} 12\eta\gamma_{100} - \varsigma\gamma_{304} - \gamma_{100}\gamma_{306} + 11\gamma_{117} - 25\gamma_{13} \\ - 25\gamma_{14} + 25\gamma_{192} - 25\gamma_{194} + 13\gamma_{228} + \gamma_{262} + \gamma_{29} \\ + \gamma_{304} + \gamma_{305} + \gamma_{306} + \gamma_{307} + \gamma_{308} - 12\gamma_{57} - \gamma_{71} \\ + 11\gamma_{74} - 13\gamma_{77} \end{pmatrix} \\ & -\gamma_{241} \begin{pmatrix} 6\gamma_{100}\gamma_{14} + \gamma_{100}\gamma_{315} - 6\gamma_{105} + 34\gamma_{107} - 9\gamma_{111} \\ - \gamma_{113}\gamma_{35} + \gamma_{125} - \gamma_{137} - 21\gamma_{138} - 48\gamma_{165} + \gamma_{166} \\ + 21\gamma_{167} + \gamma_{173} + 26\gamma_{179} + 59\gamma_{193} + 39\gamma_{218} - 7\gamma_{219} \\ - 39\gamma_{221} - 3\gamma_{226} - 48\gamma_{23} + \gamma_{243} - \gamma_{250} - 8\gamma_{27} + \gamma_{272} \\ + \gamma_{274} + \gamma_{289}\gamma_{67} - \gamma_{292} - \gamma_{293} + 13\gamma_{31} + \gamma_{310} + \gamma_{311} \\ + \gamma_{312} - \gamma_{313} - \gamma_{314} + \gamma_{317} + \gamma_{321} - 26\gamma_{46} + \gamma_{47} + \gamma_{48} \\ - 4\gamma_{49} + \gamma_{58} - 9\gamma_{66} + 4\gamma_{72} + \gamma_{73} + \gamma_{91} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A_{2,4} &= 29\eta\gamma_{31} + 34\varsigma\gamma_{100}\gamma_{21} + \varsigma\gamma_{100}\gamma_{33} + 29\varsigma\gamma_{100}\gamma_{35} \\ & - 21\gamma_{105} - \gamma_{113}\gamma_{29} - \gamma_{113}\gamma_{67} - \gamma_{118} + 21\gamma_{12}\gamma_{13} \\ & + 7\gamma_{12}\gamma_{31} - 28\gamma_{23} - \gamma_{245}\gamma_{259} - \gamma_{278} - \gamma_{294} + 7\gamma_{31} \\ & - \gamma_{313} - \gamma_{314} - \gamma_{322} - \gamma_{324} - \gamma_{326} - \gamma_{46} \end{aligned}$$

Row 3

$$A_{3,1} = -\gamma_{333}w$$

$$-\gamma_{92} \begin{pmatrix} 8\eta\gamma_{13} + 9\eta\gamma_{31} + 59\varsigma\gamma_{35} + 14\gamma_0\gamma_{21} - 18\gamma_1 + \gamma_{110} \\ - 56\gamma_{111} - 12\gamma_{127} - 27\gamma_{138} - 25\gamma_{164} + \gamma_{175} - \gamma_{203} \\ - 3\gamma_{23} - \gamma_{239} - \gamma_{271} - \gamma_{277} - 10\gamma_{29} - \gamma_{330} - \gamma_{334} \\ - \gamma_{335} + \gamma_{336} - \gamma_{337} - \gamma_{51} - 24\gamma_{67} - \gamma_{80} - \gamma_{99} \end{pmatrix}$$

$$A_{3,2} = 8\gamma_0\gamma_{12}\gamma_{331}\gamma_5\gamma_9\gamma_{93}$$

$$\begin{aligned} & -\gamma_{142} \begin{pmatrix} 12\gamma_{101} - 25\gamma_{107} + 25\gamma_{108} + 41\gamma_{111} + 14\gamma_{15} - 27\gamma_{232} \\ + \gamma_{233} + \gamma_{268} + 39\gamma_{269} + 13\gamma_{29} + \gamma_{305} + \gamma_{328} + \gamma_{338} \\ + \gamma_{339} - \gamma_{340} - \gamma_{341} + \gamma_{342} + \gamma_{343} + \gamma_{344} + \gamma_{345} \\ - 50\gamma_{71} - 37\gamma_{74} \end{pmatrix} \\ & -\gamma_{183} \begin{pmatrix} 6\varsigma\gamma_{35} + \gamma_0\gamma_{102} - \gamma_{101}\gamma_{264} + 26\gamma_{107} + \gamma_{108}\gamma_{92} \\ - 109\gamma_{111} + \gamma_{131}\gamma_{264} - \gamma_{134} - 68\gamma_{138} - 39\gamma_{165} \\ - 29\gamma_{167} + \gamma_{176} + 34\gamma_{179} + \gamma_{184} + \gamma_{211} + 4\gamma_{217} \\ - 9\gamma_{220} - 48\gamma_{221} + \gamma_{223} + \gamma_{254} + \gamma_{26} - 31\gamma_{269} - \gamma_{272} \\ + \gamma_{300} + \gamma_{303} + \gamma_{317} - \gamma_{335} + \gamma_{346} - \gamma_{347} - \gamma_{348} - \gamma_{349} \\ + \gamma_{350} + \gamma_{351} + \gamma_{353} + \gamma_{354} + \gamma_{357} + 6\gamma_{38} + 14\gamma_{63} \\ + 78\gamma_{66} + 105\gamma_{69} + 23\gamma_{73} + \gamma_{90} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A_{3,3} &= \gamma_{204} \begin{pmatrix} 15\gamma_{10} + 5\gamma_{107} + \gamma_{114} + 24\gamma_{117} - \gamma_{154} + \gamma_{158} - \gamma_{159} \\ - \gamma_{231} - 21\gamma_{268} + 7\gamma_{269} + \gamma_{279} + \gamma_{339} - \gamma_{358} - \gamma_{359} \\ + \gamma_{360} - \gamma_{361} - \gamma_{363} + \gamma_{364} - \gamma_{365} + \gamma_{368} + \gamma_{370} + \gamma_{64} \end{pmatrix} \\ & + \gamma_{241} \begin{pmatrix} 24\varsigma\gamma_{33} + 41\varsigma\gamma_{35} + 79\varsigma\gamma_0\gamma_{12} + 119\varsigma\gamma_0\gamma_{21} \\ + 23\varsigma\gamma_{100}\gamma_{33} + 10\gamma_0\gamma_{100}\gamma_{12} + 25\gamma_0\gamma_{21} - 8\gamma_{105} \\ - \gamma_{114} - \gamma_{136} - 105\gamma_{138} - 18\gamma_{164} - 29\gamma_{165} - 42\gamma_{167} \\ - 14\gamma_{217} - 24\gamma_{221} - 4\gamma_{226} - \gamma_{23} - 24\gamma_{236} - \gamma_{251} \\ + \gamma_{276} - \gamma_{299} - \gamma_{316} - \gamma_{32} - \gamma_{321} - \gamma_{327} - \gamma_{346} - \gamma_{347} \\ - \gamma_{348} + \gamma_{371} + \gamma_{372} + \gamma_{373} + \gamma_{374} + \gamma_{375} - \gamma_{376} - \gamma_{377} \\ - \gamma_{378} - \gamma_{379} - 9\gamma_{46} - \gamma_{51} - \gamma_{76} \end{pmatrix} \\ & - \gamma_{333}\gamma_{93} \end{aligned}$$

$$\begin{aligned}
A_{3,4} = & 6\gamma_{101} - 29\gamma_{107} + \gamma_{13}\gamma_{180} + 12\gamma_{138} + \gamma_{193} - \gamma_{194} \\
& + 14\gamma_{218} - 34\gamma_{232} + \gamma_{245}\gamma_{332} + \gamma_{270} + \gamma_{324} + \gamma_{344} \\
& + \gamma_{350} - \gamma_{380} - \gamma_{381} - \gamma_{382} + \gamma_{383} + \gamma_{384} + \gamma_{385} \\
& - 58\gamma_{66} - 42\gamma_{69} - 14\gamma_{71}
\end{aligned}$$

Row 4

$$A_{4,1} = -\gamma_{389}w$$

$$-\gamma_{92} \begin{pmatrix} 20\zeta\gamma_{12} + 20\zeta\gamma_{33} + 6\zeta\gamma_{12}\gamma_{13} + 6\zeta\gamma_{13}\gamma_{35} + 2\zeta\gamma_{33} \\ + 7\gamma_0\gamma_{35} + 5\gamma_{12}\gamma_{13} + 15\gamma_{12}\gamma_{31} + \gamma_{12} - \gamma_{124} - \gamma_{126} \\ - 6\gamma_{127} - \gamma_{193} - \gamma_{202} + \gamma_{21} - 18\gamma_{216} - 8\gamma_{236} - \gamma_{242} \\ - \gamma_{265} - \gamma_{27} - \gamma_{274} - \gamma_{288} + \gamma_{291} - \gamma_{297} - \gamma_{325} - \gamma_{33} \\ - \gamma_{334} - \gamma_{336} - \gamma_{340} - \gamma_{35} - \gamma_{36} - \gamma_{370} - \gamma_{390} - \gamma_{391} \\ - \gamma_{392} - \gamma_{393} - \gamma_{44} - 11\gamma_{49} - 14\gamma_{66} - \gamma_{67}\gamma_{92} - 8\gamma_{67} \\ - 14\gamma_{69} - 8\gamma_{71} - 8\gamma_{73} - \gamma_{98} \end{pmatrix}$$

$$A_{4,2} = 8\gamma_0\gamma_{12}\gamma_{386}\gamma_{387}\gamma_5\gamma_6\gamma_{93}$$

$$-\gamma_{142} \begin{pmatrix} -\zeta\gamma_{380} - \gamma_{10} + 5\gamma_{101} + \gamma_{109} - 17\gamma_{111} + \gamma_{123} + \gamma_{13} \\ + \gamma_{138} - 5\gamma_{14} - 8\gamma_{179} - \gamma_{218} + 29\gamma_{23} - \gamma_{268} + \gamma_{274} \\ + \gamma_{296} + \gamma_{308} + \gamma_{322} + \gamma_{341} - \gamma_{342} + \gamma_{378} + \gamma_{394} + \gamma_{395} \\ + \gamma_{397} + \gamma_{398} - 34\gamma_{50} - 4\gamma_{67} + \gamma_{69} + 21\gamma_{70} + 17\gamma_{71} \\ + 13\gamma_{73} \end{pmatrix}$$

$$-\gamma_{183} \begin{pmatrix} 26\eta\gamma_{31} + 18\zeta\gamma_0\gamma_{33} + \zeta\gamma_{12}\gamma_{13} + 13\zeta\gamma_{12}\gamma_{31} \\ + 70\zeta\gamma_{13}\gamma_{21} + 44\zeta\gamma_{13}\gamma_{35} + 3\zeta\gamma_{21}\gamma_{31} + 2\zeta\gamma_{27}\gamma_{35} \\ + 33\zeta\gamma_{100}\gamma_{33} + 6\zeta\gamma_{100}\gamma_{35} + 22\zeta\gamma_{100}\gamma_{42} \\ + 17\gamma_0\gamma_{100}\gamma_{21} + 17\gamma_0\gamma_{100}\gamma_{33} + 27\gamma_0\gamma_{100}\gamma_{35} \\ - 13\gamma_1 - \gamma_{101}\gamma_{31} + \gamma_{103} - 6\gamma_{107} + 82\gamma_{12}\gamma_{31} \\ + 3\gamma_{13}\gamma_{35} - \gamma_{15} + \gamma_{150} - \gamma_{151} - \gamma_{156} + \gamma_{177} - \gamma_{201} \\ + 50\gamma_{21}\gamma_{31} - 51\gamma_{226} - 73\gamma_{23} - 14\gamma_{230} - 23\gamma_{236} \\ - \gamma_{253} - \gamma_{28}\gamma_{77} - \gamma_{28} - \gamma_{281}\gamma_{31} - \gamma_{302} + \gamma_{31}\gamma_{33} - \gamma_{34} \\ - \gamma_{357} - 31\gamma_{36} - 3\gamma_{38} - \gamma_{393} - \gamma_{398} - \gamma_{399} - 4\gamma_{40} \\ - \gamma_{401} - \gamma_{402} - \gamma_{403} - \gamma_{404} - \gamma_{405} - 15\gamma_{46} - 72\gamma_{49} \\ - 30\gamma_{66} - 50\gamma_{69} - 13\gamma_{71} - 7\gamma_{72} \end{pmatrix}$$

$$A_{4,3} = \gamma_{204} \begin{pmatrix} -\zeta\gamma_{290} + 4\gamma_1 - 17\gamma_{10} - 4\gamma_{114} - 25\gamma_{117} + 13\gamma_{14} - \gamma_{158} \\ + \gamma_{16} + \gamma_{172} - 5\gamma_{192} + 4\gamma_{194} + \gamma_{206} + \gamma_{231} + \gamma_{244} \\ + 23\gamma_{268} + \gamma_{311} + \gamma_{349} + \gamma_{359} - \gamma_{360} + \gamma_{361} + \gamma_{363} \\ - \gamma_{367} + \gamma_{394} + \gamma_{406} + \gamma_{407} - \gamma_{64} + \gamma_{67} - \gamma_{73} - 4\gamma_{77} \end{pmatrix}$$

$$+ \gamma_{241} \begin{pmatrix} 57\zeta\gamma_{12}\gamma_{31} + 82\zeta\gamma_{13}\gamma_{35} + 47\zeta\gamma_{21}\gamma_{31} - 17\gamma_0\gamma_{108} \\ + 18\gamma_0\gamma_{35} + 14\gamma_{100}\gamma_{12}\gamma_{13} + 3\gamma_{100}\gamma_{12}\gamma_{31} \\ + 51\gamma_{100}\gamma_{13}\gamma_{21} + 7\gamma_{100}\gamma_{13}\gamma_{33} - 17\gamma_{100}\gamma_{23} \\ + 14\gamma_{12}\gamma_{31} - \gamma_{133} - \gamma_{147} + \gamma_{152} - 5\gamma_{193} + 11\gamma_{21}\gamma_{31} \\ - \gamma_{210} + \gamma_{224} - 4\gamma_{229} - 5\gamma_{23} - 3\gamma_{230} - 33\gamma_{232} - 4\gamma_{235} \\ - 11\gamma_{236} - 27\gamma_{237} - \gamma_{240} + \gamma_{26} - \gamma_{273} - \gamma_{301} + \gamma_{319} \\ - \gamma_{320} + \gamma_{34} - \gamma_{352} - \gamma_{354} - 15\gamma_{36} - \gamma_{371} - \gamma_{372} - \gamma_{373} \\ - \gamma_{374} - \gamma_{375} + \gamma_{376} + \gamma_{377} - \gamma_{379} - \gamma_{382} - \gamma_{39} + \gamma_{399} \\ + \gamma_{402} + \gamma_{403} - \gamma_{405} - \gamma_{408} - \gamma_{409} - \gamma_{41} - 3\gamma_{49} - \gamma_{52} \\ - 96\gamma_{66} - 142\gamma_{69} - 9\gamma_{72} + \gamma_{83} - 8\gamma_{88} \end{pmatrix}$$

$$- \gamma_{389}\gamma_{93}$$

$$\begin{aligned}
A_{4,4} = & -\eta\gamma_{187} + \gamma_{101}\gamma_{255} + 9\gamma_{107} + \gamma_{128} - 24\gamma_{138} - 21\gamma_{164} \\
& - \gamma_{166} - 9\gamma_{179} + \gamma_{206} - 29\gamma_{218} + \gamma_{24} + \gamma_{246}\gamma_{388} \\
& - 42\gamma_{269} + \gamma_{295} + \gamma_{298} + \gamma_{326} + \gamma_{369} + \gamma_{381} - \gamma_{383} \\
& - \gamma_{384} + \gamma_{390} - \gamma_{401} + \gamma_{404} + \gamma_{409} + 28\gamma_{49} + 44\gamma_{66} \\
& + 30\gamma_{69} + 28\gamma_{71} + 26\gamma_{73}
\end{aligned}$$

Row 5

$$A_{5,1} = -\gamma_{140}\gamma_{411}\gamma_{92}(\varsigma + \gamma_{141})(\zeta + \gamma_{366} + \gamma_{410} + \gamma_{77})$$

$$\begin{aligned}
A_{5,2} = & -\gamma_{140}\gamma_{183}\gamma_5 \left(\begin{aligned} & 3\varsigma\gamma_{12}\gamma_{13} + 5\varsigma\gamma_{13}\gamma_{21} + 2\zeta\gamma_{21}\gamma_{412} - \zeta\gamma_{281} \\ & + 2\zeta\gamma_{35}\gamma_{412} - \gamma_{100}\gamma_{291} + 2\gamma_{12}\gamma_{13} - \gamma_{184} - \gamma_{247} \\ & - \gamma_{285} - \gamma_{310} - \gamma_{338} + \gamma_{35}\gamma_{412} - \gamma_{355} - \gamma_{395} - \gamma_{408} \\ & - \gamma_{414} - \gamma_{71} \end{aligned} \right) \\
& - \gamma_{142} \left(\begin{aligned} & -\gamma_{106} - \gamma_{107} + 2\gamma_{108} + 5\gamma_{111} + \gamma_{116} - 3\gamma_{117} \\ & - \gamma_{15}\gamma_{412} + \gamma_{171} + \gamma_{178} + \gamma_{189} + \gamma_{328} + \gamma_{364} + \gamma_{365} \\ & + \gamma_{397} + \gamma_{407} + \gamma_{412}\gamma_{47} + \gamma_{414} + \gamma_{415} - \gamma_{416} + \gamma_{418} \\ & + \gamma_{419} + \gamma_{43}\gamma_{50} - \gamma_{75} \end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
A_{5,3} = & -\gamma_{204} \left(\begin{aligned} & -\eta\gamma_{412} - \zeta\gamma_{100} - \gamma_{10}\gamma_{412} + \gamma_{196} - 3\gamma_{228} + \gamma_{287} + \gamma_3 \\ & + \gamma_{345} + \gamma_{368} + \gamma_{392} + \gamma_{400} + \gamma_{406} + \gamma_{410} + \gamma_{413} + \gamma_{415} \\ & + \gamma_{417} + \gamma_{420} + \gamma_{422} + \gamma_{423} \end{aligned} \right) \\
& - \gamma_{241}\gamma_{411} \left(\begin{aligned} & -\gamma_{228} - 3\gamma_{268} + \gamma_{307} + 3\gamma_{362} - 3\gamma_{371} + \gamma_{412}\gamma_{416} \\ & + \gamma_{418} + \gamma_{421} + \gamma_{422} - 3\gamma_{74} \end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
A_{5,4} = & 5\varsigma\gamma_{50} + \gamma_0\gamma_{100}\gamma_{263} + \gamma_0 + \gamma_{100}\gamma_{129} - 5\gamma_{105} \\
& - \gamma_{113}\gamma_{417} + \gamma_{119} + \gamma_{120} + \gamma_{143} - \gamma_{157} - \gamma_{165} + \gamma_{186} \\
& + \gamma_{261} + \gamma_{264}\gamma_{29} + \gamma_{266}\gamma_{412} - \gamma_{272}\gamma_{412} - \gamma_{315} \\
& + \gamma_{338} + \gamma_{351} + \gamma_{396} - \gamma_{412}\gamma_{424} + 2\gamma_{417} + \gamma_{419} + \gamma_{421} \\
& + \gamma_{423} + \gamma_{424} - \gamma_{59}
\end{aligned}$$

S9 The classical deformation gradients and pressure

The parameters and coordinates are those of Section 2.2. The gradient matrices below use orthonormal cylindrical bases when a cylindrical coordinate is present. The first row is radial and the third row axial. For the homogeneous family $F = L \in \text{SL}(3, \mathbb{R})$. For the block and straightening families, respectively,

$$F_1 = \begin{pmatrix} \alpha/r & 0 & 0 \\ 0 & \beta r & 0 \\ 0 & \gamma & 1/(\alpha\beta) \end{pmatrix}, \quad F_2 = \begin{pmatrix} \alpha R & 0 & 0 \\ 0 & \beta/R & 0 \\ 0 & \gamma/R & 1/(\alpha\beta) \end{pmatrix}.$$

Their determinants are one. For the cylindrical family,

$$F_3 = \begin{pmatrix} AR/r & 0 & 0 \\ 0 & kr/R & \delta r \\ 0 & \kappa/R & \gamma \end{pmatrix}, \quad \det F_3 = A(k\gamma - \delta\kappa) = 1.$$

For the spherical family, after the fixed orthogonal map Q , the gradient has radial value $\varepsilon R^2/r^2$ and tangential values $r/R, r/R$. Its signed determinant is $\varepsilon \det Q = 1$. For the logarithmic cylindrical family,

$$F_5 = \begin{pmatrix} \alpha & 0 & 0 \\ \alpha\beta & \alpha k & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \det F_5 = \alpha^2 k \gamma = 1.$$

Thus each matrix has the prescribed signed Jacobian.

For the block and cylindrical families, $B = FF^\top$ and $c = B^{-1}$ have no radial off-diagonal entries and their physical components depend only on r . The local pressure derivative is

$$\frac{dp_W}{dr} = 2 \frac{d}{dr} (W_1 B_{rr} - W_2 c_{rr}) + \frac{2}{r} \{W_1 (B_{rr} - B_{\theta\theta}) - W_2 (c_{rr} - c_{\theta\theta})\}. \quad (\text{S9.1})$$

For the spherical family the final factor $2/r$ becomes $4/r$. For straightening, the tensors have no first-coordinate off-diagonal entries and depend only on x ; the pressure is $p_W = 2(W_1 B_{xx} - W_2 c_{xx})$ up to a constant.

The logarithmic cylindrical family has constant principal invariants. In the current cylindrical frame,

$$B = \begin{pmatrix} \alpha^2 & \alpha^2 \beta & 0 \\ \alpha^2 \beta & \alpha^2 (\beta^2 + k^2) & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix}, \quad c = \begin{pmatrix} (\beta^2 + k^2)/(\alpha^2 k^2) & -\beta/(\alpha^2 k^2) & 0 \\ -\beta/(\alpha^2 k^2) & 1/(\alpha^2 k^2) & 0 \\ 0 & 0 & \gamma^{-2} \end{pmatrix}.$$

Its local pressure one-form is

$$dp_W = 2\{W_1 (B_{rr} - B_{\theta\theta}) - W_2 (c_{rr} - c_{\theta\theta})\} d \log r + 4(W_1 B_{r\theta} - W_2 c_{r\theta}) d\theta.$$

The coefficients are constant for each fixed energy W . These formulas give the pressure outputs used in Proposition 9.7.

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Declarations

Competing interests. The author declares no competing interests.

Code availability. The SageMath source code, exact coefficient data and recorded outputs are stored at <https://github.com/Jingbei-Bai/Ericksen-complete>. The repository includes an environment lock for SageMath 10.9 and instructions for reproducing the calculations.

Author contributions (CRediT). Jingbei Bai: Conceptualization, Methodology, Formal analysis, Software, Validation, Writing – original draft, Writing – review & editing.

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