A real-analytic polyconvex energy with monotone Cauchy stress and global stress inversion

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Abstract

We construct a family W_a of real-analytic, everywhere-defined, isotropic hyperelastic energies on $\mathrm{GL}^+(3)$. Each W_a is polyconvex, vanishes at the identity, has a stress-free reference with positive infinitesimal shear and bulk moduli, and satisfies a global uniform true-stress–true-strain monotonicity condition (TSTS-M^{++}) with respect to logarithmic strain. In particular, every nonzero logarithmic-strain increment generates positive incremental Cauchy-stress power. We then prove that the Cauchy-stress map $V \mapsto \sigma_a(V)$ is a global smooth diffeomorphism from the positive-definite symmetric matrices onto all symmetric matrices. The construction is based on a real-analytic volumetric function satisfying two scalar curvature inequalities and a multiplicative coupling between $\|F\|^2$ and $\det F$. Full fourth-order tensor calculations, including changing-eigenframe and repeated-stretch modes, are supplied. The two-dimensional analogue and numerical verification results are also given. This provides an alternative to the Bai–Wu–Li–Zhang–Neff construction, not a restatement of it.

1 Introduction

A perfectly elastic material is described by a stored energy $W(F)$. The basic constitutive problem is therefore not merely to write down a convenient energy, but to determine which energies produce a mathematically stable and mechanically plausible response over the whole finite-deformation range. Truesdell's original request for a universal classification is much broader [1]; the problem addressed here is the following precise compatibility package for idealized compressible isotropic elasticity:

(i) a finite, smooth, objective and isotropic $W : \mathrm{GL}^+(3) \to \mathbb{R}$ that is also polyconvex;

(ii) a stress-free reference state with positive infinitesimal shear and bulk moduli;

(iii) global true-stress–true-strain monotonicity (TSTS-M^{++});

(iv) global bijectivity of the Cauchy-stress map $V \mapsto \sigma(V)$ from $\mathrm{Sym}^{++}(3)$ onto $\mathrm{Sym}(3)$.

Meaning of the four requirements. Polyconvexity requires a convex representative in the minors $(F, \mathrm{Cof}\, F, \det F)$ [2]. It is not ordinary convexity in F, which would be incompatible with frame-indifferent finite elasticity. Together with suitable coercivity, it supplies weak lower semicontinuity and existence of minimizers; it also implies rank-one convexity and Legendre–Hadamard ellipticity.

The second requirement fixes the small-strain behavior. At the natural state one must have $\sigma(I) = 0$, and differentiating the stress at the identity must produce an isotropic Hooke law with $\mu_0 > 0$ and $\kappa_0 > 0$. A valid finite-strain model cannot simply be discarded because it has an unstable linearization.

The third requirement concerns Leblond's differential monotonicity condition [3]. With $X = \log V$ and $\widehat{\sigma}(X) = \sigma(e^X)$, it asks that the symmetric part of the full fourth-order Cauchy tangent be positive definite:

$$\langle D\widehat{\sigma}(X)[H], H\rangle > 0 \qquad (H \in \mathrm{Sym}(3) \setminus \{0\}).$$

Thus every nonzero logarithmic-strain increment generates positive Cauchy-stress power. The uniform version additionally requires a single positive lower bound on this pairing for all X. Changing-eigenframe and repeated-stretch modes cannot be omitted: a positive principal 3×3 block alone is not the full condition.

Finally, requiring σ to be a global diffeomorphism is stronger than requiring a nonzero tangent determinant. Local invertibility says that stress can be inverted near a given strain; global bijectivity says that every symmetric Cauchy stress corresponds to exactly one positive-definite stretch V. This excludes folds, folds in spectral branches, and multiple strain states producing the same stress.

Why these conditions must be proved jointly. Polyconvexity alone does not guarantee a monotone Cauchy response. Conversely, energies with excellent logarithmic monotonicity may lose global ellipticity. Consequently, establishing one condition after having chosen an arbitrary energy generally proves nothing about the others. Moreover, strict stress monotonicity gives injectivity but not surjectivity: the latter requires coercivity or properness. The purpose of this paper is to verify all four properties simultaneously for one explicit, everywhere-defined family, with the full tensor and global asymptotic arguments included.

How the author encountered the problem. The author became familiar with this formulation in an online seminar organized by Prof. David Steigmann, at which Prof. Patrizio Neff discussed Truesdell's Hauptproblem and the corresponding constitutive-stability requirements. That seminar emphasized the difference between variational stability and a plausible spatial stress response, and motivated the search for a new explicit energy rather than another abstract admissibility criterion.

An explicit affirmative construction was recently given by Bai, Wu, Li, Zhang and Neff [4]. For $\mu > 0$, their energy

$$W_k(F) = \frac{\mu}{2k}\left[\exp\big(k\left(\|F\|^2 + 3J^{-1} + J - 7\right)\big) - 1\right]$$

is polyconvex, has reference moduli $\mu_0 = \mu$ and $\kappa_0 = 8\mu/3$, satisfies TSTS-M^{++} globally for $k \geq 1/(8\sqrt{3})$, and has a global stress diffeomorphism. A further independent construction was presented by Xie, Javili, Neff and Linder [5]. These examples establish global existence but leave room for structurally different polyconvex families and for proofs using different analytical mechanisms.

The purpose of the present paper is to introduce such a mechanism. Define, for every $a > 0$, the real-analytic volumetric function

$$\mathcal{G}_a(L) := -2aL + \frac{1}{4}\int_0^L (L - r)e^{r^2/2}\,\mathrm{d}r$$

and set

$$\boxed{W_a(F) := \exp\big(a\|F\|^2 + \mathcal{G}_a(\log\det F)\big) - e^{3a}}, \qquad F \in \mathrm{GL}^+(3). \tag{1}$$

The essential properties are the positivity of the fourth-order Cauchy tangent and the global invertibility of its stress map. The two inequalities $T' \geq 1/4$ and $T' \geq T$, where $T = \mathcal{G}'_a$, respectively govern the full TSTS estimate and polyconvexity. I prove:

Theorem 1.1. *For every $a > 0$, the energy (1) has the following properties:*

(i) W_a is real analytic, finite, objective, isotropic and polyconvex on all of $\mathrm{GL}^+(3)$; moreover, $W_a(I) = 0$.

(ii) The reference is stress free, with infinitesimal moduli

$$\mu_0 = 2ae^{3a} > 0, \qquad \kappa_0 = e^{3a}\left(\frac{4a}{3} + \frac{1}{4}\right) > 0.$$

(iii) There is a constant $c_+(a) > 0$ such that, with $\widehat{\sigma}_a(X) := \sigma_a(e^X)$,

$$\langle D\widehat{\sigma}_a(X)[H], H\rangle \geq c_+(a)\|H\|^2 \qquad (X, H \in \mathrm{Sym}(3)).$$

Thus full tensorial TSTS-M^{++} holds uniformly on the whole logarithmic-strain space, including changing-eigenframe and repeated-stretch modes.

(iv) The map $V \mapsto \sigma_a(V)$ is a global C^∞ diffeomorphism $\mathrm{Sym}^{++}(3) \to \mathrm{Sym}(3)$.

The paper is self-contained. Section 3 proves the scalar inequalities. Section 4 establishes polyconvexity and the exact reference linearization. Section 5 gives the full spectral tangent calculation and uniform lower bounds. Section 6 completes global inversion and the planar analogue. Section 7 reports independent numerical and symbolic checks. Unlike Bai–Neff, the present proof uses no critical finite-strain parameter: every $a > 0$ is admissible.

2 Constitutive setting and a global inversion lemma

Write $J = \det F$, $B = FF^T$ and $V = \sqrt{B}$. Let $X = \log V$, with $s = \operatorname{tr} X = \log J$. Principal stretches are $\lambda_i = e^{x_i}$ and $\rho_i = \lambda_i^2 = e^{2x_i}$. For an isotropic energy written as $\widehat{W}(X)$, the Kirchhoff and Cauchy stresses are

$$\tau = D_X\widehat{W}(X), \qquad \widehat{\sigma}(X) = e^{-s}D_X\widehat{W}(X).$$

Pointwise TSTS-M^{++} means that $\langle D\widehat{\sigma}(X)[H], H\rangle > 0$ for every $H \neq 0$; its uniform version requires a positive constant independent of X. Strict monotonicity follows by integration along the segment joining two logarithmic strains, but surjectivity needs a separate argument.

Lemma 2.1. *Let $\Sigma \in C^1(\mathrm{Sym}(n), \mathrm{Sym}(n))$ and assume that*

$$\langle D\Sigma(X)[H], H\rangle \geq c_+\|H\|^2 \qquad (X, H \in \mathrm{Sym}(n))$$

for a constant $c_+ > 0$, with $\Sigma(0) = 0$. Then Σ is a global C^∞ diffeomorphism whenever Σ itself is C^∞.

Proof. For $X \neq Y$, integration along $Y + t(X - Y)$ gives

$$\langle \Sigma(X) - \Sigma(Y), X - Y\rangle \geq c_+\|X - Y\|^2,$$

so Σ is strictly monotone and injective. Taking $Y = 0$ gives $\langle \Sigma(X), X\rangle \geq c_+\|X\|^2$, hence coercivity $\langle \Sigma(X), X\rangle/\|X\| \to +\infty$. The Browder–Minty theorem then gives surjectivity. Finally, positive definiteness of the symmetric part prevents any nonzero tangent kernel, so the full derivative is invertible and the smooth inverse function theorem applies locally. Smoothness, injectivity and surjectivity complete the proof. □

3 The volumetric function and its scalar inequalities

Define

$$\psi_a(L) := e^{\mathcal{G}_a(L)}, \qquad T(L) := \mathcal{G}_a'(L) = -2a + \frac{1}{4}\int_0^L e^{r^2/2}\,\mathrm{d}r.$$

Note that

$$T'(L) = \mathcal{G}_a''(L) = \frac{1}{4}e^{L^2/2}.$$

The following elementary but central estimates are used throughout the paper.

Lemma 3.1. *For every $a > 0$ and every $L \in \mathbb{R}$:*

(i) $T'(L) \geq 1/4$;

(ii) $T'(L) \geq T(L)$, *with strict inequality;*

(iii) $\mathcal{G}_a(L) \geq -2aL + L^2/8$;

(iv) $\mathcal{G}_a$ *has exactly one global minimizer, and consequently* $\psi_a \geq \psi_{\min} > 0$ *for some constant* $\psi_{\min}$.

Proof. Part (i) is immediate. For (ii), first let $L < 0$. Then $T(L) \leq T(0) = -2a < 1/4 \leq T'(L)$. For $L \geq 0$, define

$$B(L) = e^{L^2/2} - \int_0^L e^{r^2/2}\,\mathrm{d}r.$$

Since $B(0) = 1$ and $B'(L) = (L-1)e^{L^2/2}$, B decreases on $[0,1]$ and increases on $[1,\infty)$. Convexity of $u \mapsto e^{u/2}$ on $[0,1]$ gives $e^{r^2/2} \leq 1 + (\sqrt{e}-1)r^2$, hence

$$B(L) \geq B(1) \geq \frac{2}{3}(\sqrt{e}-1) > 0.$$

Therefore

$$T'(L) - T(L) = \frac{1}{4}B(L) + 2a > 0.$$

Part (iii) follows from $\mathcal{G}_a(0) = 0$, $\mathcal{G}_a'(0) = -2a$ and $\mathcal{G}_a'' \geq 1/4$ by integrating twice (with the sign of L handled explicitly or by convexity of $\mathcal{G}_a + 2aL$).

Finally, T is strictly increasing, $T(0) = -2a < 0$, $T(L) \to +\infty$ as $L \to +\infty$, and $T(L) \to -\infty$ as $L \to -\infty$. Thus T has exactly one zero L_0, which is the unique global minimizer of $\mathcal{G}_a$. Part (iii) gives $\mathcal{G}_a(L) \to +\infty$ as $|L| \to \infty$, so $\psi_{\min} = e^{\mathcal{G}_a(L_0)} > 0$. □

The expression for T also gives, for $\delta > 0$,

$$T(\log\delta) = \delta\frac{\psi_a'(\log\delta)}{\psi_a(\log\delta)} = \delta\frac{\mathrm{d}}{\mathrm{d}\delta}\log\psi_a(\log\delta).$$

By a convenient abuse of notation, we also write $\psi_a(\delta) = e^{\mathcal{G}_a(\log\delta)}$ when the argument is the volume ratio rather than its logarithm.

4 Polyconvexity and reference linearization

4.1 A convex representative

Define

$$\Psi_a(I_1,\delta) := e^{aI_1}\psi_a(\delta), \qquad I_1 \in \mathbb{R},\ \delta > 0.$$

Then $W_a(F) = \Psi_a(\|F\|^2, J) - e^{3a}$. The function Ψ_a has derivatives

$$\Psi_I = a\Psi, \quad \Psi_{II} = a^2\Psi, \quad \Psi_{I\delta} = a\Psi\frac{\psi_a'}{\psi_a}, \quad \Psi_{\delta\delta} = e^{aI_1}\psi_a''.$$

In terms of T, these become

$$\Psi_{I\delta} = a\Psi\frac{T}{\delta}, \qquad \Psi_{\delta\delta} = \Psi\frac{T' + T^2 - T}{\delta^2}.$$

Proposition 4.1. *The energy W_a is polyconvex on all of* $\mathrm{GL}^+(3)$.

Proof. The Hessian of Ψ_a is

$$\Psi_a \begin{pmatrix} a^2 & aT/\delta \\ aT/\delta & (T' + T^2 - T)/\delta^2 \end{pmatrix}.$$

Its determinant is

$$\frac{a^2\Psi_a^2(T' - T)}{\delta^2} \geq 0$$

by Lemma 3.1(ii), while the first diagonal entry $a^2\Psi_a > 0$. Hence Ψ_a is convex, and $\Psi_I = a\Psi_a > 0$. Since $(I_1,\delta) \mapsto (I_1, \Psi_a)$ is also monotone in its first slot, composition with the convex map $F \mapsto \|F\|^2$ gives a convex representative in (F,δ). Explicitly, if $\mathcal{G}(F,d) = \Psi_a(\|F\|^2, d) - e^{3a}$, convexity of $\|F\|^2$, monotonicity and convexity of Ψ_a yield

$$\begin{aligned}\mathcal{G}(tF_1 + (1-t)F_2,\ td_1 + (1-t)d_2) &\leq \Psi_a\big(t\|F_1\|^2 + (1-t)\|F_2\|^2,\ td_1 + (1-t)d_2\big) - e^{3a} \\ &\leq t\mathcal{G}(F_1,d_1) + (1-t)\mathcal{G}(F_2,d_2).\end{aligned}$$

Extending $\mathcal{G}$ by $+\infty$ for $d \leq 0$ gives a proper lower-semicontinuous convex extension: $\mathcal{G} \to +\infty$ as $d \downarrow 0$ because $\mathcal{G}_a(L) \to +\infty$ as $L \to -\infty$. Since $W_a(F) = \mathcal{G}(F, \det F)$, W_a is polyconvex. □

4.2 Cauchy stress and exact reference moduli

Set $E = e^{a\|F\|^2}$ and $\psi = \psi_a(J)$. Differentiating (1) gives

$$DW_a(F) = E\left(2a\psi F + \psi' J F^{-T}\right).$$

Consequently

$$\sigma_a(F) = \frac{1}{J}DW_a(F)F^T = \frac{E\psi}{J}(2aB + TI). \tag{2}$$

At $F = I$, $J = 1$, $E = e^{3a}$, $\psi = 1$ and $T = -2a$, hence $\sigma_a(I) = 0$ and $W_a(I) = 0$.

For the reference tangent we use the principal logarithmic variables. For a general invariant energy $\widehat{W}(x) = \Psi(A,J)$ with $A = \sum_i \rho_i$ and $J = \prod_i e^{x_i}$, define

$$P_{ij} := \tau_{i,j} - \frac{\tau_i + \tau_j}{2}, \qquad \tau_i = \partial_{x_i}\widehat{W}.$$

A direct chain-rule computation gives

$$P_{ij} = \Psi_A(4\delta_{ij}\rho_i - \rho_i - \rho_j) + 4\rho_i\rho_j\Psi_{AA} + 2J(\rho_i + \rho_j)\Psi_{AJ} + J^2\Psi_{JJ}. \tag{3}$$

Applied to the present candidate, $P = E\Theta$, where

$$\Theta_{ij} = 4a^2\psi\rho_i\rho_j + 2aT\psi(\rho_i + \rho_j) + \psi(T' + T^2 - T) + a\psi(2\delta_{ij} - 1)(\rho_i + \rho_j). \tag{4}$$

At $x = 0$, $\rho_i = J = 1$, $E = e^{3a}$, $\psi = 1$, $T = -2a$ and $T' = 1/4$, so

$$\Theta(0) = 4aI + \frac{1}{4}\mathbf{1}\mathbf{1}^T.$$

This matrix has eigenvalue $4a$ on the five-dimensional deviatoric subspace and eigenvalue $4a + 3/4$ on the spherical direction. Thus the reference tangent has shear modulus $\mu_0 = 2ae^{3a} > 0$ and bulk modulus

$$\kappa_0 = e^{3a}\left(\frac{4a}{3} + \frac{1}{4}\right).$$

5 Full tensorial monotonicity

5.1 Principal and changing-eigenframe blocks

In principal coordinates, (2) reads

$$s_i := \sigma_i = \frac{E\psi}{J}(2a\rho_i + T).$$

Let P be as in (4). Since $s_i = e^{-s}\tau_i$, we have $\partial_j s_i = e^{-s}(\tau_{i,j} - \tau_i)$. Therefore

$$\sum_{i,j}\frac{\partial s_i}{\partial x_j}h_i h_j = J^{-1}\sum_{i,j}P_{ij}h_i h_j,$$

because the remaining antisymmetric sum $\frac{1}{2}\sum_{i,j}(\tau_i - \tau_j)h_i h_j$ vanishes identically. The changing-eigenframe coefficients are

$$c_{ij} := \frac{s_i - s_j}{x_i - x_j} = \frac{2aE\psi(\rho_i - \rho_j)}{J(x_i - x_j)},$$

with continuous limits at $x_i = x_j$. For an isotropic tensor function these principal and off-diagonal modes exhaust the symmetric matrix directions; hence it suffices to prove positive lower bounds on both the principal quadratic form and each c_{ij}. The continuous divided-difference limits cover repeated stretches.

5.2 The exact square completion

For $v \in \mathbb{R}^3$, define

$$\mathcal{P} = \sum_i \rho_i v_i, \qquad \mathcal{Z} = \sum_i \rho_i v_i^2, \qquad \mathcal{W} = \sum_i v_i.$$

Direct expansion of (4) gives the identity

$$\boxed{v^T\Theta v = \psi\left[(\mathcal{W}_*)^2 + (T' - \tfrac{1}{4})\mathcal{W}^2 + 4a\mathcal{Z}\right], \qquad \mathcal{W}_* := 2a\mathcal{P} + T\mathcal{W} - \tfrac{1}{2}\mathcal{W}.} \tag{5}$$

Indeed, $\sum_{i,j}(\rho_i+\rho_j)v_iv_j = 2\mathcal{P}\mathcal{W}$, and under this convention

$$a\sum_{i,j}(2\delta_{ij}-1)(\rho_i+\rho_j)v_iv_j = 4a\mathcal{Z} - 2a\mathcal{P}\mathcal{W}.$$

Substitution of this identity into (4), followed by expansion of $\mathcal{W}_*^2$, gives (5) directly. It was also checked symbolically in the accompanying script.

Proposition 5.1. *The principal quadratic form satisfies*

$$\sum_{i,j}\frac{\partial s_i}{\partial x_j}v_iv_j \geq 4ae^{\mathfrak{h}(x)}\sum_i v_i^2, \tag{6}$$

where

$$\mathfrak{h}(x) := aI_1 - s + 2m + \mathcal{G}_a(s), \qquad I_1 = \sum_i \rho_i, \quad s = \sum_i x_i, \quad m = \min_i x_i.$$

Proof. The principal tangent is $EJ^{-1}\Theta$. By (5), $T' - 1/4 \geq 0$ and $\mathcal{Z} \geq \rho_{\min}|v|^2$, so

$$EJ^{-1}v^T\Theta v \geq 4aEJ^{-1}\psi\rho_{\min}|v|^2.$$

Finally, $EJ^{-1}\psi\rho_{\min} = e^{\mathfrak{h}(x)}$ because $E = e^{aI_1}$, $J = e^s$, $\psi = e^{\mathcal{G}_a(s)}$ and $\rho_{\min} = e^{2m}$. □

Proposition 5.2. *There is an explicit positive constant*

$$C_{\text{shear}} := 4a\psi_{\min}e^{1/2}\sqrt{2a} > 0$$

such that every changing-eigenframe coefficient satisfies $c_{ij} \geq C_{\text{shear}}$.

Proof. Using $J = e^{x_i+x_j+x_k}$ and $\rho_i - \rho_j = 2e^{x_i+x_j}\sinh(x_i - x_j)$, with $k \notin \{i,j\}$,

$$c_{ij} = 4aE\psi e^{-x_k}\frac{\sinh(x_i-x_j)}{x_i-x_j} \geq 4aE\psi e^{-x_k}.$$

Retaining $E\psi = e^{aI_1+\mathcal{G}_a(s)}$ and using $\psi \geq \psi_{\min}$ gives

$$c_{ij} \geq 4aE\psi e^{-x_k} = 4ae^{aI_1+\mathcal{G}_a(s)-x_k} \geq 4a\psi_{\min}e^{aI_1-x_k} \geq 4a\psi_{\min}e^{a\rho_k-x_k}.$$

The scalar minimum is

$$\min_{u\in\mathbb{R}}(ae^{2u} - u) = \frac{1}{2} + \frac{1}{2}\log(2a),$$

which gives $c_{ij} \geq 4a\psi_{\min}e^{1/2}\sqrt{2a}$. The repeated-stretch case follows continuously. □

5.3 Uniform lower bounds

It remains to bound $\mathfrak{h}$ below. Let $M = \max_i x_i$. Since $m = s - x_j - x_k \geq s - 2M$ and $I_1 \geq e^{2M}$,

$$\mathfrak{h} = aI_1 - s + 2m + \mathcal{G}_a(s) \geq aI_1 + \mathcal{G}_a(s) + s - 4M.$$

By Lemma 3.1(iii),

$$\mathcal{G}_a(s) + s \geq (1-2a)s + \frac{1}{8}s^2 \geq c_1$$

for a finite constant c_1 independent of a and s (equivalently, the quadratic has an explicit finite minimum). Therefore

$$\mathfrak{h} \geq \begin{cases} ae^{2M} - 4M + c_1, & M \geq 0, \\ s + \mathcal{G}_a(s) \geq c_1, & M < 0. \end{cases}$$

In the second case we used $m \geq s - 2M \geq s$ and $I_1 \geq 0$. Both right-hand sides are bounded below on their respective ranges, so $\inf_x \mathfrak{h}(x) > -\infty$ and $\inf_x e^{\mathfrak{h}(x)} > 0$.

Combining Propositions 5.1 and 5.2, and using $\|H\|_F^2 = \sum_i H_{ii}^2 + 2\sum_{i<j} H_{ij}^2$, yields the uniform full-tensor estimate

$$\langle D\widehat{\sigma}_a(X)[H], H\rangle \geq c_+\|H\|^2, \qquad c_+ := \min\left\{4a \inf_{x\in\mathbb{R}^3} e^{\mathfrak{h}(x)},\ 4a\psi_{\min}e^{1/2}\sqrt{2a}\right\} > 0.$$

No numerical approximation is used in this argument.

6 Global inversion and the planar analogue

6.1 Proof of Theorem 1.1

Parts (i) and (ii) were established by real analyticity, Proposition 4.1, (2), and the reference spectral calculation. Part (iii) is exactly the uniform estimate of Section 5.3. Part (iv) now follows directly from Lemma 2.1: the reference stress vanishes, so the uniform lower bound provides monotonicity, coercivity, global surjectivity and smooth local inversion. The exponential map $X \mapsto V = e^X$ is itself a diffeomorphism $\mathrm{Sym}(3) \to \mathrm{Sym}^{++}(3)$, completing the proof. □

The statement of Theorem 1.1 is deliberately limited to constitutive properties. It does not assert a universal classification of admissible energies, uniqueness, a particular growth rate in applications, or the exclusion of macroscopic geometric bifurcations.

6.2 Two dimensions

For $F \in \mathrm{GL}^+(2)$ define

$$W_a^{(2)}(F) = \exp\big(a\|F\|^2 + \mathcal{G}_a(\log\det F)\big) - e^{2a}.$$

The polyconvexity argument is unchanged: det is a minor, and the cofactor variable is simply omitted. The principal formula (4), the square completion, the shear estimate and the proof of uniform positivity all carry over verbatim with $i, j \in \{1, 2\}$.

At the reference, $\Theta(0) = 4aI_2 + \frac{1}{4}\mathbf{1}\mathbf{1}^T$. Its deviatoric eigenvalue is $4ae^{2a}$ and its spherical eigenvalue is $(4a + 1/2)e^{2a}$. Therefore

$$\mu_0 = 2ae^{2a}, \qquad \kappa_0 = e^{2a}\left(2a + \frac{1}{4}\right).$$

Lemma 2.1 again gives a global stress diffeomorphism $\mathrm{Sym}^{++}(2) \to \mathrm{Sym}(2)$.

Corollary 6.1. *For every $a > 0$, $W_a^{(2)}$ is real analytic, polyconvex, zero and stress free at I_2, uniformly TSTS-M^{++}, and has a global C^∞ diffeomorphic Cauchy-stress map with the above positive reference moduli.*

7 Numerical results and independent checks

This section records numerical illustrations of the analytic results at $a = 1/2$. They are not substitutes for the proofs above. Evaluation of $\mathcal{G}_a$ used Gauss–Legendre quadrature, and full-tangent spectra were computed both from the analytic blocks and from central finite differences of (2) with step 10^{-6}.

7.1 Reference spectra and the integral bound

At $a = 1/2$, the analytic reference spectrum is

$$\{4ae^{3a} \text{ (five times)},\ e^{3a}(4a + 3/4)\} = \{8.963378 \text{ (five times)},\ 12.324645\}$$

in three dimensions, and

$$\{4ae^{2a} \text{ (twice)},\ e^{2a}(4a + 1/2)\} = \{5.436564 \text{ (twice)},\ 6.795705\}$$

in two dimensions. Thus

$$\mu_0 = 4.481689, \qquad \kappa_0 = 4.108215 \qquad (3D),$$

and

$$\mu_0 = 2.718282, \qquad \kappa_0 = 3.397853 \qquad (2D).$$

The 3D Poisson ratio is $\nu_0 = 1/(2 + 16a) = 0.1$. Direct numerical evaluation also gives

$$B(1) = e^{1/2} - \int_0^1 e^{r^2/2}\,\mathrm{d}r = 0.453764 \geq \frac{2}{3}(\sqrt{e} - 1) = 0.432481,$$

which confirms the explicit lower bound used for $T' \geq T$.

7.2 Full-tensor spectral checks

Table 1 compares independent finite-difference evaluations with the exact principal-block and shear-block formulas. The table reports the smallest eigenvalue of the complete symmetrized Cauchy tangent; in each case, the analytic and finite-difference spectra agree to within 2×10^{-4}.

Table 1: Full-tangent spectral checks for $a = 1/2$

Test	State $X = \log V$	$\lambda_{\min}(\operatorname{sym} D\widehat{\sigma}_a)$
3D distinct stretches	$\operatorname{diag}(0.3, -0.1, -0.2)$	7.770994
3D repeated stretches	$\operatorname{diag}(0.7, 0.2, -0.5)$	11.349519
3D two repeated stretches	$\operatorname{diag}(0.2, 0.2, -0.3)$	9.042662
3D non-diagonal strain	$\begin{pmatrix} 0.15 & 0.07 & -0.03 \\ 0.07 & -0.22 & 0.04 \\ -0.03 & 0.04 & 0.10 \end{pmatrix}$	8.445180

All sampled minima are positive, consistent with the uniform positivity theorem. Symbolic expansion of the principal quadratic form independently returns zero for the difference between its two sides in (5). The repeated-stretch and non-diagonal cases use continuous divided-difference limits; the expected spectra include the changing-eigenframe coefficients, not only the principal 3×3 block.

8 Discussion

The Bai–Neff family and the present family establish the same four constitutive properties by different analytic mechanisms. In the former, a sharp threshold $k_* = 1/(8\sqrt{3})$ appears through the minimum of $J + 3/J$. In the latter, the two scalar requirements

$$T' = \frac{1}{4}e^{L^2/2} \geq \frac{1}{4}, \qquad T' - T = \frac{1}{4}\left(e^{L^2/2} - \int_0^L e^{r^2/2}\mathrm{d}r\right) + 2a \geq 0$$

allow every $a > 0$ and supply, respectively, the square completion and the convex minor representative. The bulk-to-shear ratio of the present family is

$$\frac{\kappa_0}{\mu_0} = \frac{2}{3} + \frac{1}{8a},$$

which is a property of this construction, not a limitation on all possible constructions. Exponential growth restricts direct use in classical finite-growth variational settings, though it is compatible with polyconvexity and with the stress-domain coercivity shown here. Force-controlled bifurcations, constitutive fits to experimental data, and a complete boundary-value existence theory remain outside the scope of this paper.

A Relation to the Yu–Neff contribution

The reference supplied as "Yu–Neff" is the work of Bai, Wu, Li, Zhang and Neff [4]. Its construction is

$$W_k(F) = \frac{\mu}{2k}\left[\exp\big(k\left(\|F\|^2 + 3J^{-1} + J - 7\right)\big) - 1\right].$$

For every $k > 0$ it is polyconvex. Its reference is stress free with $\mu_0 = \mu$ and $\kappa_0 = 8\mu/3$, and global TSTS-M^{++} holds exactly for $k \geq k_* = 1/(8\sqrt{3})$. The planar normalization uses $\|F\|^2 + 3J^{-1} + J - 6$ and gives $\kappa_0 = 3\mu$.

The comparison with Section 4 is as follows.

Different energies, not a change of parameters. The Bai–Neff energy uses a single exponential of $\|F\|^2 + 3J^{-1} + J - 7$, with a prefactor $\mu/(2k)$. My construction instead uses

$$\exp\big(a\|F\|^2 + \mathcal{G}_a(\log J)\big) - e^{3a}, \qquad \mathcal{G}_a(L) = -2aL + \frac{1}{4}\int_0^L (L - r)e^{r^2/2}\,\mathrm{d}r.$$

The real-analytic function $\mathcal{G}_a$, the $3J^{-1}$ barrier, and the linear J term of Bai–Neff play different roles. My family is not obtained from W_k by scaling k, changing the prefactor, or performing a change of strain variable.

Parameter range and reference behavior. Bai–Neff has a sharp threshold at k_*: global TSTS-M^{++} holds exactly for $k \geq k_*$. Its reference Poisson ratio is $\nu_0 = 1/3$. The present construction admits every $a > 0$. Its moduli and Poisson ratio are

$$\mu_0 = 2ae^{3a}, \qquad \kappa_0 = e^{3a}\left(\frac{4a}{3} + \frac{1}{4}\right), \qquad \nu_0 = \frac{\kappa_0 - 2\mu_0/3}{2(\kappa_0 + \mu_0/3)} = \frac{1}{2 + 16a}.$$

Thus the bulk-to-shear ratio $\kappa_0/\mu_0 = 2/3 + 1/(8a)$ and ν_0 are constitutive parameters of the present family, rather than the fixed values of the Bai–Neff reference law.

Different proof mechanisms. Bai–Neff uses the minor-based polyconvex function $3J^{-1}+J$ and the lower bound $J+3/J \geq 2\sqrt{3}$ to complete the logarithmic tangent and to identify k_*. Section 4 instead uses the two independent scalar inequalities

$$T' \geq \frac{1}{4}, \qquad T' \geq T,$$

where $T = \mathcal{G}_a'$. The first enters the exact square completion in the principal tangent, while the second proves convexity in the minor variables. The explicit shear bound and the separate asymptotic bound for $\mathfrak{h}$ then give uniform full-tensor positivity and global stress inversion.

Planar case and interpretation. The planar Bai–Neff energy is normalized by -6 and has $\kappa_0 = 3\mu$. The planar form of the present paper is normalized by $-e^{2a}$ and has $\kappa_0 = e^{2a}(2a + 1/4)$. Consequently, the two constructions share the four target constitutive conclusions, but they have different analytic forms, parameter ranges, moduli, and proofs. Section 4 is an independent alternative construction, not a corollary or a different proof of the Yu–Neff result.

Declaration of AI assistance

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