The strain-energy function for combined bending and stretching of incompressible isotropic elastic plates*

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*Dedicated to the memory of E. S. Şuhubi

Abstract The strain-energy function of a thin plate composed of an incompressible isotropic elastic material is derived. In contrast to conventional models developed for applications involving large plate deformations with small midplane strains, the present model accommodates large deformations accompanied by finite midplane strains. An extended Kirchhoff-Love hypothesis underpinning prior work on the topic is here justified on energetic grounds. The model, specialized to certain generalized neo-Hookean materials, exhibits bending-stretching coupling and reduces to the classical pure-bending energy in the absence of midplane strain.

Keywords Elastic plate theory · Incompressibility · Bending-stretching coupling

1. Introduction

Current developments in the derivation of two-dimensional plate and shell models from three-dimensional nonlinear elasticity theory emphasize deformations involving finite midsurface strains [1-3], whereas classical models of the Koiter type [5-8], for example, are confined to finite deformations accompanied by small midsurface strains. These developments are inspired by the need for models of thin structures composed of polymers or biological tissues, i.e., materials that sustain finite elastic strains, whereas the classical models are intended for application to conventional engineering materials for which elastic response is confined to the small-strain regime. A central issue addressed by the extended theories, including that presented here, is the modeling of bending-stretching coupling, absent in the constitutive equations of the classical theories.

Due to the prevalence of incompressibility in materials capable of large elastic strains, it is necessary to incorporate the isochoricity of the bulk deformation in the course of extracting a thin-plate/shell model from a suitable dimension reduction procedure. Recent efforts along these lines are based on an extended Kirchhoff-Love hypothesis [2,3], according to which material lines initially normal to the midsurface remain so after deformation while accommodating isochoricity via an appropriate expression for the thickness distension. One of our main objectives in the present work is to justify this hypothesis. We show that extended Kirchhoff-Love kinematics are energetically optimal in materials that exhibit reflection symmetry with respect to the plate midplane provided that the three-dimensional strain-energy function is strongly elliptic. Examples of such material models are the three-term Odgen energy [9,10], the neo-Hookean energy, and some of its generalizations including the Demiray [11] and Gent [12,13] models, both of which play significant roles in the modeling of bio-tissue response.

In Section 2 we recount some aspects of nonlinear three-dimensional elasticity theory that are needed in our development. Section 3 is concerned with a small-thickness estimate of the potential energy of a plate that is either dead-loaded on a part of its cylindrical generating surface and fixed on the remainder, or fixed on this entire surface and pressurized by fixed, uniform pressures applied to its major surfaces. In Section 4 we derive restrictions on certain vector fields associated with the three-dimensional deformation in the vicinity of the midplane arising from the isochoricity of the bulk deformation. These are further restricted, in Section 5, for strongly elliptic materials that exhibit reflection symmetry with respect to the midplane. In particular, for such materials we demonstrate the energetic optimality of the extended Kirchhoff-Love hypothesis proposed in [2]. Enhancements of the model *vis à vis* the three-dimensional theory are described in Section 6, and the explicit plate energy for a class of generalized neo-Hookean materials is derived in Section 7. This exhibits stretching-bending coupling and reduces to the classical plate energy for deformations that generate pure bending.

Concerning notation, we use the usual bold-face symbols for vectors and tensors. A dot between such symbols is used to denote the standard Euclidean inner product. For example, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are second-order tensors, then $\mathbf{A}_1 \cdot \mathbf{A}_2 = tr(\mathbf{A}_1\mathbf{A}_2^t)$, where $tr(\cdot)$ is the trace and the superscript ^{t} is used to denote transposition. The norm of $\mathbf{A}$ is $|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$. The linear operator $Sym(\cdot)$ delivers the symmetric part of its second-order tensor argument, and the notation $\otimes$ identifies the standard tensor product of vectors. We use Lin^+ to denote the set of second-order tensors with positive determinant. If $\mathcal{M}$ is a fourth-order tensor, then $\mathcal{M}[\mathbf{A}]$ is a linear second-order-tensor-valued function of the second-order tensor $\mathbf{A}$. $\mathcal{M}$ is said to possess major symmetry if $\mathbf{A}_1 \cdot \mathcal{M}[\mathbf{A}_2] = \mathbf{A}_2 \cdot \mathcal{M}[\mathbf{A}_1]$, and minor symmetry if $\mathbf{A}_1 \cdot \mathcal{M}[\mathbf{A}_2] = \mathbf{A}_1^t \cdot \mathcal{M}[\mathbf{A}_2]$ and $\mathbf{A}_1 \cdot \mathcal{M}[\mathbf{A}_2] = \mathbf{A}_1 \cdot \mathcal{M}[\mathbf{A}_2^t]$. Finally, scalars with bold subscripts are used to denote the derivatives of scalar-valued functions with respect to their tensor or vector arguments.

2. Three-dimensional elasticity theory for incompressible materials

2.1 Basic equations of the three-dimensional theory

We use superposed tildes or carets to identify three-dimensional quantities. Symbols appearing without these are used to denote the restrictions of these quantities to a midplane Ω of a thin plate-like body occupying the region $\kappa = \Omega \times [-\frac{h}{2}, \frac{h}{2}]$, where h is the uniform plate thickness. We assume the plate to be thin in the sense that $h/l \ll 1$, where l is any spanwise dimension of Ω.

In the purely mechanical setting considered here, the Piola stress of the three-dimensional theory is given by

$$\tilde{\mathbf{P}} = \mathcal{W}_{\tilde{\mathbf{F}}} - \tilde{q}\tilde{\mathbf{F}}^*, \tag{1}$$

where $\mathcal{W}(\tilde{\mathbf{F}})$ is the strain energy per unit volume of κ, $\tilde{q}$ is a reaction pressure associated with the constraint of incompressibility, and $\tilde{\mathbf{F}} \in Lin^+$ is the gradient of a deformation $\tilde{\boldsymbol{\chi}}(\mathbf{X})$ with respect to position $\mathbf{X} \in \kappa$, having unit determinant and cofactor $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}^{-t}$. We assume the material to be uniform in the sense that $\mathcal{W}$ does not depend explicitly on $\mathbf{X}$. The Piola traction transmitted across a surface with unit normal $\tilde{\boldsymbol{\nu}}$ in κ is

$$\tilde{\mathbf{p}} = \tilde{\mathbf{P}}\tilde{\boldsymbol{\nu}}, \tag{2}$$

and Piola's equation of equilibrium, in the absence of body force, is

$$Div\tilde{\mathbf{P}} = \mathbf{0}, \tag{3}$$

where $Div(\cdot)$ is the divergence with respect to $\mathbf{X}$, this holding at all $\mathbf{X}$ in the interior of κ.

It is advantageous to decompose the position $\mathbf{X}$ of a material point in the form

$$\mathbf{X} = \mathbf{x} + \varsigma\mathbf{k}, \tag{4}$$

where $\mathbf{k}$ is the (uniform) unit normal to the midplane Ω, $\varsigma \in [-\frac{h}{2}, \frac{h}{2}]$ is a through-thickness coordinate, and $\mathbf{x}$ is the position of the projected point on Ω. With

$$\hat{\mathbf{P}}(\mathbf{x}, \varsigma) = \tilde{\mathbf{P}}(\mathbf{x} + \varsigma\mathbf{k}) \tag{5}$$

we may then write (3) in the form

$$div(\hat{\mathbf{P}}\mathbf{1}) + \hat{\mathbf{P}}'\mathbf{k} = \mathbf{0}, \tag{6}$$

where $div(\cdot)$ is the (two-dimensional) divergence with respect to $\mathbf{x}$, $(\cdot)' = \partial(\cdot)/\partial\varsigma$, and

$$\mathbf{1} = \mathbf{I} - \mathbf{k} \otimes \mathbf{k} \tag{7}$$

is the projection onto, and the identity for, Ω', the translation (vector) space of Ω, whereas $\mathbf{I}$ is the referential identity for 3-space.

Using the projection we may express the deformation gradient in the form

$$\hat{\mathbf{F}} = \nabla\hat{\boldsymbol{\chi}} + \hat{\boldsymbol{\chi}}' \otimes \mathbf{k}, \tag{8}$$

where $\hat{\boldsymbol{\chi}}(\mathbf{x}, \varsigma) = \tilde{\boldsymbol{\chi}}(\mathbf{X})$, $\hat{\mathbf{F}}(\mathbf{x}, \varsigma) = \tilde{\mathbf{F}}(\mathbf{X})$ and $\nabla(\cdot)$ is the (two-dimensional) gradient with respect to $\mathbf{x}$. We will encounter $\hat{\mathbf{F}}'$ and $\hat{\mathbf{F}}''$ in the course of our further development, where

$$\hat{\mathbf{F}}' = \nabla\hat{\boldsymbol{\chi}}' + \hat{\boldsymbol{\chi}}'' \otimes \mathbf{k} \quad \text{and} \quad \hat{\mathbf{F}}'' = \nabla\hat{\boldsymbol{\chi}}'' + \hat{\boldsymbol{\chi}}''' \otimes \mathbf{k}. \tag{9}$$

For points on the midplane these reduce to

$$\mathbf{F} = \nabla\boldsymbol{\chi} + \boldsymbol{\chi}' \otimes \mathbf{k}, \quad \mathbf{F}' = \nabla\boldsymbol{\chi}' + \boldsymbol{\chi}'' \otimes \mathbf{k} \quad \text{and} \quad \mathbf{F}'' = \nabla\boldsymbol{\chi}'' + \boldsymbol{\chi}''' \otimes \mathbf{k}, \tag{10}$$

where, for a generic function $\hat{f}(\mathbf{x},\varsigma)$,

$$f^{(n)}(\mathbf{x}) = \frac{\partial^n}{\partial\varsigma^n}\hat{f}_{|\varsigma=0}. \tag{11}$$

The equilibrium equation for such points is

$$div(\mathbf{P1}) + \mathbf{P}'\mathbf{k} = \mathbf{0}. \tag{12}$$

2.2 Strong ellipticity

To ease the notation, in the present subsection we drop the superposed tildes and carets on three-dimensional quantities. Thus, let $\mathbf{F}(u)$ be a curve on the constraint manifold, i.e., $J(u) = 1$, where $J(u) = \det\mathbf{F}(u)$. The associated strain energy is $\mathcal{W}(\mathbf{F}(u))$. Its derivative with respect to the parameter u is

$$\dot{\mathcal{W}} = \mathcal{W}_{\mathbf{F}} \cdot \dot{\mathbf{F}}, \tag{13}$$

in which $\dot{\mathbf{F}}$ is such that

$$0 = \dot{J} = J_{\mathbf{F}} \cdot \dot{\mathbf{F}} = \mathbf{F}^* \cdot \dot{\mathbf{F}}, \quad \text{with} \quad \mathbf{F}^* = \mathbf{F}^{-t} \tag{14}$$

on the constraint manifold. Here we regard $\mathcal{W}_{\mathbf{F}}$ as the derivative of an extended energy defined on Lin^+, evaluated *post facto* on the constraint manifold. Without loss of generality we choose this extended energy to coincide with the actual energy (defined on the manifold) but with Lin^+ as domain [14].

Continuing, we have

$$\ddot{\mathcal{W}} = \mathcal{W}_{\mathbf{FF}}[\dot{\mathbf{F}}] \cdot \dot{\mathbf{F}} + \mathcal{W}_{\mathbf{F}} \cdot \ddot{\mathbf{F}}, \tag{15}$$

where $\mathcal{W}_{\mathbf{FF}}$ is the second derivative of the extended energy, again evaluated on the constraint manifold, with $\dot{\mathbf{F}}$ restricted by (14) and with $\ddot{\mathbf{F}}$ such that

$$0 = \ddot{J} = \mathbf{F}^* \cdot \ddot{\mathbf{F}} + (\mathbf{F}^*)^{\cdot} \cdot \dot{\mathbf{F}}. \tag{16}$$

Noting that $(14)_2$ holds on the manifold, we differentiate $\mathbf{F}^t\mathbf{F}^{-t} = \mathbf{I}$, the referential identity, to obtain

$$(\mathbf{F}^*)^{\cdot} = -\mathbf{F}^*\dot{\mathbf{F}}^t\mathbf{F}^* \tag{17}$$

on the manifold. Accordingly, $\ddot{\mathbf{F}}$ is such that

$$\mathbf{F}^* \cdot \ddot{\mathbf{F}} = \mathbf{F}^*\dot{\mathbf{F}}^t\mathbf{F}^* \cdot \dot{\mathbf{F}}. \tag{18}$$

Invoking the identity $\mathbf{AB} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{CB}^t$, with $\mathbf{A} = \mathbf{F}^*\dot{\mathbf{F}}^t$, $\mathbf{B} = \mathbf{F}^*$ and $\mathbf{C} = \dot{\mathbf{F}}$, we derive

$$\mathbf{F}^* \cdot \ddot{\mathbf{F}} = \mathbf{G} \cdot \mathbf{G}^t = tr(\mathbf{G}^2), \quad \text{where} \quad \mathbf{G} = \mathbf{F}^*\dot{\mathbf{F}}^t. \tag{19}$$

For incompressible materials, the strong ellipticity condition is

$$\mathcal{W}_{\mathbf{FF}}[\mathbf{a}\otimes\mathbf{b}] \cdot \mathbf{a}\otimes\mathbf{b} > 0 \quad \text{for all} \quad \mathbf{a}\otimes\mathbf{b} \neq \mathbf{0}, \quad \text{with} \quad \mathbf{a}\cdot\mathbf{F}^*\mathbf{b} = 0 \tag{20}$$

in accordance with (14) [15].

2.3 Strong ellipticity for generalized neo-Hookean materials

We consider so-called *generalized neo-Hookean materials* for which

$$\mathcal{W}(\mathbf{F}) = F(I), \tag{21}$$

for particular functions F, where

$$I = tr(\mathbf{F}\mathbf{F}^t) = \mathbf{F} \cdot \mathbf{F}. \tag{22}$$

Examples include the conventional neo-Hookean material [9], with

$$F(I) = \tfrac{1}{2}\mu(I - 3); \tag{23}$$

the Demiray material [11], with

$$F(I) = \tfrac{1}{2\gamma}\mu\{\exp[\gamma(I - 3)] - 1\}; \tag{24}$$

and the Gent material [12,13], with

$$F(I) = -\tfrac{1}{2}\mu J_m \ln(1 - \tfrac{I-3}{J_m}), \quad \text{provided that} \quad I - 3 < J_m. \tag{25}$$

Here μ, γ and J_m are (positive) material parameters, with μ representing the shear modulus associated with infinitesimal strains. As is well known, the neo-Hookean model furnishes a faithful description of the response of rubber up to moderate strains, whereas the Gent model is valid over a wider range of strain, subject to the stated inequality constraint on I associated with locking of the material. Further, the Gent and Demiray models furnish useful descriptions of the elastic response of bio-tissues.

From the perspective of the present work, the salient features of these models are that

$$F'(I) > 0 \quad \text{and} \quad F''(I) \geq 0 \tag{26}$$

for all admissible I.

For these materials we have

$$\mathcal{W}_{\mathbf{F}} \cdot \dot{\mathbf{F}} = F'(I)\dot{I} = 2F'(I)\mathbf{F} \cdot \dot{\mathbf{F}} \tag{27}$$

for all $\dot{\mathbf{F}}$ such that (14) is satisfied. This follows by differentiating (21) and (22). Thus, from (15),

$$\mathcal{W}_{\mathbf{FF}}[\dot{\mathbf{F}}] \cdot \dot{\mathbf{F}} + \mathcal{W}_{\mathbf{F}} \cdot \ddot{\mathbf{F}} = 2F'(I)\left|\dot{\mathbf{F}}\right|^2 + 4F''(I)(\mathbf{F} \cdot \dot{\mathbf{F}})^2 + 2F'(I)\mathbf{F} \cdot \ddot{\mathbf{F}} \tag{28}$$

for all $\dot{\mathbf{F}}$ and $\ddot{\mathbf{F}}$ as stated.

Setting $\dot{\mathbf{F}} = \mathbf{a} \otimes \mathbf{b}$ in (18), we obtain $\mathbf{F}^* \cdot \ddot{\mathbf{F}} = (\mathbf{a} \cdot \mathbf{F}^*\mathbf{b})^2$, and the restriction on $\mathbf{a}, \mathbf{b}$ in (20) yields $\mathbf{F}^* \cdot \ddot{\mathbf{F}} = 0$. Thus (27) holds with $\dot{\mathbf{F}}$ replaced by $\ddot{\mathbf{F}}$ and this, together with (28), implies that

$$\mathcal{W}_{\mathbf{FF}}[\mathbf{a} \otimes \mathbf{b}] \cdot \mathbf{a} \otimes \mathbf{b} = 2F'(I)\left|\mathbf{a}\right|^2 \left|\mathbf{b}\right|^2 + 4F''(I)(\mathbf{a} \cdot \mathbf{F}\mathbf{b})^2. \tag{29}$$

Inequalities (26) thus ensure that (20) is satisfied without qualification, i.e., that the considered strain-energy functions are strongly elliptic at all admissible deformations.

3. Small-thickness estimate of the energy

The strain energy of the thin plate is

$$\mathcal{S} = \int_{\kappa} \mathcal{W}(\tilde{\mathbf{F}}(\mathbf{X}))dv = \int_{\Omega} W(\mathbf{x})da, \tag{30}$$

where

$$W(\mathbf{x}) = \int_{-h/2}^{h/2} \mathcal{W}(\hat{\mathbf{F}}(\mathbf{x},\varsigma))d\varsigma \tag{31}$$

is the areal strain energy density and h is the (uniform) plate thickness. We assume that $h/l \ll 1$, where l is any spanwise dimension of the plate. To ease the notation we adopt l as the length scale; thus, $l = 1$ and $h \ll 1$. Regarding W as a function of h at fixed $\mathbf{x} \in \Omega$, we may estimate it for small h by combining the Leibniz rule with a Taylor expansion to obtain (see [8], Sect. 4.2.1)

$$W = h\mathcal{W}(\mathbf{F}) + \tfrac{1}{24}h^3\mathcal{W}'' + o(h^3), \tag{32}$$

where

$$\mathcal{W}'' = \mathcal{W}_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}}(\mathbf{F})[\mathbf{F}'] \cdot \mathbf{F}' + \mathcal{W}_{\tilde{\mathbf{F}}}(\mathbf{F}) \cdot \mathbf{F}'', \tag{33}$$

the latter following by differentiating $\frac{\partial}{\partial\varsigma}\mathcal{W}(\hat{\mathbf{F}}(\mathbf{x},\varsigma)) = \mathcal{W}_{\tilde{\mathbf{F}}}(\hat{\mathbf{F}}) \cdot \frac{\partial}{\partial\varsigma}\hat{\mathbf{F}}$ with respect to ς and evaluating the result at $\varsigma = 0$. This estimate suffices to model membrane and leading-order bending effects.

Here (cf. (10)),

$$\mathbf{F} = \nabla\mathbf{r} + \mathbf{d} \otimes \mathbf{k}, \quad \mathbf{F}' = \nabla\mathbf{d} + \mathbf{g} \otimes \mathbf{k} \quad \text{and} \quad \mathbf{F}'' = \nabla\mathbf{g} + \mathbf{h} \otimes \mathbf{k}, \tag{34}$$

where

$$\mathbf{r} = \boldsymbol{\chi}, \quad \mathbf{d} = \boldsymbol{\chi}', \quad \mathbf{g} = \boldsymbol{\chi}'' \quad \text{and} \quad \mathbf{h} = \boldsymbol{\chi}'''. \tag{35}$$

These are the coefficient vectors in the small - ς expansion

$$\hat{\boldsymbol{\chi}}(\mathbf{x},\varsigma) = \mathbf{r}(\mathbf{x}) + \varsigma\mathbf{d}(\mathbf{x}) + \tfrac{1}{2}\varsigma^2\mathbf{g}(\mathbf{x}) + \tfrac{1}{6}\varsigma^3\mathbf{h}(\mathbf{x}) + \mathbf{o}(\varsigma^3) \tag{36}$$

of the three-dimensional deformation of a point near the midplane Ω, with $|\mathbf{o}(\epsilon)| = o(\epsilon)$. These vectors are independent functions of $\mathbf{x}$.

We are interested in conservative problems for which a load potential $\mathcal{L}$ exists. Among these we consider dead loading on the edge $\partial\Omega_n \times [-\frac{h}{2}, \frac{h}{2}]$, where $\partial\Omega_n \subset \partial\Omega$, with position assigned on $\partial\Omega_e \times [-\frac{h}{2}, \frac{h}{2}]$, where $\partial\Omega_e = \partial\Omega \backslash\, \partial\Omega_n$; or, lateral pressure loading on the major surfaces of the plate with position assigned on $\partial\Omega \times [-\frac{h}{2}, \frac{h}{2}]$.

For dead-loaded plates, we have

$$\mathcal{L} = \int_{\partial\Omega_n} \psi ds, \quad \text{where} \quad \psi = \int_{-h/2}^{h/2} \hat{\mathbf{p}}(\mathbf{x},\varsigma) \cdot \hat{\boldsymbol{\chi}}(\mathbf{x},\varsigma)d\varsigma, \tag{37}$$

in which $\hat{\mathbf{p}}(\mathbf{x},\varsigma)$ is assigned. Proceeding as in the reduction of (31) to (32), we obtain (see [8], Sect. 6.1.2)

$$\psi = \mathbf{p}_r \cdot \mathbf{r} + \mathbf{p}_d \cdot \mathbf{d} + \mathbf{p}_g \cdot \mathbf{g} + o(h^3) \tag{38}$$

where

$$\mathbf{p}_r = h\mathbf{p} + \tfrac{1}{24}h^3\mathbf{p}'', \quad \mathbf{p}_d = \tfrac{1}{12}h^3\mathbf{p}' \quad \text{and} \quad \mathbf{p}_g = \tfrac{1}{24}h^3\mathbf{p}. \tag{39}$$

Because position $\hat{\boldsymbol{\chi}}(\mathbf{x},\varsigma)$ is assigned on $\partial\Omega_e \times [-\frac{h}{2},\frac{h}{2}]$ it follows, by differentiation with respect to ς and evaluation at $\varsigma = 0$, that $\mathbf{r},\mathbf{d},\mathbf{g},\mathbf{h},...$ are fixed on $\partial\Omega_e$. Moreover, $\mathbf{p},\mathbf{p}'$ and $\mathbf{p}''$ are assigned on $\partial\Omega_n$.

In the case of a pressurized plate, we suppose the major surfaces to be subjected to the fixed uniform pressures $p^{\pm}$ at $\varsigma = \pm h/2$, with $p^{\pm} = h^3 P^{\pm} + o(h^3)$ and $P^{\pm} = O(1)$. With position assigned on $\partial\Omega \times [-\frac{h}{2},\frac{h}{2}]$ (and hence with $\mathbf{r},\mathbf{d},\mathbf{g},\mathbf{h},...$ fixed on $\partial\Omega$) the associated load potential is [16]

$$\mathcal{L} = h^3(\Delta P)V + o(h^3), \tag{40}$$

where $\Delta P = P^- - P^+$ and

$$V = \tfrac{1}{3}\int_\Omega j\mathbf{r}\cdot\mathbf{n}da, \tag{41}$$

with

$$j\mathbf{n} = \mathbf{F}^*\mathbf{k}, \tag{42}$$

where $\mathbf{F}^*$ is the cofactor of $\mathbf{F}$, defined by $\mathbf{F}^*(\mathbf{a}\times\mathbf{b}) = \mathbf{F}\mathbf{a}\times\mathbf{F}\mathbf{b}$ for all vectors $\mathbf{a},\mathbf{b}$. According to the Piola-Nanson formula $\mathbf{n}$ is a unit normal to the tangent plane $T_\omega(p)$ of the deformed image ω of the midplane Ω at the material point p associated with $\mathbf{x}\in\Omega$, whereas $j = |\mathbf{F}^*\mathbf{k}|$ is the local areal dilation of the surface ω induced by the deformation field $\mathbf{r}(\mathbf{x})$. The latter is positive because $\det\mathbf{F}^* = J^2$; accordingly, $\mathbf{F}^*\mathbf{k}\neq\mathbf{0}$.

The potential energy of the plate is

$$\mathcal{E} = \mathcal{S} - \mathcal{L}. \tag{43}$$

Retaining terms through order $O(h^3)$, we approximate this potential energy by

$$\mathcal{E} = E + o(h^3), \tag{44}$$

with

$$E = S - L, \tag{45}$$

where

$$S = h\int_\Omega \mathcal{W}(\mathbf{F})da + \tfrac{1}{24}h^3\int_\Omega\{\mathcal{W}_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}}(\mathbf{F})[\mathbf{F}']\cdot\mathbf{F}' + \mathcal{W}_{\tilde{\mathbf{F}}}(\mathbf{F})\cdot\mathbf{F}''\}da, \tag{46}$$

and, in the case of dead edge loading,

$$L = h\int_{\partial\Omega_n}\mathbf{p}\cdot\mathbf{r}ds + \tfrac{1}{24}h^3\int_{\partial\Omega_n}(\mathbf{p}''\cdot\mathbf{r} + 2\mathbf{p}'\cdot\mathbf{d} + \mathbf{p}\cdot\mathbf{g})ds; \tag{47}$$

or, in the case of conservative pressure loading,

$$L = \tfrac{1}{3}h^3(\Delta P)\int_\Omega j\mathbf{r}\cdot\mathbf{n}da. \tag{48}$$

4. Restrictions on the fields d, g and h imposed by isochoricity

Isochoricity of the bulk deformation entails the constraint $\tilde{J}(\mathbf{X}) = 1$, where $\tilde{J} = \det\tilde{\mathbf{F}}$. Writing $\hat{J}(\mathbf{x},\varsigma) = \tilde{J}(\mathbf{x}+\varsigma\mathbf{k})$, differentiating twice with respect to ς and evaluating the results at $\varsigma = 0$ gives

$$J = 1, \quad J' = 0 \quad\text{and}\quad J'' = 0, \tag{49}$$

the first of these implying that

$$1 = \mathbf{Fk} \cdot \mathbf{Fi} \times \mathbf{Fj} = \mathbf{Fk} \cdot \mathbf{F}^*\mathbf{k}, \tag{50}$$

where $\mathbf{i}, \mathbf{j} \in \Omega'$ are orthonormal vectors such that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. From $(34)_1$ and the Piola-Nanson formula we conclude that

$$\mathbf{d} \cdot \mathbf{n} = j^{-1}. \tag{51}$$

Writing $(\nabla\mathbf{r})\mathbf{e}$ for the projection of $\mathbf{d}$ onto $T_\omega(p)$, where $\mathbf{e} \in \Omega'$ is a 2-vector, we then have

$$\mathbf{d} = j^{-1}\mathbf{n} + (\nabla\mathbf{r})\mathbf{e} \tag{52}$$

for arbitrary $\mathbf{e} \in \Omega'$.

Proceeding next from $(34)_2$ and $(49)_2$, we have

$$0 = J' = \mathbf{F}^* \cdot \mathbf{F}' = \mathbf{F}^* \cdot \nabla\mathbf{d} + \mathbf{g} \cdot \mathbf{F}^*\mathbf{k} = \mathbf{F}^* \cdot \nabla\mathbf{d} + j\mathbf{g} \cdot \mathbf{n}. \tag{53}$$

Thus,

$$\mathbf{g} = (\nabla\mathbf{r})\mathbf{f} - j^{-1}(\mathbf{F}^* \cdot \nabla\mathbf{d})\mathbf{n}, \tag{54}$$

for arbitrary $\mathbf{f} \in \Omega'$.

Finally, the last of (49) implies, as in the calculation leading to (19), that

$$\mathbf{F}^* \cdot \mathbf{F}'' = tr(\mathbf{A}^2), \quad \text{where} \quad \mathbf{A} = \mathbf{F}^*(\mathbf{F}')^t, \tag{55}$$

and hence, with $(34)_3$, that

$$j\mathbf{h} \cdot \mathbf{n} = tr(\mathbf{A}^2) - \mathbf{F}^* \cdot \nabla\mathbf{g}. \tag{56}$$

Accordingly,

$$\mathbf{h} = (\mathbf{h} \cdot \mathbf{n})\mathbf{n} + (\nabla\mathbf{r})\mathbf{l} \tag{57}$$

for arbitrary $\mathbf{l} \in \Omega'$.

5. Reflection symmetry with respect to the midplane

Writing the midplane strain energy $\mathcal{W}(\mathbf{F})$ as a function $U(\mathbf{C})$, where $\mathbf{C} = \mathbf{F}^t\mathbf{F}$ is the right Cauchy-Green deformation tensor for points on the midplane Ω, reflection symmetry with respect to this plane is defined by

$$U(\mathbf{C}) = U(\mathbf{Q}^t\mathbf{CQ}) \tag{58}$$

for all admissible $\mathbf{C}$, where

$$\mathbf{Q} = \mathbf{I} - 2\mathbf{k} \otimes \mathbf{k} = \mathbf{1} - \mathbf{k} \otimes \mathbf{k}. \tag{59}$$

This is trivially satisfied in the case of isotropy relative to the configuration κ, as (58) then holds for all orthogonal $\mathbf{Q}$ and hence for reflections in particular.

5.1 Stationarity of the energy with respect to e and f

From (52), with $(\nabla\mathbf{r})^t\mathbf{n} = \mathbf{0}$ (because $(\nabla\mathbf{r})^t$ maps $T_\omega(p)$ to Ω') we have that admissible midplane Cauchy-Green tensors are of the form

$$\mathbf{C} = \mathbf{c} + \boldsymbol{\gamma}\otimes\mathbf{k} + \mathbf{k}\otimes\boldsymbol{\gamma} + (j^{-2} + \mathbf{e}\cdot\mathbf{ce})\mathbf{k}\otimes\mathbf{k}, \quad \text{where} \quad \boldsymbol{\gamma} = \mathbf{ce} \quad \text{and} \quad \mathbf{c} = (\nabla\mathbf{r})^t\nabla\mathbf{r}\colon \Omega' \to \Omega' \tag{60}$$

is the surfacial Cauchy-Green tensor. Accordingly,

$$\mathbf{Q}^t\mathbf{C}\mathbf{Q} = \mathbf{c} - \boldsymbol{\gamma}\otimes\mathbf{k} - \mathbf{k}\otimes\boldsymbol{\gamma} + (j^{-2} + \mathbf{e}\cdot\mathbf{ce})\mathbf{k}\otimes\mathbf{k}, \tag{61}$$

and it follows that the strain energy is an even function of $\mathbf{e}$:

$$G(\mathbf{e}) = G(-\mathbf{e}), \tag{62}$$

where

$$G(\mathbf{e}) = U(\mathbf{C}(\mathbf{e})), \tag{63}$$

in which all quantities other than $\mathbf{e}$ are fixed in (60).

Thus there is a function $H(\boldsymbol{\varepsilon})$, say, with $\boldsymbol{\varepsilon} = \mathbf{e}\otimes\mathbf{e}$, such that (Appendix A)

$$G(\mathbf{e}) = H(\boldsymbol{\varepsilon}). \tag{64}$$

Differentiating with respect to the parameter on a one-parameter family $\mathbf{e}(u) \in \Omega'$, we have

$$G_{\mathbf{e}}\cdot\dot{\mathbf{e}} = H_{\boldsymbol{\varepsilon}}\cdot\dot{\boldsymbol{\varepsilon}} = 2(H_{\boldsymbol{\varepsilon}})\mathbf{e}\cdot\dot{\mathbf{e}}, \tag{65}$$

where $(\cdot)^{\cdot} = \frac{d}{du}(\cdot)$ and the symmetry $H_{\boldsymbol{\varepsilon}} = (H_{\boldsymbol{\varepsilon}})^t$ has been invoked. Thus,

$$G_{\mathbf{e}} = 2(H_{\boldsymbol{\varepsilon}})\mathbf{e}, \tag{66}$$

and the energy is stationary with respect to $\mathbf{e}$ at $\mathbf{e} = \mathbf{0}$, yielding

$$\mathbf{C} = \mathbf{c} + j^{-2}\mathbf{k}\otimes\mathbf{k}. \tag{67}$$

We note that

$$\det\mathbf{c} = j^2, \tag{68}$$

where $\det\mathbf{c}$ is a 2×2 determinant. This is true whether or not $\mathbf{e}$ vanishes or the bulk deformation is isochoric.

From (52) and (63) we also have that

$$G_{\mathbf{e}}\cdot\dot{\mathbf{e}} = \mathcal{W}_{\mathbf{F}}\cdot\dot{\mathbf{F}} = \mathcal{W}_{\mathbf{F}}\cdot(\nabla\mathbf{r})\dot{\mathbf{e}}\otimes\mathbf{k}, \tag{69}$$

i.e.,

$$G_{\mathbf{e}} = (\nabla\mathbf{r})^t(\mathcal{W}_{\mathbf{F}})\mathbf{k}. \tag{70}$$

Before proceeding we pause to note, in view of $(34)_3$ and (57), that the part of the energy involving the vector field $\mathbf{l}$ in (46) is given by

$$\tfrac{1}{24}h^3\mathcal{W}_{\tilde{\mathbf{F}}}(\mathbf{F})\cdot(\nabla\mathbf{r})\mathbf{l}\otimes\mathbf{k} = \tfrac{1}{24}h^3\mathbf{l}\cdot G_{\mathbf{e}}, \tag{71}$$

where we have used $\mathcal{W}_{\tilde{\mathbf{F}}}(\mathbf{F}) = \mathcal{W}_{\mathbf{F}}(\mathbf{F})$, this following from our interpretation of $\mathcal{W}$ as the extended energy, i.e., from differentiation of $\mathcal{W}(\tilde{\mathbf{F}}) = \mathcal{W}(\mathbf{F})$ on a one-parameter family $\tilde{\mathbf{F}}(u) = \mathbf{F}(u)$ with $\dot{\mathbf{F}}$ arbitrary. Accordingly, if $\mathbf{e}$ renders the midplane strain energy $\mathcal{W}$ stationary then the term involving $\mathbf{l}$ in the order - h^3 energy vanishes. In this case no generality is lost by taking $\mathbf{h} = (\mathbf{h} \cdot \mathbf{n})\mathbf{n}$ in (57), with $\mathbf{h} \cdot \mathbf{n}$ given by (56). In particular, this situation obtains if $\mathbf{e} = \mathbf{0}$, in which case (52) yields $\mathbf{d} = j^{-1}\mathbf{n}$ and $(34)_1$ reduces to

$$\mathbf{F} = \nabla\mathbf{r} + \lambda\mathbf{n} \otimes \mathbf{k}, \quad \text{where} \quad \lambda = j^{-1}, \tag{72}$$

is the leading-order thickness distension, whereas $(34)_2$ reduces to

$$\mathbf{F}' = \lambda\nabla\mathbf{n} + \mathbf{n} \otimes \nabla\lambda + \mathbf{g} \otimes \mathbf{k}. \tag{73}$$

Next, we establish, again if $\mathbf{e}$ vanishes, that the term $\mathcal{W}_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}}(\mathbf{F})[\mathbf{F}'] \cdot \mathbf{F}'$ in the energy (46) is an even function of $\mathbf{f}$ and hence stationary at $\mathbf{f} = \mathbf{0}$, where $\mathbf{f}$ determines the tangential part of $\mathbf{g}$ (see (54)). To prove this claim we first differentiate $\mathcal{W}(\mathbf{F}) = U(\mathbf{C})$ on the one-parameter family $\mathbf{F}(u)$, obtaining $\mathcal{W}_{\mathbf{F}}(\mathbf{F}) \cdot \dot{\mathbf{F}} = U_{\mathbf{C}}(\mathbf{C}) \cdot \dot{\mathbf{C}}$ with

$$\dot{\mathbf{C}} = 2Sym(\mathbf{F}^t\dot{\mathbf{F}}). \tag{74}$$

With the interpretation of $\mathcal{W}$ and U as extended energies, we may regard $\mathbf{F}(u)$ as an arbitrary path in Lin^+. We suppose that $\mathbf{F}(0)$ belongs to the constraint manifold and differentiate again on a straight-line path ($\ddot{\mathbf{F}} = \mathbf{0}$) in an open ball centered at $\mathbf{F}(0)$ - a convex set - obtaining

$$\mathcal{W}_{\mathbf{FF}}(\mathbf{F})[\dot{\mathbf{F}}] \cdot \dot{\mathbf{F}} = U_{\mathbf{CC}}(\mathbf{C})[\dot{\mathbf{C}}] \cdot \dot{\mathbf{C}} + U_{\mathbf{C}}(\mathbf{C}) \cdot \ddot{\mathbf{C}}, \tag{75}$$

with $\mathbf{F} = \mathbf{F}(0)$ and

$$\ddot{\mathbf{C}} = 2Sym(\dot{\mathbf{F}}^t\dot{\mathbf{F}}). \tag{76}$$

Thus, on invoking the symmetry of $U_{\mathbf{C}}$ and the minor symmetries of $U_{\mathbf{CC}}$, we arrive at

$$\mathcal{W}_{\mathbf{FF}}(\mathbf{F})[\dot{\mathbf{F}}] \cdot \dot{\mathbf{F}} = 4U_{\mathbf{CC}}(\mathbf{C})[\mathbf{F}^t\dot{\mathbf{F}}] \cdot \mathbf{F}^t\dot{\mathbf{F}} + 2U_{\mathbf{C}}(\mathbf{C}) \cdot \dot{\mathbf{F}}^t\dot{\mathbf{F}} \tag{77}$$

for all $\dot{\mathbf{F}}$. On noting that $\mathcal{W}_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}}(\mathbf{F}) = \mathcal{W}_{\mathbf{FF}}(\mathbf{F})$, the relevant term in the energy (46) reduces to

$$\mathcal{W}_{\mathbf{FF}}(\mathbf{F})[\mathbf{F}'] \cdot \mathbf{F}' = 4U_{\mathbf{CC}}(\mathbf{C})[\mathbf{F}^t\mathbf{F}'] \cdot \mathbf{F}^t\mathbf{F}' + 2U_{\mathbf{C}}(\mathbf{C}) \cdot (\mathbf{F}')^t\mathbf{F}'. \tag{78}$$

We now differentiate the symmetry condition (58) on the same straight-line path $\mathbf{F}(u)$, reaching $U_{\mathbf{C}}(\mathbf{C}) \cdot \dot{\mathbf{C}} = U_{\mathbf{C}}(\mathbf{Q}^t\mathbf{C}\mathbf{Q}) \cdot \mathbf{Q}^t\dot{\mathbf{C}}\mathbf{Q}$ and

$$U_{\mathbf{CC}}(\mathbf{C})[\dot{\mathbf{C}}] \cdot \dot{\mathbf{C}} + U_{\mathbf{C}}(\mathbf{C}) \cdot \ddot{\mathbf{C}} = U_{\mathbf{CC}}(\mathbf{Q}^t\mathbf{C}\mathbf{Q})[\mathbf{Q}^t\dot{\mathbf{C}}\mathbf{Q}] \cdot \mathbf{Q}^t\dot{\mathbf{C}}\mathbf{Q} + U_{\mathbf{C}}(\mathbf{Q}^t\mathbf{C}\mathbf{Q}) \cdot \mathbf{Q}^t\ddot{\mathbf{C}}\mathbf{Q}, \tag{79}$$

with $\dot{\mathbf{C}}$ and $\ddot{\mathbf{C}}$ given respectively by (74) and (76) in which $\dot{\mathbf{F}}$ is arbitrary. If $\mathbf{e} = \mathbf{0}$ we have, from (72) and with $\mathbf{Q}$ given by (59), that

$$\mathbf{C} = \mathbf{c} + \lambda^2\mathbf{k} \otimes \mathbf{k} \quad \text{and} \quad \mathbf{Q}^t\mathbf{C}\mathbf{Q} = \mathbf{C}, \tag{80}$$

and hence,

$$4U_{\mathbf{CC}}(\mathbf{C})[\mathbf{F}^t\mathbf{F}'] \cdot \mathbf{F}^t\mathbf{F}' + 2U_{\mathbf{C}}(\mathbf{C}) \cdot (\mathbf{F}')^t\mathbf{F}' = 4U_{\mathbf{CC}}(\mathbf{C})[\mathbf{Q}^t\mathbf{F}^t\mathbf{F}'\mathbf{Q}] \cdot \mathbf{Q}^t\mathbf{F}^t\mathbf{F}'\mathbf{Q} + 2U_{\mathbf{C}}(\mathbf{C}) \cdot \mathbf{Q}^t(\mathbf{F}')^t\mathbf{F}'\mathbf{Q}. \tag{81}$$

To examine the implications of this condition we combine $(34)_2$ and (72). After a fairly lengthy but straightforward calculation we reach

$$Sym(\mathbf{F}^t\mathbf{F}') = \lambda\boldsymbol{\kappa} + \lambda(\mathbf{n}\cdot\mathbf{g})\mathbf{k}\otimes\mathbf{k} + Sym(\mathbf{cf}\otimes\mathbf{k}) \tag{82}$$

and

$$(\mathbf{F}')^t\mathbf{F}' = \lambda^2(\nabla\mathbf{n})^t\nabla\mathbf{n} + |\mathbf{g}|^2\,\mathbf{k}\otimes\mathbf{k} + 2\lambda Sym(\boldsymbol{\kappa}\mathbf{f}\otimes\mathbf{k}), \tag{83}$$

where

$$\boldsymbol{\kappa} = (\nabla\mathbf{r})^t\nabla\mathbf{n} \tag{84}$$

is the bending strain, and

$$|\mathbf{g}|^2 = \mathbf{f}\cdot\mathbf{cf} + (\mathbf{n}\cdot\mathbf{g})^2, \tag{85}$$

in which we have suppressed terms involving $\nabla\lambda$. In Section 6 we show that such terms may be neglected without impairing the order - h^3 accuracy of the expression (46) for the energy. In the course of deriving (82) and (83) we have used the fact that $(\nabla\mathbf{r})^t\mathbf{n} = \mathbf{0}$, as already noted, together with $(\nabla\mathbf{n})^t\mathbf{n} = \mathbf{0}$ (the differential form of $\mathbf{n}\cdot\mathbf{n} = 1$).

The symmetric bending strain tensor $\boldsymbol{\kappa}$: $\Omega' \to \Omega'$ is effectively the pullback of the curvature tensor $\mathbf{b}$: $T_\omega(p) \to T_\omega(p)$ defined by $d\mathbf{n} = -\mathbf{b}d\mathbf{r}$. The symmetry of $\mathbf{b}$ follows from the Weingarten equations of surface theory [18]. Using $d\mathbf{r} = (\nabla\mathbf{r})d\mathbf{x}$ and $d\mathbf{n} = (\nabla\mathbf{n})d\mathbf{x}$ we have $\nabla\mathbf{n} = -\mathbf{b}(\nabla\mathbf{r})$ and

$$\boldsymbol{\kappa} = -(\nabla\mathbf{r})^t\mathbf{b}(\nabla\mathbf{r}). \tag{86}$$

With these expressions in hand and with $\mathbf{Q}$ given by (59) we arrive at

$$Sym(\mathbf{Q}^t\mathbf{F}^t\mathbf{F}'\mathbf{Q}) = \lambda\boldsymbol{\kappa} + \lambda(\mathbf{n}\cdot\mathbf{g})\mathbf{k}\otimes\mathbf{k} - Sym(\mathbf{cf}\otimes\mathbf{k}) \tag{87}$$

and

$$\mathbf{Q}^t(\mathbf{F}')^t\mathbf{F}'\mathbf{Q} = \lambda^2(\nabla\mathbf{n})^t\nabla\mathbf{n} + |\mathbf{g}|^2\,\mathbf{k}\otimes\mathbf{k} - 2\lambda Sym(\boldsymbol{\kappa}\mathbf{f}\otimes\mathbf{k}). \tag{88}$$

Combining with (78) and (81)-(83), we finally conclude that

$$\mathcal{W}_{\mathbf{FF}}(\mathbf{F})[\mathbf{F}']\cdot\mathbf{F}' \quad \text{is an even function of} \quad \mathbf{f}, \tag{89}$$

and hence, as in the discussion leading to (66), that it is stationary with respect to $\mathbf{f}$ at $\mathbf{f} = \mathbf{0}$.

5.2 Uniqueness and optimality of the stationarity conditions $\mathbf{e} = \mathbf{0}$ and $\mathbf{f} = \mathbf{0}$

Let $\sigma(u) = G(\mathbf{e}(u))$, where G is defined in (63) and $\mathbf{e}(u) \in \Omega'$ is a one-parameter family of 2-vectors. Then,

$$\dot\sigma = G_{\mathbf{e}}\cdot\dot{\mathbf{e}} \quad \text{and} \quad \ddot\sigma = G_{\mathbf{e}}\cdot\ddot{\mathbf{e}} + (G_{\mathbf{e}})^{\cdot}\cdot\dot{\mathbf{e}}, \quad \text{where} \quad (G_{\mathbf{e}})^{\cdot} = (G_{\mathbf{ee}})\dot{\mathbf{e}}. \tag{90}$$

From $(34)_1$ and (52) we have that

$$(G_{\mathbf{e}})^{\cdot} = (\nabla\mathbf{r})^t\{\mathcal{W}_{\mathbf{FF}}[(\nabla\mathbf{r})\dot{\mathbf{e}}\otimes\mathbf{k}]\}\mathbf{k} \tag{91}$$

and

$$\dot{\mathbf{e}}\cdot(G_{\mathbf{ee}})\dot{\mathbf{e}} = (\nabla\mathbf{r})\dot{\mathbf{e}}\otimes\mathbf{k}\cdot\mathcal{W}_{\mathbf{FF}}[(\nabla\mathbf{r})\dot{\mathbf{e}}\otimes\mathbf{k}]. \tag{92}$$

Here we note that $(\nabla\mathbf{r})\dot{\mathbf{e}}\cdot\mathbf{F}^*\mathbf{k} = j(\nabla\mathbf{r})\dot{\mathbf{e}}\cdot\mathbf{n}$. This vanishes for all $\dot{\mathbf{e}}$, and the positive definiteness of $G_{\mathbf{ee}}$ follows from the strong-ellipticity condition (20).

Consider the straight-line path $\mathbf{e}(u) = u\mathbf{e}_2 + (1-u)\mathbf{e}_1$ with $\mathbf{e}_2 \neq \mathbf{e}_1$ and $u \in [0,1]$. Then, $\dot{\mathbf{e}} = \mathbf{e}_2 - \mathbf{e}_1$ is non-zero and $\ddot{\sigma} = \dot{\mathbf{e}}\cdot(G_{\mathbf{ee}})\dot{\mathbf{e}} > 0$ on this path. It follows by integration that $\dot{\sigma}(u) > \dot{\sigma}(0)$ for $u \in (0,1]$ and $\sigma(1) - \sigma(0) > \dot{\sigma}(0)$, i.e., that $G(\mathbf{e})$ is convex:

$$G(\mathbf{e}_2) - G(\mathbf{e}_1) > G_{\mathbf{e}}(\mathbf{e}_1)\cdot(\mathbf{e}_2 - \mathbf{e}_1); \quad \mathbf{e}_2 \neq \mathbf{e}_1. \tag{93}$$

Because such functions have unique stationary points we conclude that the solution $\mathbf{e} = \mathbf{0}$ to the equation $G_{\mathbf{e}} = \mathbf{0}$ is unique and furnishes the global minimum of the energy: $G(\mathbf{e}) > G(\mathbf{0})$ for all $\mathbf{e} \neq \mathbf{0}$.

Next, consider the function

$$B(\mathbf{f}) = \tfrac{1}{2}\mathcal{W}_{\mathbf{FF}}(\mathbf{F})[\mathbf{F}']\cdot\mathbf{F}' \tag{94}$$

in which all quantities other than $\mathbf{f}$ are fixed in $(34)_2$ and (54). Differentiating on a curve $\mathbf{f}(u) \in \Omega'$, we have

$$B_{\mathbf{f}}\cdot\dot{\mathbf{f}} = \mathcal{W}_{\mathbf{FF}}(\mathbf{F})[\mathbf{F}']\cdot(\nabla\mathbf{r})\dot{\mathbf{f}}\otimes\mathbf{k}; \quad \text{hence,} \quad B_{\mathbf{f}} = (\nabla\mathbf{r})^t(\mathcal{W}_{\mathbf{FF}}[\mathbf{F}'])\mathbf{k}, \tag{95}$$

and

$$\dot{\mathbf{f}}\cdot(B_{\mathbf{ff}})\dot{\mathbf{f}} = (\nabla\mathbf{r})\dot{\mathbf{f}}\otimes\mathbf{k}\cdot\mathcal{W}_{\mathbf{FF}}[(\nabla\mathbf{r})\dot{\mathbf{f}}\otimes\mathbf{k}]. \tag{96}$$

Strong ellipticity thus implies that $B_{\mathbf{ff}}$ is positive definite.

Proceeding as above, let $\sigma(u) = B(\mathbf{f}(u))$ with $\mathbf{f}(u) = u\mathbf{f}_2 + (1-u)\mathbf{f}_1$, $\mathbf{f}_2 \neq \mathbf{f}_1$ and $u \in [0,1]$. Then $\ddot{\sigma} = \dot{\mathbf{f}}\cdot(B_{\mathbf{ff}})\dot{\mathbf{f}} > 0$, implying, as above, that $B(\mathbf{f})$ is also convex. Thus $B(\mathbf{f})$ has a unique stationary point that furnishes the global minimum of $\mathcal{W}_{\mathbf{FF}}(\mathbf{F})[\mathbf{F}']\cdot\mathbf{F}'$ with respect to $\mathbf{f}$. If $\nabla\lambda$ is negligible, as shown in Section 6 below, then (89) follows and $\mathbf{f} = \mathbf{0}$ is the unique minimizer.

These results, together with the remarks following (71), justify the extended Kirchhoff-Love hypothesis

$$\hat{\boldsymbol{\chi}}(\mathbf{x},\varsigma) = \mathbf{r}(\mathbf{x}) + \phi(\mathbf{x},\varsigma)\mathbf{n}(\mathbf{x}) + \mathbf{o}(\varsigma^3) \tag{97}$$

proposed in [2], with (cf. (36), (52), (54) and (57))

$$\phi = \mathbf{n}\cdot(\varsigma\mathbf{d} + \tfrac{1}{2}\varsigma^2\mathbf{g} + \tfrac{1}{6}\varsigma^3\mathbf{h}). \tag{98}$$

With some effort, detailed in Appendix B, we may show that

$$\phi = j^{-1}[\varsigma + j^{-1}H\varsigma^2 + \tfrac{1}{3}j^{-2}\varsigma^3(6H^2 - K)], \tag{99}$$

where

$$H = \tfrac{1}{2}tr\mathbf{b} \quad \text{and} \quad K = \det\mathbf{b} \tag{100}$$

are the mean and Gaussian curvatures of ω, respectively. This agrees with the result stated in [2] if due account is taken of our sign convention for H.

6. Refinements based on the three-dimensional theory

We seek an order - h^3 estimate of the potential energy that is as accurate as possible by the standard of three-dimensional elasticity theory. Accordingly we incorporate information from that theory to refine our approximation (46) to the plate energy. For example, the tractions acting at the major surfaces of the plate with exterior unit normals $\pm\mathbf{k}$ are

$$\tilde{\mathbf{p}}^{\pm} = \hat{\mathbf{p}}(\mathbf{x}, \pm\tfrac{h}{2}) = \pm\mathbf{P}\mathbf{k} + \tfrac{h}{2}\mathbf{P}'\mathbf{k} \pm \tfrac{h^2}{8}\mathbf{P}''\mathbf{k} + \mathbf{O}(h^3), \tag{101}$$

where $|\mathbf{O}(\epsilon)| = O(\epsilon)$, yielding the net lateral traction and traction difference

$$\tilde{\mathbf{p}}^{+} + \tilde{\mathbf{p}}^{-} = h\mathbf{P}'\mathbf{k} + \mathbf{O}(h^3) \quad \text{and} \quad \tilde{\mathbf{p}}^{+} - \tilde{\mathbf{p}}^{-} = 2\mathbf{P}\mathbf{k} + \mathbf{O}(h^2), \tag{102}$$

respectively. Thus, if $\tilde{\mathbf{p}}^{\pm}$ vanish, as they do for edge-loaded plates, or if $|\tilde{\mathbf{p}}^{\pm}| = O(h^3)$, as we have stipulated for pressurized plates, then

$$\mathbf{P}\mathbf{k} = \mathbf{O}(h^2) \quad \text{and} \quad \mathbf{P}'\mathbf{k} = \mathbf{O}(h^2). \tag{103}$$

Accordingly, we may impose

$$\mathbf{P}\mathbf{k} = \mathbf{0} \quad \text{and} \quad \mathbf{P}'\mathbf{k} = \mathbf{0} \tag{104}$$

in the coefficient of h^3 in the expression (46) without impairing the accuracy of the approximate energy.

To examine the implications of the first restriction, we combine it with (1) and the Piola-Nanson formula (42), obtaining

$$(\mathcal{W}_{\mathbf{F}})\mathbf{k} = qj\mathbf{n}, \tag{105}$$

where q is the midplane value of the constraint pressure. This furnishes

$$(\nabla\mathbf{r})^t(\mathcal{W}_{\mathbf{F}})\mathbf{k} = \mathbf{0}, \tag{106}$$

which, as we have seen, implies that $\mathbf{e} = \mathbf{0}$ furnishes the optimal midplane deformation gradient, given by (72). This also furnishes the optimum value of $\mathcal{W}$ for a given midplane deformation field $\mathbf{r}(\mathbf{x})$. Accordingly, we put $\mathbf{e} = \mathbf{0}$ in all terms of the approximation (46) to the energy. The associated constraint pressure is given by

$$q = \lambda\mathbf{n}\cdot(\mathcal{W}_{\mathbf{F}})\mathbf{k}; \quad \lambda = j^{-1}. \tag{107}$$

The second restriction in (104) is

$$(\mathcal{W}_{\mathbf{FF}}[\mathbf{F}'])\mathbf{k} - q'\mathbf{F}^*\mathbf{k} - q(\mathbf{F}^*)'\mathbf{k} = \mathbf{0}, \tag{108}$$

where (cf. (17)) $(\mathbf{F}^*)' = -\mathbf{F}^*(\mathbf{F}')^t\mathbf{F}^*$, which, together with the Piola-Nanson formula, implies that

$$(\mathcal{W}_{\mathbf{FF}}[\mathbf{F}'])\mathbf{k} - q'j\mathbf{n} + qj\mathbf{F}^*(\mathbf{F}')^t\mathbf{n} = \mathbf{0}, \tag{109}$$

this further yielding

$$(\nabla\mathbf{r})^t(\mathcal{W}_{\mathbf{FF}}[\mathbf{F}'])\mathbf{k} + qj(\nabla\mathbf{r})^t\mathbf{F}^*(\mathbf{F}')^t\mathbf{n} = \mathbf{0}, \tag{110}$$

where, with reference to (95) and the remarks following (96), the first term vanishes at the value of $\mathbf{f}$ that optimizes $\mathcal{W}_{\mathbf{FF}}(\mathbf{F})[\mathbf{F}'] \cdot \mathbf{F}'$. We may thus impose

$$qj(\nabla \mathbf{r})^t \mathbf{F}^* (\mathbf{F}')^t \mathbf{n} = \mathbf{0} \tag{111}$$

in the order - h^3 energy without adversely affecting the accuracy of the approximation.

On combining

$$\mathbf{F}^* = (\nabla \mathbf{r})^{-t} + j\mathbf{n} \otimes \mathbf{k} \tag{112}$$

with $(\nabla \mathbf{r})^t (\nabla \mathbf{r})^{-t} = \mathbf{1}$, $(\nabla \mathbf{r})^t \mathbf{n} = \mathbf{0}$, $(\nabla \mathbf{n})^t \mathbf{n} = \mathbf{0}$ and (see (73)) $(\mathbf{F}')^t \mathbf{n} = \nabla \lambda + (\mathbf{g} \cdot \mathbf{n})\mathbf{k}$, we arrive at $(\nabla \mathbf{r})^t \mathbf{F}^* (\mathbf{F}')^t \mathbf{n} = \nabla \lambda$ and thus reduce (111) to $qj\nabla\lambda = \mathbf{0}$. As q is generally non-zero, we conclude, from $(104)_2$, that $\nabla\lambda = \mathbf{0}$, or, more precisely, from $(103)_2$, that

$$|\nabla \lambda| = O(h^2), \tag{113}$$

and hence that terms involving $\nabla\lambda$ may be suppressed in the order - h^3 contribution to the energy without impairing the accuracy of the approximation. This result justifies our prior suppression of $\nabla\lambda$ in the course of deriving (82)-(85).

Moving on to the term $\mathcal{W}_{\tilde{\mathbf{F}}}(\mathbf{F}) \cdot \mathbf{F}''(= \mathcal{W}_{\mathbf{F}}(\mathbf{F}) \cdot \mathbf{F}'')$ in (46), we use (1), (7), (12), $(34)_3$ and (104) to write

$$\mathcal{W}_{\mathbf{F}}(\mathbf{F}) \cdot \mathbf{F}'' = \mathbf{P} \cdot \mathbf{F}'' + q\mathbf{F}^* \cdot \mathbf{F}'', \tag{114}$$

with

$$\begin{aligned} \mathbf{P} \cdot \mathbf{F}'' &= \mathbf{P1} \cdot \nabla \mathbf{g} + \mathbf{h} \cdot \mathbf{Pk} \\ &= div[(\mathbf{P1})^t \mathbf{g}] - \mathbf{g} \cdot div(\mathbf{P1}) \\ &= div[(\mathbf{P1})^t \mathbf{g}] + \mathbf{g} \cdot \mathbf{P}'\mathbf{k} \\ &= div[(\mathbf{P1})^t \mathbf{g}], \end{aligned} \tag{115}$$

i.e, $\mathbf{P} \cdot \mathbf{F}'' = div[(\mathbf{P1})^t \mathbf{g}] + O(h^2)$. Thus, we maintain order - h^3 accuracy by imposing

$$\int_\Omega \mathcal{W}_{\tilde{\mathbf{F}}}(\mathbf{F}) \cdot \mathbf{F}'' da = \int_\Omega q\mathbf{F}^* \cdot \mathbf{F}'' da + \int_{\partial\Omega} \mathbf{P}\boldsymbol{\nu} \cdot \mathbf{g} ds \tag{116}$$

in (46), where $\boldsymbol{\nu} \in \Omega'$ is the exterior unit normal to the edge $\partial\Omega$.

On substituting into (46) and (47) we note that according to the three-dimensional theory, the contribution to the final integral from $\partial\Omega_n$ cancels the term $\mathbf{p} \cdot \mathbf{g}$ in the load potential for the dead-load problem, leaving a residual integral over $\partial\Omega_e = \partial\Omega \setminus \partial\Omega_n$, whereas $\partial\Omega_e = \partial\Omega$ in the case of pressure loading. According to (1) and (107), this residual integral is determined by the restrictions of $\mathbf{d}, \mathbf{g}$ and $\nabla \mathbf{r}$ to $\partial\Omega_e$. Here we invoke the normal-tangential decomposition

$$\nabla \mathbf{r}_{|\partial\Omega_e} = \mathbf{r}_s \otimes \boldsymbol{\tau} + \mathbf{r}_\nu \otimes \boldsymbol{\nu}, \tag{117}$$

where $\boldsymbol{\tau} = \mathbf{k} \times \boldsymbol{\nu}$ is a unit tangent to the boundary, and $\mathbf{r}_s$ and $\mathbf{r}_\nu$ respectively are the tangential and normal derivatives of $\mathbf{r}$ on the boundary. Recalling that $\mathbf{r}, \mathbf{d}$ and $\mathbf{g}$ are assigned on $\partial\Omega_e$, and noting that $\mathbf{r}_s$ is determined by $\mathbf{r}_{|\partial\Omega_e}$, it follows that the residual integral is controlled by $\mathbf{r}_\nu$. If this is also assigned on $\partial\Omega_e$, i.e., if this part of the boundary is *clamped*, then the residual integral is fixed. Its variational derivative then vanishes, and thus makes no contribution to the mechanics of the plate. Assuming the plate to be clamped on $\partial\Omega_e$ we may therefore suppress the final integral in (116) entirely.

Alternatively, on noting that $\mathbf{P} = \mathbf{FS}$, where $\mathbf{S}$ is the symmetric Piola-Kirchhoff stress, we infer that $(104)_1$ is equivalent to $\mathbf{Sk} = \mathbf{0}$. Then, $(34)_1$ yields $\mathbf{P} = (\nabla\mathbf{r})\mathbf{S}$ and $\mathbf{P}\boldsymbol{\nu} \cdot \mathbf{g} = \mathbf{S}\boldsymbol{\nu} \cdot (\nabla\mathbf{r})^t\mathbf{g} = \mathbf{S}\boldsymbol{\nu} \cdot \mathbf{cf}$. This vanishes because $\mathbf{f}$ vanishes. The residual integral over $\partial\Omega_e$ then vanishes identically whether or not $\partial\Omega_e$ is clamped.

Remark: These statements presume that the assigned boundary values of $\mathbf{d}$ and $\mathbf{g}$ on $\partial\Omega_e$ agree with the continuous extensions to $\partial\Omega_e$ of their interior values, as determined in the foregoing. This is atypical, however. It is therefore necessary to use the three-dimensional theory (or a further refinement of the present model) in a region adjoining the boundary. One then attempts to match its predictions to those of the present model in the interior. This situation arises in all two-dimensional plate theories. Here, we proceed as in the usual treatments of plate theory and regard the present model as applying on the closure of Ω.

Summarizing, we have

$$E = \int_\Omega W da - L, \tag{118}$$

with

$$W = h\mathcal{W}(\mathbf{F}) + \tfrac{1}{24}h^3\{\mathcal{W}_{\mathbf{FF}}(\mathbf{F})[\mathbf{F}'] \cdot \mathbf{F}' + q\mathbf{F}^* \cdot \mathbf{F}''\}, \tag{119}$$

wherein (Appendix B)

$$\mathbf{F} = \nabla\mathbf{r} + \lambda\mathbf{n} \otimes \mathbf{k}, \quad \mathbf{F}' = \lambda\nabla\mathbf{n} + 2\lambda^2 H\mathbf{n} \otimes \mathbf{k} \quad \text{and} \quad \mathbf{F}^* \cdot \mathbf{F}'' = \lambda^2(|\mathbf{b}|^2 + 4H^2), \tag{120}$$

with L given by

$$L = h\int_{\partial\Omega_n} \mathbf{p} \cdot \mathbf{r} ds + \tfrac{1}{24}h^3 \int_{\partial\Omega_n} (\mathbf{p}'' \cdot \mathbf{r} + 2\lambda\mathbf{p}' \cdot \mathbf{n}) ds \tag{121}$$

in the case of dead loading or by (48) in the case of pressure loading, and with q given by (107).

These results apply to all strongly elliptic strain energies that exhibit reflection symmetry with respect to the midplane Ω.

7. The explicit energy for generalized neo-Hookean materials

For generalized neo-Hookean materials we have $\mathcal{W}(\mathbf{F}) = F(I)$ with $I = \mathbf{F} \cdot \mathbf{F}$. Then,

$$\mathcal{W}' = F'(I)I', \quad \text{with} \quad I' = 2\mathbf{F} \cdot \mathbf{F}' \tag{122}$$

and

$$\mathcal{W}'' = F'(I)I'' + F''(I)(I')^2, \quad \text{with} \quad I'' = 2\mathbf{F}' \cdot \mathbf{F}' + 2\mathbf{F} \cdot \mathbf{F}''. \tag{123}$$

On comparison with (33), and with our interpretation of $\mathcal{W}$ as the extended energy, it follows that

$$\mathcal{W}_{\mathbf{F}} = 2F'(I)\mathbf{F} \quad \text{and} \quad \mathcal{W}_{\mathbf{FF}}[\mathbf{F}'] \cdot \mathbf{F}' = 2F'(I)\mathbf{F}' \cdot \mathbf{F}' + 4F''(I)(\mathbf{F} \cdot \mathbf{F}')^2. \tag{124}$$

We note in passing that the midplane stress (cf. (1)) is

$$\mathbf{P} = 2F'(I)(\mathbf{F} - \lambda^2 \mathbf{F}^*). \tag{125}$$

These formulae may be used to confirm the stationarity conditions $\mathbf{e} = \mathbf{0}$ and $\mathbf{f} = \mathbf{0}$ obtained in Section 5. For example, (69) reduces to

$$\begin{aligned} G_{\mathbf{e}} \cdot \dot{\mathbf{e}} &= 2F'(I)(\nabla \mathbf{r})\dot{\mathbf{e}} \cdot \mathbf{Fk} \\ &= 2F'(I)\dot{\mathbf{e}} \cdot (\nabla \mathbf{r})^t \mathbf{d} \\ &= 2F'(I)\dot{\mathbf{e}} \cdot \mathbf{ce}, \end{aligned} \tag{126}$$

where $\mathbf{c} = (\nabla \mathbf{r})^t (\nabla \mathbf{r})$ is the surfacial Cauchy-Green deformation tensor. Thus,

$$G_{\mathbf{e}} = 2F'(I)\mathbf{ce}. \tag{127}$$

Because $F'(I) > 0$ for the models considered, and because $\det \mathbf{c} (= j^2) > 0$, it follows that $\mathbf{e} = \mathbf{0}$ is the unique stationary point of G, as claimed.

In the same way, from (95) we have

$$B_{\mathbf{f}} \cdot \dot{\mathbf{f}} = 2F'(I)\mathbf{F}' \cdot \dot{\mathbf{F}}' + 4F''(I)(\mathbf{F} \cdot \mathbf{F}')\mathbf{F} \cdot \dot{\mathbf{F}}', \quad \text{where} \quad \dot{\mathbf{F}}' = (\nabla \mathbf{r})\dot{\mathbf{f}} \otimes \mathbf{k}. \tag{128}$$

With $\mathbf{F} \cdot \dot{\mathbf{F}}' = (\nabla \mathbf{r})\dot{\mathbf{f}} \cdot \mathbf{d} = \dot{\mathbf{f}} \cdot \mathbf{ce} = 0$ and $\mathbf{F}' \cdot \dot{\mathbf{F}}' = (\nabla \mathbf{r})\dot{\mathbf{f}} \cdot \mathbf{g} = \dot{\mathbf{f}} \cdot \mathbf{cf}$, we obtain

$$B_{\mathbf{f}} = 2F'(I)\mathbf{cf}, \tag{129}$$

and conclude, as claimed, that $\mathbf{f} = \mathbf{0}$ is the unique stationary point of B.

The final formula for the plate energy W is obtained from (107) and (119). Thus,

$$W = hF(I) + \tfrac{1}{12}h^3[F'(I)\mathbf{F}' \cdot \mathbf{F}' + 2F''(I)(\mathbf{F} \cdot \mathbf{F}')^2 + \lambda^4 F'(I)(|\mathbf{b}|^2 + 4H^2)],$$

where, from (22) and (120),

$$I = \nabla \mathbf{r} \cdot \nabla \mathbf{r} + \lambda^2, \quad \mathbf{F}' \cdot \mathbf{F}' = \lambda^2(\nabla \mathbf{n} \cdot \nabla \mathbf{n} + 4\lambda^2 H^2) \quad \text{and} \quad \mathbf{F} \cdot \mathbf{F}' = \lambda(tr\boldsymbol{\kappa} + 2\lambda^2 H). \tag{130}$$

To better exhibit the bending-stretching coupling in this energy we express it in terms of $\mathbf{c}$ and $\boldsymbol{\kappa}$. Thus, noting that

$$2H = -tr(\mathbf{c}^{-1}\boldsymbol{\kappa}), \quad \nabla \mathbf{n} \cdot \nabla \mathbf{n} = tr(\mathbf{c}^{-1}\boldsymbol{\kappa}^2) \quad \text{and} \quad |\mathbf{b}|^2 = tr[(\mathbf{c}^{-1}\boldsymbol{\kappa})^2], \tag{131}$$

we obtain

$$\begin{aligned} W &= hF(I) \\ &\quad + \tfrac{1}{12}h^3\lambda^2\{F'(I)[tr(\mathbf{c}^{-1}\boldsymbol{\kappa}^2) + 2\lambda^2(tr(\mathbf{c}^{-1}\boldsymbol{\kappa}))^2 + \lambda^2 tr(\mathbf{c}^{-1}\boldsymbol{\kappa})^2] + 2F''(I)[tr\boldsymbol{\kappa} - \lambda^2 tr(\mathbf{c}^{-1}\boldsymbol{\kappa})]^2\}, \end{aligned} \tag{132}$$

with

$$I = tr\mathbf{c} + \lambda^2, \quad \lambda^2 = (\det \mathbf{c})^{-1} \quad \text{and} \quad \mathbf{c}^{-1} = \lambda^2[(tr\mathbf{c})\mathbf{1} - \mathbf{c}], \tag{133}$$

the last of these following from the Cayley-Hamilton formula. Here, $tr(\mathbf{c}^{-1}\boldsymbol{\kappa}^2) = \left|\mathbf{c}^{-1/2}\boldsymbol{\kappa}\right|^2 \geq 0$, with equality if and only if $\mathbf{c}^{-1/2}\boldsymbol{\kappa} = \mathbf{0}$ and, thus, if and only if $\boldsymbol{\kappa} = \mathbf{0}$. Inequalities (26) then imply that this energy is a positive-definite homogeneous quadratic function of the bending strain $\boldsymbol{\kappa}$.

7.1 The Legendre-Hadamard inequality

The Weingarten equations of differential geometry may be used to express $\boldsymbol{\kappa}$ in terms of $\nabla\mathbf{r}$ and $\nabla\nabla\mathbf{r}$. The explicit expression is $\boldsymbol{\kappa} = \kappa_{\alpha\beta}\mathbf{e}_\alpha\otimes\mathbf{e}_\beta$ with $\kappa_{\alpha\beta} = -\mathbf{n}\cdot\mathbf{r}_{,\alpha\beta}$, where $\mathbf{e}_\alpha \in \Omega'$ $(\alpha, \beta = 1, 2)$ are orthonormal vectors associated with a Cartesian coordinate system x_α and the comma preceded by a subscript signifies partial differentiation with respect to the indicated coordinate. In particular, $\nabla\nabla\mathbf{r} = \mathbf{r}_{,\alpha\beta}\otimes\mathbf{e}_\alpha\otimes\mathbf{e}_\beta$. The operative Legendre-Hadamard inequality for the energy E (see (118)) - a necessary condition satisfied by energy minimizing deformations $\mathbf{r}(\mathbf{x})$ - is the requirement that the coefficient of h^3 in the expression for W be non-negative if $\nabla\nabla\mathbf{r}$ is replaced by $\mathbf{a}\otimes\mathbf{z}\otimes\mathbf{z}$, with $\mathbf{a}$ an arbitrary 3-vector and $\mathbf{z}\in\Omega'$ an arbitrary 2-vector [18]. With some effort we may reduce this requirement to

$$(\mathbf{a}\cdot\mathbf{n})^2\{F'(I)[|\mathbf{z}|^2(\mathbf{z}\cdot\mathbf{c}^{-1}\mathbf{z}) + 3\lambda^2(\mathbf{z}\cdot\mathbf{c}^{-1}\mathbf{z})^2] + 2F''(I)(\lambda^2\mathbf{z}\cdot\mathbf{c}^{-1}\mathbf{z} - |\mathbf{z}|^2)^2\} \geq 0. \tag{134}$$

Inequalities (26) ensure that this is satisfied for all $\mathbf{a}$ and $\mathbf{z}$, with equality if and only if either $\mathbf{z} = \mathbf{0}$ or $\mathbf{a}\cdot\mathbf{n} = 0$. Moreover, W is a convex function of $\nabla\nabla\mathbf{r}$ but is neither strictly convex nor coercive because it involves only the normal part $\mathbf{n}\cdot\mathbf{r}_{,\alpha\beta}$ of $\mathbf{r}_{,\alpha\beta}$. Precisely the same situation is encountered in Koiter's shell theory. In [19] this is addressed by estimating the Koiter energy in terms of a suitable polyconvex function for which an existence theorem is proved. We anticipate that a similar procedure could be adapted to the present energy.

In connection with this issue we note that a term $\epsilon|\nabla\lambda|^2$, say, with ϵ an arbitrary positive constant of order unity, may be added to the coefficient of h^3 in (132) without impairing the order- h^3 accuracy of the energy (see (113)). We state without proof that this yields $\nabla\lambda \to -\lambda[\mathbf{a}\cdot(\nabla\mathbf{r})^{-t}\mathbf{z}]\mathbf{z}$ and hence that the proposed modification adds the term $\epsilon\lambda^2[\mathbf{a}\cdot(\nabla\mathbf{r})^{-t}\mathbf{z}]^2|\mathbf{z}|^2$ to the left-hand side of (134). This modification (or any similar one based on $\nabla\lambda$) thus achieves only a partial regularization of the energy because equality in (134) follows if $\mathbf{a}\in T_\omega(p)$ is orthogonal to $(\nabla\mathbf{r})^{-t}\mathbf{z}$. The modified energy is again convex in $\nabla\nabla\mathbf{r}$ but not strictly so.

We remark, however, that there is no reason to expect minimizers of the parent three-dimensional energy to minimize its truncated approximation (118). Nevertheless it is clearly desirable from the viewpoint of analysis that the truncated energy should admit minimizers.

7.2 Pure bending

Pure bending corresponds to deformations for which $\mathbf{c} = \mathbf{1}$, identically, and hence $\mathbf{C} = \mathbf{I}$ at all points on the closure of Ω. With $F(3) = 0$ and $F'(3) = \frac{1}{2}\mu$ for the considered family of generalized neo-Hookean materials, the energy (132) reduces to

$$W_b = \tfrac{1}{12}h^3\mu[tr(\boldsymbol{\kappa}^2) + (tr\boldsymbol{\kappa})^2], \tag{135}$$

in precise agreement with the classical bending energy for plates composed of incompressible materials having a Poisson ratio $\nu = \frac{1}{2}$ (see [8], eq. (4.136)).

For such deformations both $\mathbf{F}$ and $\mathbf{F}^*$ reduce to the rotation field $\mathbf{R}(\mathbf{x})$ in the polar decomposition of the deformation gradient. Then (125) furnishes $\mathbf{P} = \mathbf{0}$ and $\mathbf{p}(= \mathbf{P}\boldsymbol{\nu}) = \mathbf{0}$, and the pure-bending energy of an edge-loaded plate reduces to

$$E_b = \int_\Omega W_b da - \tfrac{1}{24}h^3 \int_{\partial\Omega_n} (\mathbf{p}'' \cdot \mathbf{r} + 2\mathbf{p}' \cdot \mathbf{n}) ds. \tag{136}$$

Gauss's *Theorema Egregium* [17] implies that nonlinear pure-bending theory is limited to situations in which a deformed surface ω has vanishing Gaussian curvature. It is therefore of limited utility. Our purpose in discussing it here is to promote confidence in the expression (132) for the bending-stretching energy.

Appendix A: Even scalar-valued functions of vectors

We establish that $G(-\mathbf{e}) = G(\mathbf{e})$ if and only if there is a function H such that $G(\mathbf{e}) = H(\mathbf{e} \otimes \mathbf{e})$. Sufficiency is obvious. To prove necessity we show that $G(\mathbf{e})$ is determined by $\mathbf{e} \otimes \mathbf{e}$; that is,

$$G(\mathbf{a}) = G(\mathbf{b}) \quad \text{whenever} \quad \mathbf{a} \otimes \mathbf{a} = \mathbf{b} \otimes \mathbf{b}. \tag{137}$$

The second of these yields

$$a^2\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b} \quad \text{and} \quad b^2\mathbf{b} = (\mathbf{a} \cdot \mathbf{b})\mathbf{a}, \tag{138}$$

where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$, and therefore $a = b$ and $a^2b^2 = (\mathbf{a} \cdot \mathbf{b})^2$. But there is $\theta \in \mathbb{R}$ such that $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$. Thus, $\cos\theta = \pm 1$ and either of (138) implies that $\mathbf{b} = \pm\mathbf{a}$ and $G(\mathbf{b}) = G(\pm\mathbf{a})$, so that if G is insensitive to the choice of sign, as assumed, then (137) follows.

Appendix B: Detailed calculations

Apart from a term involving $\nabla\lambda$, the suppression of which is justified in Section 6, we have, from (52) with $\mathbf{e} = \mathbf{0}$, that $\nabla\mathbf{d} = \lambda\nabla\mathbf{n}$. On combining (54) with

$$\mathbf{F}^* = (\nabla\mathbf{r})^{-t} + j\mathbf{n} \otimes \mathbf{k} \tag{139}$$

we then obtain

$$\begin{aligned} j\mathbf{g} \cdot \mathbf{n} &= -\lambda[(\nabla\mathbf{r})^{-t} + j\mathbf{n} \otimes \mathbf{k}] \cdot \nabla\mathbf{n} \\ &= -\lambda(\nabla\mathbf{r})^{-t} \cdot \nabla\mathbf{n} - (\nabla\mathbf{n})^t\mathbf{n} \cdot \mathbf{k} \\ &= \lambda\mathbf{b}(\nabla\mathbf{r}) \cdot (\nabla\mathbf{r})^{-t} \\ &= \lambda tr[\mathbf{b}(\nabla\mathbf{r})(\nabla\mathbf{r})^{-1}] \\ &= \lambda tr\mathbf{b}, \end{aligned} \tag{140}$$

on noting that $(\nabla\mathbf{r})(\nabla\mathbf{r})^{-1}$ is the identity on $T_\omega(p)$. Thus,

$$\mathbf{g} \cdot \mathbf{n} = 2j^{-2}H, \quad \text{where} \quad 2H = tr\mathbf{b}. \tag{141}$$

Next, with $g_n = \mathbf{g}\cdot\mathbf{n}$ we have

$$\begin{aligned}\mathbf{F}^*\cdot\nabla\mathbf{g} &= [(\nabla\mathbf{r})^{-t} + j\mathbf{n}\otimes\mathbf{k}]\cdot(\mathbf{n}\otimes\nabla g_n + g_n\nabla\mathbf{n})\\ &= (\nabla\mathbf{r})^{-1}\mathbf{n}\cdot\nabla g_n + g_n(\nabla\mathbf{r})^{-t}\cdot\nabla\mathbf{n} + j\mathbf{k}\cdot\nabla g_n + jg_n(\nabla\mathbf{n})^t\mathbf{n}\cdot\mathbf{k}\\ &= g_n(\nabla\mathbf{r})^{-t}\cdot\nabla\mathbf{n}\\ &= -g_n\mathbf{b}(\nabla\mathbf{r})\cdot(\nabla\mathbf{r})^{-t}\\ &= -g_n tr\mathbf{b}\\ &= -4j^{-2}H^2, \end{aligned}\tag{142}$$

From $(55)_2$ we also have

$$\begin{aligned}\mathbf{A} &= \mathbf{F}^*[\lambda(\nabla\mathbf{n})^t + \mathbf{k}\otimes\mathbf{g}]\\ &= \lambda[(\nabla\mathbf{r})^{-t} + j\mathbf{n}\otimes\mathbf{k}](\nabla\mathbf{n})^t + j\mathbf{n}\otimes\mathbf{g}\\ &= \lambda(\nabla\mathbf{r})^{-t}(\nabla\mathbf{n})^t + \mathbf{n}\otimes(\nabla\mathbf{n})\mathbf{k} + j\mathbf{n}\otimes\mathbf{g}\\ &= -\lambda\mathbf{b} + j\mathbf{n}\otimes\mathbf{g}. \end{aligned}\tag{143}$$

Thus,

$$\begin{aligned}\mathbf{A}^2 &= \lambda^2\mathbf{b}^2 - g_n\mathbf{n}\otimes\mathbf{b}\mathbf{n} + j^2 g_n\mathbf{n}\otimes\mathbf{g}\\ &= \lambda^2\mathbf{b}^2 + j^2 g_n\mathbf{n}\otimes\mathbf{g}, \end{aligned}\tag{144}$$

giving (see $(55)_1$)

$$\begin{aligned}\mathbf{F}^*\cdot\mathbf{F}'' &= tr(\mathbf{A}^2)\\ &= \lambda^2 tr(\mathbf{b}^2) + j^2 g_n^2\\ &= j^{-2}(|\mathbf{b}|^2 + 4H^2), \end{aligned}\tag{145}$$

and (56) delivers

$$\begin{aligned}j\mathbf{h}\cdot\mathbf{n} &= tr(\mathbf{A}^2) - \mathbf{F}^*\cdot\nabla\mathbf{g}\\ &= j^{-2}(|\mathbf{b}|^2 + 8H^2). \end{aligned}\tag{146}$$

Finally, the Cayley-Hamilton formula

$$\mathbf{b}^2 = (tr\mathbf{b})\mathbf{b} - (\det\mathbf{b})\boldsymbol{\iota},\tag{147}$$

where $\boldsymbol{\iota}$ is the identity on $T_\omega(p)$, furnishes $|\mathbf{b}|^2 = tr(\mathbf{b}^2) = 4H^2 - 2K$, where $K = \det\mathbf{b}$, yielding

$$\mathbf{h}\cdot\mathbf{n} = 2j^{-3}(6H^2 - K).\tag{148}$$

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