Rational Mechanics of Material Strength in Brittle Solids

Arash Yavari*[1,2] and Aditya Kumar[1]

[1] *School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*
[2] *The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*

May 5, 2026

Abstract

Material strength is a classical concept that has recently found renewed applications in fracture mechanics, especially in models for crack nucleation in brittle solids. In this paper, we formulate material strength in the setting of finite elasticity and examine its geometric, constitutive, and symmetry-theoretic foundations. We show that spatial covariance requires a strength function to depend on both stress and the corresponding strain measure, so that strength is not controlled by stress alone, but by the pair (stress, strain). Only in this case can a strength function written in terms of one stress measure be consistently rewritten in terms of another, while classical stress-based strength criteria are recovered as a special case in which the strain dependence is suppressed. We discuss the covariance of strength functions under arbitrary spatial diffeomorphisms and use this to relate representations in terms of the first Piola–Kirchhoff, second Piola–Kirchhoff, and Cauchy stresses. Restricting attention to the stress-based criteria that appear in the existing literature, we define the associated strength hypersurface as a subset of the constitutively admissible stress manifold, distinguish constitutive admissibility from fracture, and analyze the geometric and topological properties of the corresponding safe domain. We show that, for stress-based strength functions satisfying standard regularity conditions and the requirement that sufficiently large stresses are inadmissible, the strength surface is a smooth compact hypersurface of the constitutively admissible stress manifold. For isotropic solids, we study the symmetry of strength surfaces in principal stress space and show that the safe domain is star-shaped under a natural proportional-reduction hypothesis. We then extend the formulation to anelastic brittle solids and examine the effects of residual stresses and anelastic distortions on material strength. We also discuss the action of material symmetry on strength functions for anisotropic solids. Finally, we discuss material strength in the setting of linear elasticity and show how the general theory reduces to the classical stress-based criteria in this limit.

Keywords: Material strength, brittle solids, finite elasticity, covariance, strength hypersurface, anisotropy, residual stress, anelasticity.

Contents

*Corresponding author, e-mail: arash.yavari@ce.gatech.edu

1 Introduction

Brittle solids offer in principle the simplest setting to formulate a theory of fracture. Brittle solids are those that are nominally elastic and only dissipate energy through creation of new surfaces [Griffith, 1921]. Despite this apparent simplicity, developing a comprehensive theory that is predictive across all brittle materials and loading conditions has proved to be challenging. Brittle solids encompass not only classical linear elastic brittle materials, such as glass and ceramics, which fail at infinitesimal strains, but also elastomers that can undergo large deformations before brittle failure [Rivlin and Thomas, 1953]. Brittle behavior is often regarded as an idealization, as microscopic crack-growth mechanisms typically introduce a degree of inelasticity prior to failure. However, the influence of such inelastic effects diminishes with increasing structural length scale. Moreover, brittle materials can exhibit measurable inelasticity under compressive loading. Nevertheless, a rigorous understanding of the idealized brittle state remains foundational to the development of fracture theories in more complex settings.

Experimental studies of brittle solids emphasize two canonical problems: (i) failure of the solids under uniform stress/strain, and (ii) crack growth and failure from a pre-existing large crack. The study of the former

dates back at least to the time of Galileo Galilei (1564-1642) [Yu, 2002], but picked up steam in the 19th century with the experimental efforts of Lamé and Clapeyron [1833] and others. The first study of the latter problem is largely credited to Griffith [1921]. These two experimental problems are used to define the two key material properties that dictate the fracture mechanics of brittle solids: strength and toughness. Out of these two properties, strength has proved to be both a harder property to measure and define in so much that a clear definition is rarely laid out and often implicitly misunderstood. The goal of this work is to present a more complete understanding of strength.

Strength has been defined in two subtly distinct ways as a material property. The conventional view treats it as the elastic limit in an elastic brittle material. Peak or maximum attainable stress is used to define this elastic limit in conventional hard brittle materials [Yu, 2004], whereas in elastomers, the peak strain or strain energy is often utilized [Hamdi et al., 2006; Chen et al., 2017]. The response remains elastic below this threshold and stress or strain exceeding it are deemed inadmissible. However, this definition is frequently conflated with a failure criterion, although the two are distinct, even in the absence of pre-existing cracks or sharp notches that would induce stress singularities. To illustrate, consider a structure containing a U-shaped notch. The associated stress concentration depending on the radius of the notch causes the local elastic limit to be reached at relatively low applied loads; yet, the onset of a macroscopic crack may occur only at substantially higher loads. Another illustrative example is the Brazilian fracture test, commonly used to estimate the tensile strength of rock-like materials [Li and Wong, 2013]. In this test, a thin circular disk is compressed between two rigid platens until it fractures. A widespread misinterpretation is that failure occurs when the maximum tensile stress reaches the material's tensile strength. However, more careful analyses reveal that, due to the non-uniformity of the stress field, fracture can occur at significantly higher loads [Kumar et al., 2024]. Thus, a point-wise violation of the strength limit does not, by itself, signify material failure, and certainly does not imply structural failure. In reality, the material is often presumed to undergo a softening process, wherein its load-carrying capacity degrades progressively until sufficient energy is available for crack formation. However, this post-peak softening behavior is generally not directly accessible from experiments. Consequently, defining strength purely as an elastic limit does not lead to a robust definition of a material property.

A second definition of strength is obtained by identifying it with the onset of failure. Experiments show that this occurs in homogeneous bodies subjected to spatially uniform stress states. Motivated by this observation, Kumar and Lopez-Pamies [2020] and Kumar et al. [2020] defined strength as follows: "the strength of an elastic brittle material is the set of all critical stresses at which the material fractures when it is subjected to a state of monotonically increasing, spatially uniform, but otherwise arbitrary stress." This is a more meaningful definition as it relates directly to experimental observations of failure. However, there are two aspects of this definition that need careful scrutiny.

The first concerns the exclusive characterization of strength as a function of stress, rather than as a function of strain, strain energy, or a combined stress–strain description. Stress-based definitions of strength, represented in form of a strength surface, are well accepted in the literature and are experimentally consistent. However, as stated before, experimental practice in elastomers often favors strain- or energy-based measures. This preference is largely pragmatic: due to the pronounced strain-stiffening exhibited by elastomers at large deformations, it is often more reliable to identify a critical stretch or strain energy than the corresponding critical stress. Moreover, there is an outsized focus in experimental literature on the uniaxial tensile state. Nevertheless, Kumar et al. [2018] and Kumar et al. [2020] presented a counterexample illustrating the limitations of strain- or energy-based definitions. In the case of cavitation in an incompressible elastomer subjected to uniform hydrostatic tension, the internal constraint prevents any increase in strain or strain energy, while the stress continues to grow until failure occurs (see Example 3.4). Motivated by such examples, and by the success of stress-based strength theories in hard brittle materials, they advocated for a definition of strength based solely on stress. However, in finite elasticity, the choice of stress measure is not unique. Consequently, it is not immediately clear which stress measure is most appropriate for defining strength surfaces and whether purely stress-based definitions of strength are appropriate in finite elasticity. Clarifying this issue constitutes one of the central objectives of the present work.

The second aspect concerns the notion of spatially uniform stress. In linear elasticity, the meaning of this is usually unambiguous, but in finite elasticity one must distinguish carefully between different stress measures and between homogeneous stress and homogeneous deformation. In particular, a homogeneous deformation induces a spatially uniform stress field in a homogeneous compressible hyperelastic (or even Cauchy elastic) solid, but the converse is not true, in general. Thus, the phrase "spatially uniform stress" requires further clarification

before it can be used as the basis for a mathematical definition of material strength. We discuss this further in this work.

The present study of strength is timely. Recent developments in computational fracture mechanics, in particular the phase-field method [Bourdin et al., 2000; Kumar et al., 2020], has provided unprecedented ability to simulate precise crack physics. However, these developments also underscore the need to reconcile strength-based and toughness-based descriptions of fracture within a unified framework and with it the need to understand strength as a material property. Strength and toughness theories of fracture have seen parallel development over the last century. In many materials communities, strength has been effectively sidelined, to the extent that toughness-based approaches are often referred to as *the fracture mechanics theory* [Anderson, 2005], directly contrasted with strength-based viewpoints. This perspective is, in part, pragmatically justified: in some materials, failure is typically governed by pre-existing microscopic defects introduced during processing, rendering toughness the controlling parameter. Yet this is not universally applicable across all brittle materials. More fundamentally, any comprehensive theory of fracture must account for both the nucleation and propagation of cracks. This has been implicitly realized with the development of cohesive zone models [Barenblatt, 1959; Dugdale, 1960] that account for strength. However, such approaches are often treated as distinct frameworks, rather than as models that bridge and unify strength and toughness. There have been efforts recently to understand strength from a bottom-up approach [Bonacci et al., 2026; Lamont et al., 2025]. Strength is the macroscopic manifestation of the presence of microscopic defects, i.e., the "weakest links" in the material. This makes strength inherently stochastic; recent work has explored incorporation of strength as a stochastic parameter in the phase field models [Zeng et al., 2025].

Contributions of this paper. In this paper, we formulate material strength for brittle solids within the framework of finite elasticity and examine its geometric, constitutive, and symmetry-theoretic foundations. We show that spatial covariance requires a strength function to depend on both stress and the corresponding strain measure, so that strength is not determined by stress alone but by the pair (stress, strain). We then analyze the implications of this observation for classical strength criteria and their geometric interpretation. The main contributions of this work can be summarized as follows:

- We provide a covariant formulation of material strength in nonlinear elasticity and show that a strength function must depend on both stress and strain in order to be representation-independent under changes of stress measure.
- We define the strength hypersurface as a subset of the manifold of constitutively admissible stresses and distinguish clearly between constitutive admissibility and fracture.
- We analyze the geometric and topological properties of the strength hypersurface and show that, under a proportional-reduction hypothesis, the associated safe domain is star-shaped.
- For isotropic solids, we derive invariant representations of strength functions in terms of stress invariants and study the symmetry properties of the corresponding strength surfaces in principal stress space.
- We extend the formulation to anelastic brittle solids and show that eigenstrains, through their modification of the material metric, alter the stress invariants and simultaneously affect the manifold of constitutively admissible stresses.
- We analyze the action of material symmetry on strength functions for anisotropic solids and provide invariant representations for several symmetry classes.
- We show that classical stress-based strength criteria of linear elasticity arise as leading-order approximations in the small-strain regime and are not obtained by direct linearization of a general nonlinear strength function.

This paper is organized as follows. In §2, we review the necessary background from nonlinear elasticity, including material symmetry, anisotropic elasticity, and constitutive equations of implicit elasticity. §3 formulates the notion of material strength in finite elasticity, discusses the choice of stress measure, loading conditions, the strength function and the associated strength hypersurface, and examines their topological properties, as well as the role of internal constraints. §4 specializes the discussion to isotropic solids and studies several classical isotropic strength surfaces and the geometry of their safe domains. §5 discusses material strength in the presence of residual stresses and introduces the corresponding formulation in anelasticity. §6 turns to anisotropic solids and discusses how material symmetry acts on strength functions. §7 discusses material strength in the setting of linear elasticity and shows how the general theory reduces to the classical stress-based criteria in this limit. Finally, §8 summarizes the main conclusions and comments on possible directions for future work.

2 Nonlinear Elasticity

This section reviews the basic framework of nonlinear elasticity used throughout this work. We begin with the kinematics of deformation, followed by commonly used stress measures and material symmetry, and conclude with the framework of implicit elasticity.

2.1 Kinematics

Consider a body whose reference configuration is identified with an embedded submanifold $\mathcal{B}$ of a Euclidean ambient space $\mathcal{S}$. We denote by $\mathbf{g}$ the Euclidean metric of $\mathcal{S}$ and by $\mathbf{G} = \mathbf{g}\big|_{\mathcal{B}}$ the induced metric on the body in its undeformed configuration. A deformation is a map $\varphi : \mathcal{B} \to \mathcal{C} \subset \mathcal{S}$, where $\mathcal{C} = \varphi(\mathcal{B})$ is the current configuration. Its tangent map is the deformation gradient $\mathbf{F} = T\varphi$, which is independent of any metric structure. At each material point $X \in \mathcal{B}$, the deformation gradient is a linear map $\mathbf{F}(X) : T_X\mathcal{B} \to T_{\varphi(X)}\mathcal{C}$ where $T_X\mathcal{B}$ and $T_{\varphi(X)}\mathcal{C}$ are the tangent spaces of $\mathcal{B}$ and $\mathcal{C}$ at $X \in \mathcal{B}$ and $\varphi(X) \in \mathcal{C}$, respectively. If $\{X^A\}$ and $\{x^a\}$ are coordinate charts on $\mathcal{B}$ and $\mathcal{C}$, respectively, then the components of the deformation gradient are $F^a{}_A = \frac{\partial \varphi^a}{\partial X^A}$. Deformation gradient has the following local coordinate representation

$$\mathbf{F} = F^a{}_A \, \frac{\partial}{\partial x^a} \otimes dX^A \,. \tag{2.1}$$

Its adjoint $\mathbf{F}^\star : T^*_{\varphi(X)}\mathcal{C} \to T^*_X\mathcal{B}$, where $T^*_X\mathcal{B}$ and $T^*_{\varphi(X)}\mathcal{C}$ are cotangent spaces of $\mathcal{B}$ and $\mathcal{C}$ at $X \in \mathcal{B}$ and $\varphi(X) \in \mathcal{C}$, respectively,[1] is defined such that $\langle \mathbf{F}^\star \boldsymbol{\alpha}, \mathbf{U} \rangle = \langle \boldsymbol{\alpha}, \mathbf{F}\mathbf{U} \rangle$, $\forall \boldsymbol{\alpha} \in T^*_{\varphi(X)}\mathcal{C}\,, \mathbf{U} \in T_X\mathcal{B}$, where $\langle .,. \rangle$ is the natural paring of covectors (1-forms) and vectors. The adjoint deformation gradient has the following local coordinate representation

$$\mathbf{F}^\star = F^a{}_A \, dX^A \otimes \frac{\partial}{\partial x^a} \,. \tag{2.2}$$

While deformation gradient maps a vector in the reference tangent space to a vector in the deformed tangent space, adjoint deformation gradient maps a covector in the deformed cotangent space to a covector in the reference cotangent space. It should be noted that the adjoint operator is metric independent.

The deformation gradient transpose $\mathbf{F}^\mathsf{T} : T_{\varphi(X)}\mathcal{C} \to T_X\mathcal{B}$ is defined such that $\langle\!\langle \mathbf{F}\mathbf{U}, \mathbf{u} \rangle\!\rangle_{\mathbf{g}} = \langle\!\langle \mathbf{U}, \mathbf{F}^\mathsf{T}\mathbf{u} \rangle\!\rangle_{\mathbf{G}}$, $\forall \mathbf{U} \in T_X\mathcal{B}\,, \mathbf{u} \in T_{\varphi(X)}\mathcal{C}$, where $\langle\!\langle .,. \rangle\!\rangle_{\mathbf{g}}$ and $\langle\!\langle .,. \rangle\!\rangle_{\mathbf{G}}$ are the inner products induced by $\mathbf{g}$ and $\mathbf{G}$, respectively. Thus, $\mathbf{F}^\mathsf{T} = \mathbf{G}^\sharp \mathbf{F}^\star \, \mathbf{g}$ and in components are $(F^\mathsf{T})^A{}_a = G^{AB} \, g_{ab} \, F^b{}_B$.

The right Cauchy–Green strain tensor is defined as $\mathbf{C} = \mathbf{F}^\mathsf{T}\mathbf{F}$. In components, $C^A{}_B = (F^\mathsf{T})^A{}_a \, F^a{}_B$, and hence $C_{AB} = (g_{ab} \circ \varphi) \, F^a{}_A \, F^b{}_B$. Thus, the covariant right Cauchy–Green tensor is simply the pull-back of the spatial metric, i.e., $\mathbf{C}^\flat = \varphi^* \mathbf{g}$, where $\flat$ denotes lowering indices with $\mathbf{G}$. The left Cauchy–Green strain tensor is defined by $\mathbf{B}^\sharp = \varphi^*(\mathbf{g}^\sharp)$. Its components are $B^{AB} = F^{-A}{}_a \, F^{-B}{}_b \, g^{ab}$, where $F^{-A}{}_a$ are the components of $\mathbf{F}^{-1}$. The spatial counterparts of these tensors are obtained by pushing forward the reference metric and its inverse. In particular, the tensor $\mathbf{c}^\flat = \varphi_* \mathbf{G} = \mathbf{F}^{-\star} \mathbf{G} \mathbf{F}^{-1}$ has components $c_{ab} = F^{-A}{}_a \, F^{-B}{}_b \, G_{AB}$, while $\mathbf{b}^\sharp = \varphi_*(\mathbf{G}^\sharp)$ has components $b^{ab} = F^a{}_A \, F^b{}_B \, G^{AB}$. Recall that $\mathbf{b} = \mathbf{c}^{-1}$. The tensors $\mathbf{C}$ and $\mathbf{b}$ (with components $C^A{}_B$ and $b^a{}_b$) have the same principal invariants. These are [Ogden, 1997; Marsden and Hughes, 1983]

$$I_1 = \operatorname{tr} \mathbf{b} = b^{ab} \, g_{ab} \,, \qquad I_2 = \frac{1}{2} \left(I_1^2 - \operatorname{tr} \mathbf{b}^2 \right) = \frac{1}{2} \left(I_1^2 - b^{ab} b^{cd} \, g_{ac} \, g_{bd} \right) \,, \qquad I_3 = \det \mathbf{b} \,. \tag{2.3}$$

Nanson's formula is written as [Marsden and Hughes, 1983]

$$\mathbf{n}^\flat \, da = J \, \mathbf{F}^{-\star} \, \mathbf{N}^\flat \, dA \,, \qquad n_a \, da = J \, F^{-A}{}_a \, N_A \, dA \,, \tag{2.4}$$

where the Jacobian of deformation is defined by $dv = J \, dV$ and is explicitly given as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \, \det \mathbf{F} \,. \tag{2.5}$$

Raising the spatial index, one obtains $n^a \, da = J \, g^{ab} \, F^{-B}{}_b \, G_{BA} \, N^A \, dA$. Equivalently, Nanson's formula can be written in vector form as

$$\mathbf{n} \, da = J \, \mathbf{F}^{-\mathsf{T}} \, \mathbf{N} \, dA \,. \tag{2.6}$$

[1] A cotangent space (space of covectors or 1-forms) is dual to its corresponding tangent space.

The polar decomposition of the deformation gradient is written as

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}\,, \tag{2.7}$$

where $\mathbf{U}$ and $\mathbf{V}$ denote the material and spatial stretch tensors, respectively, and should not be confused with the material velocity. The tensor $\mathbf{R} : T\mathcal{B} \to T\mathcal{C}$ is a $(\mathbf{G}, \mathbf{g})$-orthogonal tensor field [Simo and Marsden, 1984], i.e.,[2]

$$\mathbf{R}^\star(\mathbf{g} \circ \varphi)\, \mathbf{R} = \mathbf{G}\,. \tag{2.8}$$

In components, this relation reads $R^a{}_A\, (g_{ab} \circ \varphi)\, R^b{}_B = G_{AB}$. The polar decomposition itself is written in components as $F^a{}_A = R^a{}_B\, U^B{}_A = V^a{}_b\, R^b{}_A$. Equation (2.8) implies that $(\det \mathbf{R})^2 \det \mathbf{g} = \det \mathbf{G}$, while (2.7) implies that $\det \mathbf{U} = \det \mathbf{V}$. From (2.5) one can see that $J = \det \mathbf{U} = \det \mathbf{V}$. The material stretch tensor $\mathbf{U} : T_X\mathcal{B} \to T_X\mathcal{B}$ and the spatial stretch tensor $\mathbf{V} : T_x\mathcal{C} \to T_x\mathcal{C}$ are related to the right and left Cauchy–Green deformation tensors by

$$\begin{aligned} \mathbf{C} = \mathbf{F}^\mathsf{T}\mathbf{F} &= (\mathbf{R}\mathbf{U})^\mathsf{T}\mathbf{R}\mathbf{U} = \mathbf{G}^\sharp(\mathbf{R}\mathbf{U})^\star\mathbf{g}\mathbf{R}\mathbf{U} = \mathbf{G}^\sharp\mathbf{U}^\star\mathbf{R}^\star\mathbf{g}\mathbf{R}\mathbf{U} = \mathbf{G}^\sharp\mathbf{U}^\star\mathbf{G}\mathbf{U} = \mathbf{U}^2\,, \\ \mathbf{b} = \mathbf{F}\mathbf{F}^\mathsf{T} &= \mathbf{V}\mathbf{R}(\mathbf{V}\mathbf{R})^\mathsf{T} = \mathbf{V}\mathbf{R}\mathbf{G}^\sharp(\mathbf{V}\mathbf{R})^\star\mathbf{g} = \mathbf{V}\mathbf{R}\mathbf{G}^\sharp\mathbf{R}^\star\mathbf{V}^\star\mathbf{g} = \mathbf{V}\mathbf{g}^\sharp\mathbf{V}^\star\mathbf{g} = \mathbf{V}^2\,. \end{aligned} \tag{2.9}$$

Equivalently, one may write

$$\mathbf{C}^\flat = \mathbf{U}^\star\mathbf{G}\mathbf{U}\,, \qquad \mathbf{b}^\sharp = \mathbf{V}\mathbf{g}^\sharp\mathbf{V}^\star\,. \tag{2.10}$$

In components, these relations read $C_{AB} = U^M{}_A\, G_{MN}\, U^N{}_B$ and $b^{ab} = V^a{}_m\, g^{mn}\, V^b{}_n$. The relations (2.9) are commonly written as $\mathbf{U} = \sqrt{\mathbf{C}}$ and $\mathbf{V} = \sqrt{\mathbf{b}}$. It is straightforward to show that $U_{AB} = G_{AM}U^M{}_B = U_{BA}$.

Let $\{\overset{1}{\mathbf{N}}, \overset{2}{\mathbf{N}}, \overset{3}{\mathbf{N}}\}$ be a $\mathbf{G}$-orthonormal eigenbasis of $\mathbf{U}$, with corresponding principal stretches λ_j, $j = 1, 2, 3$. Using the spectral decomposition of $\mathbf{C}$ and $\mathbf{U}$, one has [Ogden, 1997]

$$\mathbf{C}^\sharp = \lambda_1^2\, \overset{1}{\mathbf{N}} \otimes \overset{1}{\mathbf{N}} + \lambda_2^2\, \overset{2}{\mathbf{N}} \otimes \overset{2}{\mathbf{N}} + \lambda_3^2\, \overset{3}{\mathbf{N}} \otimes \overset{3}{\mathbf{N}}\,, \qquad \mathbf{U}^\sharp = \lambda_1\, \overset{1}{\mathbf{N}} \otimes \overset{1}{\mathbf{N}} + \lambda_2\, \overset{2}{\mathbf{N}} \otimes \overset{2}{\mathbf{N}} + \lambda_3\, \overset{3}{\mathbf{N}} \otimes \overset{3}{\mathbf{N}}\,, \tag{2.11}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches and $\overset{1}{\mathbf{N}}, \overset{2}{\mathbf{N}}, \overset{3}{\mathbf{N}}$ are the corresponding principal directions. Recall that $\overset{1}{\mathbf{N}} \otimes \overset{1}{\mathbf{N}} + \overset{2}{\mathbf{N}} \otimes \overset{2}{\mathbf{N}} + \overset{3}{\mathbf{N}} \otimes \overset{3}{\mathbf{N}} = \mathbf{G}^\sharp$. The representation $(2.11)_2$ is equivalent to

$$\mathbf{U} = \lambda_1\, \overset{1}{\mathbf{N}} \otimes \overset{1}{\mathbf{N}}{}^\flat + \lambda_2\, \overset{2}{\mathbf{N}} \otimes \overset{2}{\mathbf{N}}{}^\flat + \lambda_3\, \overset{3}{\mathbf{N}} \otimes \overset{3}{\mathbf{N}}{}^\flat\,, \tag{2.12}$$

and therefore

$$\mathbf{U}\overset{j}{\mathbf{N}} = \lambda_j \overset{j}{\mathbf{N}}\,, \ \text{(no summation on } j)\,. \tag{2.13}$$

Using the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, one obtains

$$\mathbf{F}\overset{j}{\mathbf{N}} = \mathbf{R}\mathbf{U}\overset{j}{\mathbf{N}} = \lambda_j \mathbf{R}\overset{j}{\mathbf{N}} = \mathbf{V}\mathbf{R}\overset{j}{\mathbf{N}}\,, \qquad j = 1, 2, 3\,. \tag{2.14}$$

Thus, if $\{\overset{1}{\mathbf{n}}, \overset{2}{\mathbf{n}}, \overset{3}{\mathbf{n}}\}$ is the eigenbasis of $\mathbf{V}$, then [Ogden, 1997]

$$\overset{j}{\mathbf{n}} = \mathbf{R}\overset{j}{\mathbf{N}}\,, \qquad j = 1, 2, 3\,. \tag{2.15}$$

Knowing that the eigenvalues of $\mathbf{b}$ and $\mathbf{V}$ are λ_j^2 and λ_j, respectively, one concludes that the Finger and spatial stretch tensors admit the spectral representations

$$\mathbf{b}^\sharp = \lambda_1^2\, \overset{1}{\mathbf{n}} \otimes \overset{1}{\mathbf{n}} + \lambda_2^2\, \overset{2}{\mathbf{n}} \otimes \overset{2}{\mathbf{n}} + \lambda_3^2\, \overset{3}{\mathbf{n}} \otimes \overset{3}{\mathbf{n}}\,, \qquad \mathbf{V}^\sharp = \lambda_1\, \overset{1}{\mathbf{n}} \otimes \overset{1}{\mathbf{n}} + \lambda_2\, \overset{2}{\mathbf{n}} \otimes \overset{2}{\mathbf{n}} + \lambda_3\, \overset{3}{\mathbf{n}} \otimes \overset{3}{\mathbf{n}}\,. \tag{2.16}$$

Note that

$$\sum_{j=1}^{3} \overset{j}{\mathbf{n}} \otimes \overset{j}{\mathbf{n}} = \sum_{j=1}^{3} \mathbf{R}\overset{j}{\mathbf{N}} \otimes \mathbf{R}\overset{j}{\mathbf{N}} = \mathbf{R}\Big(\sum_{j=1}^{3} \overset{j}{\mathbf{N}} \otimes \overset{j}{\mathbf{N}}\Big)\mathbf{R}^\star = \mathbf{R}\mathbf{G}^\sharp\mathbf{R}^\star = \mathbf{g}^\sharp\,. \tag{2.17}$$

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth monotone function such that $f(1) = 0$ and $f'(1) = 1$. Hill's strain measures are defined by applying f to the principal stretches of $\mathbf{V}$, namely [Hill, 1968, 1970, 1978]

$$f(\mathbf{V}^\sharp) = f(\lambda_1)\, \overset{1}{\mathbf{n}} \otimes \overset{1}{\mathbf{n}} + f(\lambda_2)\, \overset{2}{\mathbf{n}} \otimes \overset{2}{\mathbf{n}} + f(\lambda_3)\, \overset{3}{\mathbf{n}} \otimes \overset{3}{\mathbf{n}}\,. \tag{2.18}$$

[2] Equivalently, one may write $\mathbf{G}^\sharp\mathbf{R}^\star(\mathbf{g} \circ \varphi)\, \mathbf{R} = \mathbf{R}^\mathsf{T}\mathbf{R} = \mathrm{id}_{T\mathcal{B}}$.

A particularly important example is Hencky's logarithmic strain, which is given by

$$\mathbf{h}^\sharp = \log \mathbf{V}^\sharp = \log\lambda_1 \overset{1}{\hat{\mathbf{n}}} \otimes \overset{1}{\hat{\mathbf{n}}} + \log\lambda_2 \overset{2}{\hat{\mathbf{n}}} \otimes \overset{2}{\hat{\mathbf{n}}} + \log\lambda_3 \overset{3}{\hat{\mathbf{n}}} \otimes \overset{3}{\hat{\mathbf{n}}} \,. \tag{2.19}$$

The material analogue of Hencky's logarithmic strain is defined using the spectral decomposition of $\mathbf{U}$. Using $(2.11)_2$, one defines

$$\log \mathbf{U}^\sharp = \log\lambda_1 \overset{1}{\hat{\mathbf{N}}} \otimes \overset{1}{\hat{\mathbf{N}}} + \log\lambda_2 \overset{2}{\hat{\mathbf{N}}} \otimes \overset{2}{\hat{\mathbf{N}}} + \log\lambda_3 \overset{3}{\hat{\mathbf{N}}} \otimes \overset{3}{\hat{\mathbf{N}}} \,. \tag{2.20}$$

Equivalently, one may write

$$\log \mathbf{U} = \log\lambda_1 \overset{1}{\hat{\mathbf{N}}} \otimes \overset{1}{\hat{\mathbf{N}}}^\flat + \log\lambda_2 \overset{2}{\hat{\mathbf{N}}} \otimes \overset{2}{\hat{\mathbf{N}}}^\flat + \log\lambda_3 \overset{3}{\hat{\mathbf{N}}} \otimes \overset{3}{\hat{\mathbf{N}}}^\flat \,. \tag{2.21}$$

Thus, $\log \mathbf{U}$ has the same eigenvectors as $\mathbf{U}$, and its eigenvalues are $\log\lambda_j$, $j = 1, 2, 3$.

2.2 Measures of stress

We briefly discuss a few commonly used stress tensors in nonlinear elasticity. Although continuum mechanics admits infinitely many stress measures, the tensors reviewed here are among the most useful in applications.

Consider an area element da in the deformed configuration $\mathcal{C}$ with $\mathbf{g}$-unit normal $\mathbf{n}$, i.e., $\langle\!\langle \mathbf{n}, \mathbf{n} \rangle\!\rangle_{\mathbf{g}} = 1$. The corresponding traction vector is defined by $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}^\flat$, where $\boldsymbol{\sigma}$ is the Cauchy stress and $\mathbf{n}^\flat = \mathbf{g}\mathbf{n}$. Hence, the force acting on this area element is $\mathbf{f} = \mathbf{t}\, da$. In components, one has $t^a = \sigma^{ab} n_b$, where $n_b = g_{bc} n^c$. Let dA be the corresponding area element in the reference configuration $\mathcal{B}$ with $\mathbf{G}$-unit normal $\mathbf{N}$, i.e., $\langle\!\langle \mathbf{N}, \mathbf{N} \rangle\!\rangle_{\mathbf{G}} = 1$. The first Piola–Kirchhoff stress tensor $\mathbf{P}$ is defined by requiring that

$$\mathbf{t}\, da = \mathbf{P}\, \mathbf{N}^\flat\, dA \,. \tag{2.22}$$

Using Nanson's formula (2.4), one obtains

$$\mathbf{P} = J\, \boldsymbol{\sigma}\, \mathbf{F}^{-\star} \,. \tag{2.23}$$

In components, this reads $P^{aA} = J\, \sigma^{ab}\, F^{-A}{}_b$.

Pulling back the force $\mathbf{f}$ to the reference configuration, one next defines the second Piola–Kirchhoff stress tensor $\mathbf{S}$ by $\mathbf{F}^{-1}\mathbf{t}\, da = \mathbf{S}\, \mathbf{N}^\flat\, dA$. Thus,

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{P} = J\, \mathbf{F}^{-1}\, \boldsymbol{\sigma}\, \mathbf{F}^{-\star} \,. \tag{2.24}$$

In components, one has $S^{AB} = F^{-A}{}_a\, P^{aB} = J\, F^{-A}{}_a\, \sigma^{ab}\, F^{-B}{}_b$. The balance of angular momentum implies that $\mathbf{S} = \mathbf{S}^\star$ or in components $S^{AB} = S^{BA}$.

The Kirchhoff stress is defined as $\boldsymbol{\tau} = J\, \boldsymbol{\sigma}$, while the convected stress is defined as $\boldsymbol{\Sigma} = \varphi_t^* \boldsymbol{\sigma} = \mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\star} = J^{-1}\mathbf{S}$. The rotated stress tensor is defined as [Green and Naghdi, 1965]

$$\overset{R}{\boldsymbol{\sigma}} = \mathbf{R}^*\boldsymbol{\sigma} = \mathbf{R}^{-1}\boldsymbol{\sigma}\mathbf{R}^{-\star} \,, \tag{2.25}$$

i.e., the pull-back of the Cauchy stress by the orthogonal tensor $\mathbf{R}$. In components, this reads $\overset{R}{\sigma}{}^{AB} = R^A{}_a\, \sigma^{ab}\, R^B{}_b$.

Recall that $\mathbf{R} : T_X\mathcal{B} \to T_x\mathcal{C}$ is a $(\mathbf{G}, \mathbf{g})$-orthogonal tensor field with components $R^a{}_A$. The components of its inverse $\mathbf{R}^{-1} : T_x\mathcal{C} \to T_X\mathcal{B}$ are denoted by $R^A{}_a$ and are given by $R^A{}_a = G^{AM}\, R^m{}_M\, g_{ma}$.

The Biot stress $\overset{(1)}{\mathbf{S}}$ is the material stress tensor that is work-conjugate to the material stretch tensor $\mathbf{U}$. Note that

$$\frac{1}{2}\mathbf{S}{:}\dot{\mathbf{C}}^\flat = \frac{1}{2}\left(\mathbf{S}{:}\mathbf{G}\dot{\mathbf{U}}\mathbf{U} + \mathbf{S}{:}\mathbf{G}\mathbf{U}\dot{\mathbf{U}}\right) = \frac{1}{2}\left(\mathbf{U}\mathbf{S} + \mathbf{S}\mathbf{U}^\star\right){:}\dot{\mathbf{U}}^\flat = \overset{(1)}{\mathbf{S}}{:}\dot{\mathbf{U}}^\flat \,, \tag{2.26}$$

where

$$\begin{aligned} \overset{(1)}{\mathbf{S}} &= \operatorname{sym}(\mathbf{U}\mathbf{S}) = \frac{1}{2}\Big(\mathbf{U}\mathbf{S} + (\mathbf{U}\mathbf{S})^\star\Big) = \frac{1}{2}\left(\mathbf{U}\mathbf{S} + \mathbf{S}^\star\mathbf{U}^\star\right) = \frac{1}{2}\left(\mathbf{U}\mathbf{S} + \mathbf{S}\mathbf{U}^\star\right) \,, \\ \overset{(1)}{S}{}^{AB} &= \frac{1}{2}\left(U^A{}_M\, S^{MB} + S^{AM}\, U^B{}_M\right) \,. \end{aligned} \tag{2.27}$$

Using the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$ and the relation $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$, one concludes that $\mathbf{U}\mathbf{S} = \mathbf{U}\mathbf{F}^{-1}\mathbf{P} = \mathbf{R}^{-1}\mathbf{P}$. Therefore,

$$\overset{(1)}{\mathbf{S}} = \frac{1}{2}\left(\mathbf{R}^{-1}\mathbf{P} + \mathbf{P}^\star\mathbf{R}^{-\star}\right) = \operatorname{sym}\left(\mathbf{R}^{-1}\mathbf{P}\right) \,, \qquad \overset{(1)}{S}{}^{AB} = \frac{1}{2}\left(R^A{}_a\, P^{aB} + R^B{}_a\, P^{aA}\right) \,. \tag{2.28}$$

Eigenvalues of Biot stress for isotropic solids. Let $\{\mathbf{E}_A\}$ be a $\mathbf{G}$-orthonormal eigenbasis of $\mathbf{U}$, i.e.,

$$\mathbf{U}\mathbf{E}_A = \lambda_A\,\mathbf{E}_A\,,\ \text{(no summation on } A)\,, \qquad \langle\!\langle \mathbf{E}_A, \mathbf{E}_B\rangle\!\rangle_{\mathbf{G}} = \delta_{AB}\,. \tag{2.29}$$

For an isotropic solid $\mathbf{U}$ and $\mathbf{S}$ are coaxial, i.e., share the same eigenvectors; this property does not hold in general for anisotropic solids. Hence,

$$\mathbf{S}\mathbf{E}_A = S_A\,\mathbf{E}_A\,,\ \text{(no summation on } A)\,. \tag{2.30}$$

Knowing that $\mathbf{U}$ is $\mathbf{G}$-symmetric, i.e., $U_{AB} = U_{BA}$, one has

$$\langle\!\langle \mathbf{U}\mathbf{X}, \mathbf{Y}\rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \mathbf{X}, \mathbf{U}\mathbf{Y}\rangle\!\rangle_{\mathbf{G}}\,, \qquad \forall\,\mathbf{X},\mathbf{Y}\in T_X\mathcal{B}\,. \tag{2.31}$$

Hence,

$$\begin{aligned}\langle\!\langle \mathbf{S}\mathbf{U}^\star\mathbf{E}_A, \mathbf{E}_B\rangle\!\rangle_{\mathbf{G}} &= \langle\!\langle \mathbf{U}^\star\mathbf{E}_A, \mathbf{S}\mathbf{E}_B\rangle\!\rangle_{\mathbf{G}} = S_B\,\langle\!\langle \mathbf{U}^\star\mathbf{E}_A, \mathbf{E}_B\rangle\!\rangle_{\mathbf{G}} = S_B\,\langle\!\langle \mathbf{E}_A, \mathbf{U}\mathbf{E}_B\rangle\!\rangle_{\mathbf{G}}\\ &= S_B\lambda_B\,\langle\!\langle \mathbf{E}_A, \mathbf{E}_B\rangle\!\rangle_{\mathbf{G}} = S_B\lambda_B\,\delta_{AB}\,,\ \text{(no summation on } A \text{ or } B)\,.\end{aligned} \tag{2.32}$$

Therefore,

$$\mathbf{S}\mathbf{U}^\star\mathbf{E}_A = \lambda_A S_A\,\mathbf{E}_A\,,\ \text{(no summation on } A)\,. \tag{2.33}$$

Since also

$$\mathbf{U}\mathbf{S}\mathbf{E}_A = \lambda_A S_A\,\mathbf{E}_A\,,\ \text{(no summation on } A)\,, \tag{2.34}$$

it follows that

$$\overset{(1)}{\mathbf{S}}\mathbf{E}_A = \frac{1}{2}(\mathbf{U}\mathbf{S}+\mathbf{S}\mathbf{U}^\star)\mathbf{E}_A = \lambda_A S_A\,\mathbf{E}_A\,,\ \text{(no summation on } A)\,. \tag{2.35}$$

Hence, $\mathbf{U}$, $\mathbf{S}$, and $\overset{(1)}{\mathbf{S}}$ are coaxial and share the same eigenvectors, and the eigenvalues of the Biot stress are given by

$$\beta_A = \lambda_A S_A\,,\ \text{(no summation on } A)\,. \tag{2.36}$$

Moreover, the eigenvalues of the Cauchy stress are related to those of the Biot stress as follows. Using $\boldsymbol{\sigma} = J^{-1}\,\mathbf{F}\mathbf{S}\mathbf{F}^\star = J^{-1}\,\mathbf{R}\mathbf{U}\mathbf{S}\mathbf{U}^\star\mathbf{R}^\star$, one sees that $\boldsymbol{\sigma}$ and $J^{-1}\mathbf{U}\mathbf{S}\mathbf{U}^\star$ have the same eigenvalues. Since

$$\mathbf{U}\mathbf{S}\mathbf{U}^\star\mathbf{E}_A = \lambda_A^2 S_A\,\mathbf{E}_A\,,\ \text{(no summation on } A)\,, \tag{2.37}$$

it follows that

$$\sigma_A = J^{-1}\lambda_A^2 S_A = J^{-1}\lambda_A\beta_A\,,\ \text{(no summation on } A)\,. \tag{2.38}$$

Another useful stress measure is the Mandel stress $\mathbf{M} = \mathbf{C}\mathbf{S}$. It is work-conjugate to the material Hencky strain $\log\mathbf{U}$ [Hoger, 1987; Xiao et al., 1997] in the sense that the stress power may be written as $\mathbf{M} : \overset{\circ}{\overline{\log\mathbf{U}}}$, where for any material tensor $\mathbf{A}$ the corotational rate is defined by $\mathring{\mathbf{A}} := \dot{\mathbf{A}} + \boldsymbol{\Omega}\mathbf{A} - \mathbf{A}\boldsymbol{\Omega}$, and $\boldsymbol{\Omega} = \mathbf{R}^{-1}\dot{\mathbf{R}}$ is the material spin associated with the rotation tensor $\mathbf{R}$. Although the Mandel stress is widely used in finite-strain plasticity, we will not use it in this work.

2.3 Material symmetry group

For a hyperelastic solid, consider a stored energy density of the form $W = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g})$, where $\mathbf{g}$ is the metric of the Euclidean ambient space and $\mathbf{G}$ is the induced metric on the body in the reference configuration. In the absence of eigenstrains, $\mathbf{G}$ is the material metric.

The material symmetry group at a point $X \in \mathcal{B}$, relative to the Euclidean reference configuration $(\mathcal{B}, \mathbf{G})$, is the subgroup $\mathcal{G}_X \leqslant \mathrm{Orth}(\mathbf{G})$ consisting of all linear maps $\mathbf{K} : T_X\mathcal{B} \to T_X\mathcal{B}$ such that

$$W(X, \mathbf{F}\mathbf{K}, \mathbf{G}, \mathbf{g}) = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g})\,, \qquad \forall\,\mathbf{F}\,,\ \forall\,\mathbf{K}\in\mathcal{G}_X \leqslant \mathrm{Orth}(\mathbf{G})\,, \tag{2.39}$$

where

$$\mathrm{Orth}(\mathbf{G}) = \left\{\mathbf{Q} : T_X\mathcal{B} \to T_X\mathcal{B} \;\middle|\; \mathbf{Q}^\star\mathbf{G}\mathbf{Q} = \mathbf{G}\right\}\,. \tag{2.40}$$

Thus, the stored energy is invariant under the action of $\mathcal{G}_X$.

Isotropic solids. For an isotropic solid $\mathcal{G}_X = \mathrm{Orth}(\mathbf{G})$, the stored energy depends only on the principal invariants I_1, I_2, I_3, and hence $W = W(I_1, I_2, I_3)$.

2.4 Anisotropic elasticity

For a hyperelastic anisotropic solid, the stored energy density per unit reference volume is written as

$$W = \hat{W}(\mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n)\,, \tag{2.41}$$

where $\boldsymbol{\zeta}_i$, $i = 1, \dots, n$, are structural tensors characterizing the material symmetry group. The role of the structural tensors is to represent the energy as an isotropic scalar-valued function of its arguments. By Hilbert's theorem, for any finite collection of tensors there exists a finite integrity basis for the algebra of isotropic invariants generated by that collection. Therefore, if I_j, $j = 1, \dots, m$, is an integrity basis associated with the arguments in (2.41), then the energy may be written as $W = W(X, I_1, \dots, I_m)$. The Doyle–Ericksen formula [Doyle and Ericksen, 1956; Marsden and Hughes, 1983; Yavari et al., 2006] then gives us

$$\mathbf{S} = 2\frac{\partial \hat{W}}{\partial \mathbf{C}^\flat} = \sum_{j=1}^{m} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat}\,, \qquad W_j = \frac{\partial W}{\partial I_j}\,, \qquad j = 1, \dots, m\,. \tag{2.42}$$

Transversely isotropic solids. A transversely isotropic solid has, at each material point, a single preferred direction orthogonal to the plane of isotropy. Let $\mathbf{N}(X)$ denote this preferred direction at $X \in \mathcal{B}$. A choice for structural tensor is $\mathbf{A} = \mathbf{N} \otimes \mathbf{N}$, and the energy is written as $W = W(\mathbf{G}, \mathbf{C}^\flat, \mathbf{A})$ [Doyle and Ericksen, 1956; Spencer, 1982; Lu and Papadopoulos, 2000]. In this case the energy depends on five independent invariants,

$$I_1 = \mathrm{tr}\,\mathbf{C}\,, \qquad I_2 = \det\mathbf{C}\,\mathrm{tr}\,\mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \qquad I_4 = \mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}\,, \qquad I_5 = \mathbf{N}\cdot\mathbf{C}^2\cdot\mathbf{N}\,, \tag{2.43}$$

i.e., $W = W(I_1, \dots, I_5)$.

Orthotropic solids. An orthotropic solid possesses reflection symmetry with respect to three mutually orthogonal planes. Let $\mathbf{N}_1(X)$, $\mathbf{N}_2(X)$, and $\mathbf{N}_3(X)$ be $\mathbf{G}$-orthonormal vectors specifying the orthotropic axes at X. One may choose the structural tensors $\mathbf{A}_1 = \mathbf{N}_1 \otimes \mathbf{N}_1$, $\mathbf{A}_2 = \mathbf{N}_2 \otimes \mathbf{N}_2$, and $\mathbf{A}_3 = \mathbf{N}_3 \otimes \mathbf{N}_3$. Since $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I} = \mathrm{id}_{T_X\mathcal{B}}$, only two of these are independent. Accordingly, one writes $W = W(\mathbf{G}, \mathbf{C}^\flat, \mathbf{A}_1, \mathbf{A}_2)$ [Doyle and Ericksen, 1956; Spencer, 1982; Lu and Papadopoulos, 2000]. A convenient integrity basis is given by the following seven invariants

$$\begin{aligned} &I_1 = \mathrm{tr}\,\mathbf{C}\,, && I_2 = \det\mathbf{C}\,\mathrm{tr}\,\mathbf{C}^{-1}\,, && I_3 = \det\mathbf{C}\,,\\ &I_4 = \mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_1\,, && I_5 = \mathbf{N}_1\cdot\mathbf{C}^2\cdot\mathbf{N}_1\,, && I_6 = \mathbf{N}_2\cdot\mathbf{C}\cdot\mathbf{N}_2\,, \quad I_7 = \mathbf{N}_2\cdot\mathbf{C}^2\cdot\mathbf{N}_2\,. \end{aligned} \tag{2.44}$$

Thus, $W = W(I_1, \dots, I_7)$.

Monoclinic solids. A monoclinic solid is characterized by three material preferred directions represented by three unit vectors $\{\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$, where $\mathbf{N}_1 \cdot \mathbf{N}_2 \neq 0$ and $\mathbf{N}_3$ is normal to the plane spanned by $\mathbf{N}_1$ and $\mathbf{N}_2$ [Merodio and Ogden, 2020]. The energy function of a monoclinic solid depends on nine invariants [Spencer, 1986]. Seven of these coincide with the orthotropic invariants in (2.44), while the remaining two are

$$I_8 = \mathfrak{g}\,\mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_2, \qquad I_9 = \mathfrak{g}^2\,, \tag{2.45}$$

where $\mathfrak{g} = \mathbf{N}_1 \cdot \mathbf{N}_2$. Thus, $W = W(I_1, \dots, I_9)$.

2.5 Constitutive equations of implicit elasticity

In the literature of elasticity it is often tacitly assumed that stress is given explicitly as a function of strain, as in Cauchy elasticity [Cauchy, 1828; Yavari and Goriely, 2025] and hyperelasticity [Truesdell, 1952] (Green elasticity [Green, 1838, 1839; Spencer, 2015]). In these theories one writes constitutive equations of the form $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{b}, \mathbf{g})$. However, this is only one possible constitutive setting. One may also consider elastic materials

for which strain is determined by stress. In his work on *controllable states of stress*, Carroll [1973a] considered constitutive laws of the form

$$\mathbf{b}^\sharp = \xi_0\,\mathbf{g}^\sharp + \xi_1\boldsymbol{\sigma} + \xi_2\boldsymbol{\sigma}^2\,. \tag{2.46}$$

Here, the scalar response functions ξ_0, ξ_1, and ξ_2 depend on the principal invariants of the Cauchy stress. Carroll defined a stress field to be controllable if it satisfies the equilibrium equations in the absence of body forces and if the associated strain is compatible for arbitrary choices of the response functions ξ_0, ξ_1, and ξ_2. He proved that, for isotropic elastic solids with constitutive equations of the form (2.46), every controllable stress field is necessarily homogeneous.[3] At the same time, he observed that, for a given material in this class, not every homogeneous stress field is constitutively admissible as its corresponding $\mathbf{b}^\sharp$ may not be compatible.

More generally, one may consider elastic materials whose constitutive response is described by an implicit relation between stress and strain measures. This broader class is usually referred to as implicit elasticity [Morgan, 1966; Rajagopal, 2003, 2007]. In such theories, the constitutive equation is not written in the explicit form $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{b},\mathbf{g})$ or $\mathbf{b} = \hat{\mathbf{b}}(\boldsymbol{\sigma},\mathbf{g})$, but rather as an implicit tensor equation of the following form

$$\mathbf{f}(\boldsymbol{\sigma},\mathbf{b},\mathbf{g}) = \mathbf{0}\,. \tag{2.47}$$

Cauchy elastic and Green elastic solids are special cases of this more general framework.

Remark 2.1. This observation is important for the notion of material strength. If strength is defined as the set of critical homogeneous stresses at which fracture nucleates (see §3), then its definition does not require stress to be an explicit function of strain. What is required instead is that the stress state be constitutively admissible. In an implicit theory this means that the pair $(\boldsymbol{\sigma},\mathbf{b})$ must lie on the constitutive stress-strain manifold. Thus, the notion of material strength extends naturally beyond explicit constitutive theories. At the same time, this suggests that a general notion of strength should not be restricted to purely stress-based descriptions, since fracture is expected to depend on both stress and strain. In the following sections we will see that covariance considerations naturally lead to strength functions that depend on both stress and strain measures.

One may ask whether there are any examples of implicit elastic materials. At this time, there are no real elastic materials that have been convincingly described by such more general implicit constitutive equations. In this sense, implicit elasticity is still a relatively new development compared with the classical and well-established theories of hyperelasticity and Cauchy elasticity. These models should be viewed as constitutive generalizations of existing elastic theories. It is important to keep in mind that, in elasticity, stress and strain are on the same footing. In other words, it does not make sense to say that strain causes stress or that stress causes strain. This is the basic premise of implicit elasticity. One simply postulates a constitutive equation as an implicit relation between stress and strain measures. This viewpoint, together with the consequences of spatial covariance, also motivates our formulation of material strength. In general, strength need not be stress-based. Rather, strength is controlled by the pair (stress, strain).

Example 2.1 (Rajagopal bar)**.** Motivated by the examples provided by Rajagopal [2003], consider a one-dimensional body whose response is described by an elastic bar connected in parallel with an inextensible string. Let λ denote the stretch, and let $\hat{\sigma}(\lambda)$ be the elastic response of the bar. Let $\lambda_c > 1$ denote the limiting stretch at which the inextensible string becomes taut, and define $\sigma_c = \hat{\sigma}(\lambda_c)$. The constitutive relation is written as (we assume that for $\lambda < 1$ the body deforms elastically, neglecting buckling)

$$\begin{cases} \sigma = \hat{\sigma}(\lambda)\,, & \lambda < \lambda_c\,, \\ \lambda = \lambda_c\,, & \sigma \geq \sigma_c\,. \end{cases} \tag{2.48}$$

An equivalent implicit representation of this response is

$$(\lambda - \lambda_c)\big(\sigma - \hat{\sigma}(\lambda)\big) = 0\,, \qquad \lambda \leq \lambda_c\,, \quad \sigma \geq \sigma_c\,. \tag{2.49}$$

For $\lambda < \lambda_c$ the string is slack, and the response is governed solely by the elastic relation $\sigma = \hat{\sigma}(\lambda)$. Once the stretch reaches λ_c, the string becomes taut and enforces the constraint $\lambda = \lambda_c$. In this regime, the stretch remains fixed while the stress can increase or change arbitrarily, subject only to $\sigma \geq \sigma_c$. Thus, at $\lambda = \lambda_c$ there

[3] See Carroll [1973b] for similar discussions for incompressible solids.

is a single admissible stretch but infinitely many admissible stresses. This response is non-dissipative, but the stress cannot be expressed globally as a single-valued function of the stretch. Hence, it does not fall within the classical framework of Cauchy elasticity or Green elasticity, and is instead viewed as belonging to the broader class of implicit constitutive theories. Upon unloading from a state with $\lambda = \lambda_c$ and $\sigma > \sigma_c$, the response is uniquely determined by the requirement of non-dissipation. The body first unloads along the vertical branch $\lambda = \lambda_c$ until $\sigma = \sigma_c$. At this point the inextensible string becomes slack, and the constraint is no longer active. Further unloading follows the elastic relation $\sigma = \hat{\sigma}(\lambda)$. Thus, the loading and unloading paths coincide, and the response is fully reversible with no hysteresis. This highlights that, although the stress-stretch relation is not single-valued, the admissible path in any process is uniquely determined by the requirement that the elastic branch be followed whenever it is accessible.

2.6 Covariance, objectivity, and material symmetry in continuum mechanics

In this section we define the notion of covariance in continuum mechanics. As we will see in §3.3, covariance plays a central role in relating strength functions corresponding to different measures of stress.

The modern notion of covariance in physics is most closely associated with Einstein's theory of relativity, where the requirement that the governing equations of a given field theory preserve their form under a prescribed class of transformations became a central organizing principle. In particular, general covariance emerged in the development of general relativity as invariance under arbitrary smooth changes of coordinates, although the underlying mathematical ideas trace back to nineteenth-century differential geometry and tensor calculus. The historical and conceptual status of covariance, especially its relation to symmetry and background independence, has been discussed extensively in the literature [Norton, 1993; Giulini, 2007].

The notion of covariance in continuum mechanics appears in several closely related contexts involving invariance under transformation groups acting on the spatial and material configurations. Three important manifestations are the following.

- When the ambient space is Euclidean, constitutive equations are postulated to be invariant under isometries of Euclidean space, a requirement usually referred to as objectivity or material frame indifference [Truesdell and Noll, 2004]. For a non-flat ambient space, however, there may be no nontrivial isometries. In this case, the natural extension of material frame indifference is the spatial covariance of the energy function, namely invariance under arbitrary diffeomorphisms of the ambient space [Hughes and Marsden, 1977; Marsden and Hughes, 1983].
- In the classical setting of Euclidean ambient spaces, Green and Rivlin [1964] showed that the balance laws of nonlinear elasticity may be derived by postulating the balance of energy together with its invariance under superposed rigid motions of the ambient space, while Noll [1963] gave a related passive interpretation in terms of time-dependent changes of spatial frame. This idea was later extended to hyperelasticity in Riemannian ambient spaces by Hughes and Marsden [1977], who postulated covariance of the energy balance under arbitrary ambient space diffeomorphisms and showed that this gives the balance laws of hyperelasticity together with the Doyle–Ericksen formula. The covariance-based formulation was later extended to Cauchy elasticity [Yavari and Goriely, 2025]; see also [Marsden and Hughes, 1983; Yavari et al., 2006; Yavari and Marsden, 2012].
- Material covariance of constitutive equations is a generalization of material isotropy. In the classical Euclidean setting, isotropy is understood as invariance of the constitutive equation under isometries of the material manifold. For a general Riemannian material manifold, however, there may be no nontrivial isometries, and this classical characterization is no longer adequate. In that case, the natural generalization of isotropy is covariance of the constitutive equation under arbitrary diffeomorphisms of the material manifold [Marsden and Hughes, 1983]. See also [Lu and Papadopoulos, 2000; Lu, 2012; Yavari and Sozio, 2023] for extensions of material covariance to anisotropic elasticity and anelasticity.

A diffeomorphism is a map $\varphi : \mathcal{B} \to \mathcal{C}$ that is smooth, invertible, and whose inverse $\varphi^{-1} : \mathcal{C} \to \mathcal{B}$ is also smooth. Equivalently, it is a smooth bijection with a smooth inverse. In continuum mechanics, a diffeomorphism represents an admissible deformation, i.e., a smooth one-to-one mapping that excludes interpenetration of matter, tearing, and the formation of voids.

The configuration space of nonlinear elasticity is the set of all embeddings of the body $\mathcal{B}$ into the ambient space $\mathcal{S}$. We denote this set by $\mathcal{E} = \mathrm{Emb}(\mathcal{B}, \mathcal{S})$ [Ebin and Marsden, 1970; Simo et al., 1988]. Let $\xi : \mathcal{S} \to \mathcal{S}$

and $\Xi : \mathcal{B} \to \mathcal{B}$ be spatial and material diffeomorphisms, respectively. The collections of spatial and material diffeomorphisms form groups, denoted by $\mathrm{Diff}(\mathcal{S})$ and $\mathrm{Diff}(\mathcal{B})$, respectively.[4]

Remark 2.2. Since $\mathcal{B} \subset \mathcal{S}$ is an embedded submanifold, let $\iota : \mathcal{B} \hookrightarrow \mathcal{S}$ denote the inclusion map. For any $\Xi \in \mathrm{Diff}(\mathcal{B})$, $\Xi : \mathcal{B} \to \mathcal{B}$, and hence $\iota \circ \Xi : \mathcal{B} \to \mathcal{S}$. Since Ξ is a diffeomorphism, $\iota \circ \Xi$ is an embedding, and thus $\iota \circ \Xi \in \mathcal{E} = \mathrm{Emb}(\mathcal{B}, \mathcal{S})$. Therefore, $\iota \circ \mathrm{Diff}(\mathcal{B}) \subset \mathcal{E}$. For any $\xi \in \mathrm{Diff}(\mathcal{S})$, the restriction $\xi\big|_{\mathcal{B}} : \mathcal{B} \to \mathcal{S}$ is an embedding, and hence $\xi\big|_{\mathcal{B}} \in \mathcal{E}$. In particular, any deformation $\varphi : \mathcal{B} \to \mathcal{C} \subset \mathcal{S}$ is an embedding, and thus $\varphi \in \mathcal{E}$. Since $\mathcal{B}$ is compact, any embedding $\varphi \in \mathcal{E}$ can be extended to a diffeomorphism $\xi \in \mathrm{Diff}(\mathcal{S})$ such that $\varphi = \xi\big|_{\mathcal{B}}$. This is a consequence of the isotopy extension theorem [Hirsch, 1976].

Let us define the left translation of $\varphi \in \mathcal{E}$ as

$$\begin{aligned} L_\xi : \mathrm{Diff}(\mathcal{S}) \times \mathcal{E} &\longrightarrow \mathcal{E} \\ (\xi, \varphi) &\longmapsto L_\xi(\varphi) = \xi \circ \varphi \,. \end{aligned} \tag{2.50}$$

Similarly, the right translation of $\varphi \in \mathcal{E}$ is defined as

$$\begin{aligned} R_\Xi : \mathcal{E} \times \mathrm{Diff}(\mathcal{B}) &\longrightarrow \mathcal{E} \\ (\xi, \varphi) &\longmapsto R_\Xi(\varphi) = \varphi \circ \Xi \,. \end{aligned} \tag{2.51}$$

In order to define the left and right actions on spatial and material tensor fields, one first computes the tangent maps associated with L_ξ and R_Ξ. To this end, consider a 1-parameter family of maps $\varphi_\epsilon \in \mathcal{E}$ such that $\varphi_\epsilon\big|_{\epsilon=0} = \varphi$. Then

$$\delta\varphi(X) \coloneqq \frac{d}{d\epsilon}\varphi_\epsilon(X)\Big|_{\epsilon=0} \,, \tag{2.52}$$

where $\delta\varphi$ is a vector field along φ.[5] If ϵ is interpreted as time, then $\delta\varphi$ coincides with the material velocity field $\mathbf{V}$. Now, the tangent map of L_ξ is computed as

$$(T_\varphi L_\xi) \cdot \delta\varphi(X) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} L_\xi \varphi_\epsilon(X) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \xi(\varphi_\epsilon(X)) = T\xi(\varphi(X)) \cdot \delta\varphi(X) \,, \tag{2.53}$$

where in the last equality the chain rule was used. Note that $\delta\varphi(X) \in T_{\varphi(X)}\mathcal{S}$, while $T\xi(\varphi(X)) : T_{\varphi(X)}\mathcal{S} \to T_{\xi(\varphi(X))}\mathcal{S}$. Therefore, $(T_\varphi L_\xi) \cdot \delta\varphi$ is a vector field along $\xi \circ \varphi$, and one may write

$$(T_\varphi L_\xi) \cdot \delta\varphi = (T\xi \circ \varphi) \cdot \delta\varphi \,, \qquad \forall \xi \in \mathrm{Diff}(\mathcal{S}) \,. \tag{2.54}$$

Similarly, the tangent map of R_Ξ is computed as

$$(T_\varphi R_\Xi) \cdot \delta\varphi(X) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} R_\Xi \varphi_\epsilon(X) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi_\epsilon(\Xi(X)) = \delta\varphi(\Xi(X)) \,. \tag{2.55}$$

Note that $\delta\varphi(\Xi(X)) \in T_{\varphi(\Xi(X))}\mathcal{S}$. Therefore, $(T_\varphi R_\Xi) \cdot \delta\varphi$ is a vector field along $\varphi \circ \Xi$, and one may write

$$(T_\varphi R_\Xi) \cdot \delta\varphi = \delta\varphi \circ \Xi \,, \qquad \forall \Xi \in \mathrm{Diff}(\mathcal{B}) \,. \tag{2.56}$$

The geometric meaning of the left and right actions defined above is illustrated in Fig. 1. A deformation φ is transformed by a material relabeling Ξ and a spatial reparametrization ξ to give $\varphi' = \xi \circ \varphi \circ \Xi^{-1}$, where both mappings describe the same physical deformation process. This diagram clarifies the roles of Ξ and ξ and the distinction between material and spatial covariance.

An energy function $W = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \varphi)$ is *spatially covariant* if it is left invariant under the action of the group $\mathrm{Diff}(\mathcal{S})$. This can be informally written as $L_\xi W = W$. More formally, one has

$$W(X, \xi_*\mathbf{F}, \mathbf{G}, \xi_*\mathbf{g}) = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \xi \circ \varphi) \,. \tag{2.57}$$

[4] For example, $\mathrm{Diff}(\mathcal{S})$ is a group under composition: the identity element is $\mathrm{id}_\mathcal{S}$, the inverse of any diffeomorphism ξ is ξ^{-1}, which is well-defined and unique, and associativity follows from the associativity of composition. The same holds for $\mathrm{Diff}(\mathcal{B})$.

[5] A vector field along φ is a map $\mathbf{Z} : \mathcal{B} \to T\mathcal{S}$ such that $\mathbf{Z}(X) \in T_{\varphi(X)}\mathcal{S}$ for every $X \in \mathcal{B}$, i.e., it assigns to each material point a vector based at its image under φ.

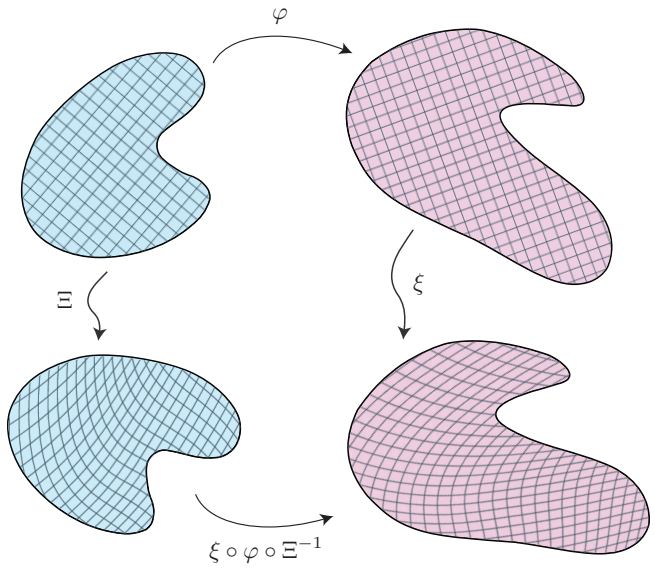

Figure 1: *Schematic illustration of spatial and material covariance. The mappings φ and φ' describe the same physical deformation process, and are related by a material relabeling Ξ and a spatial reparametrization ξ. The blue configurations represent the undeformed, stress-free body, while the red configurations represent the same deformed, stressed body. Restricting to $\Xi = \mathrm{id}_{\mathcal{B}}$ gives spatial covariance, while restricting to $\xi = \mathrm{id}_{\mathcal{S}}$ gives material covariance. The map ξ may be viewed as a warped lens through which the same deformed body is described. Covariance of a continuum field theory means that any tensor field, including scalar and vector fields, transforms under these maps via push-forward or pull-back.*

This is more specifically written as

$$W\big(X, T\xi \cdot \mathbf{F}, \mathbf{G}, (T\xi)^{-\star} \cdot \mathbf{g} \cdot (T\xi)^{-1}\big) = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \xi \circ \varphi)\,. \tag{2.58}$$

Note that under a spatial diffeomorphism material tensors remain unchanged.

Next we show that, by choosing $\xi \in \mathrm{Diff}(\mathcal{S})$ to be a diffeomorphic extension of φ^{-1}, spatial covariance implies that the dependence of the energy function on $\mathbf{F}$ and $\mathbf{g}$ reduces to that on $\mathbf{C}^\flat$. Since $\varphi : \mathcal{B} \to \mathcal{C} \subset \mathcal{S}$ is an embedding and $\mathcal{B}$ is compact, φ is a diffeomorphism onto its image $\mathcal{C}$. Hence, $\varphi^{-1} : \mathcal{C} \to \mathcal{B}$ is a diffeomorphism. Because $\mathcal{C}$ is compact, the isotopy extension theorem (see Remark 2.2) implies that there exists $\xi \in \mathrm{Diff}(\mathcal{S})$ such that $\xi\big|_{\mathcal{C}} = \varphi^{-1}$. Thus, in the spatial covariance relation one may choose ξ to be any diffeomorphic extension of φ^{-1} to $\mathcal{S}$. Therefore,

$$W(X, \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \varphi) = W\big(X, \mathbf{F}^{-1} \cdot \mathbf{F}, \mathbf{G}, \mathbf{F}^\star \mathbf{g}\, \mathbf{F}\big) = W\big(X, \mathrm{id}_{T_X\mathcal{B}}, \mathbf{G}, \mathbf{C}^\flat\big) = \hat{W}\big(X, \mathbf{G}, \mathbf{C}^\flat\big)\,, \tag{2.59}$$

where

$$\hat{W}\big(X, \mathbf{G}, \mathbf{C}^\flat\big) \coloneqq W\big(X, \mathrm{id}_{T_X\mathcal{B}}, \mathbf{G}, \mathbf{C}^\flat\big)\,. \tag{2.60}$$

An energy function $W = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \varphi)$ is *materially covariant* if it is right invariant under the action of the group $\mathrm{Diff}(\mathcal{B})$. This can be informally written as $R_\Xi W = W$. More formally, one has

$$W(\Xi(X), \Xi_*\mathbf{F}, \Xi_*\mathbf{G}, \mathbf{g} \circ \varphi \circ \Xi) = W\big(\Xi(X), \mathbf{F} \cdot T\Xi, (T\Xi)^{-\star}\mathbf{G}\,(T\Xi)^{-1}, \mathbf{g} \circ \varphi \circ \Xi\big) = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \varphi)\,. \tag{2.61}$$

Using the representation resulting from spatial covariance, the material covariance can be expressed as

$$\hat{W}\big(\Xi(X), \Xi_*\mathbf{G}, \Xi_*\mathbf{C}^\flat\big) = \hat{W}\big(\Xi(X), (T\Xi)^{-\star}\mathbf{G}\,(T\Xi)^{-1}, (T\Xi)^{-\star}\mathbf{C}^\flat\,(T\Xi)^{-1}\big) = \hat{W}\big(X, \mathbf{G}, \mathbf{C}^\flat\big)\,. \tag{2.62}$$

Next we show that, by choosing $\Xi \in \mathrm{Diff}(\mathcal{B})$ to be the material diffeomorphism induced by φ, material covariance implies that the dependence of $\hat{W}$ on $\mathbf{G}$ and $\mathbf{C}^\flat$ reduces to that on $\mathbf{c}^\flat$ and $\mathbf{g}$, i.e., material isotropy.[6] As $\varphi : \mathcal{B} \to \mathcal{C} \subset \mathcal{S}$ is an embedding and $\mathcal{B}$ is compact, φ is a diffeomorphism onto its image $\mathcal{C}$. Therefore, one

[6]Dependence on $\mathbf{c}^\flat$ and $\mathbf{g}$ implies that the energy is expressed purely in terms of spatial tensors and their invariants, and is therefore independent of any preferred material directions, which is precisely isotropy.

may use φ itself as the material relabeling map in the covariance relation, i.e., one may choose $\Xi = \varphi$. From material covariance one obtains

$$\hat{W}\big(X, \mathbf{G}, \mathbf{C}^\flat\big) = \hat{W}\big(\varphi(X), \varphi_*\mathbf{G}, \varphi_*\mathbf{C}^\flat\big) = \hat{W}\big(\varphi(X), \mathbf{F}^{-\star}\mathbf{G}\mathbf{F}^{-1}, \varphi_*\varphi^*\mathbf{g}\big) = \hat{W}\big(x, \mathbf{c}^\flat, \mathbf{g}\big)\,. \tag{2.63}$$

In the next section we show that spatial covariance plays a central role in rewriting a strength function in terms of different stress measures.

3 Material Strength

Fracture nucleation and evolution in brittle solids under quasi-static loading are governed by three macroscopic (continuum) material properties: (i) the elastic response of the material, characterized by elastic constants in linear elasticity and by a strain-energy function in hyperelasticity,[7] (ii) the fracture toughness, or equivalently a critical energy release rate, and (iii) the material strength. The present discussion precludes dynamic loading which would require additional analysis.

As stated in the Introduction, we adopt the following definition for material strength [Kumar and Lopez-Pamies, 2020; Kumar et al., 2020]: "the strength of an elastic brittle material is the set of all critical stresses $\mathbf{S}$ at which the material fractures when it is subjected to a state of monotonically increasing,[8] spatially uniform, but otherwise arbitrary stress." In other words, strength surface controls when crack formation occurs under uniform stresses. However, in general, the strength surface violation acts as a necessary but not sufficient condition for crack evolution. Material toughness plays an important role under non-uniform stresses. A complete mathematical understanding of how strength and toughness couple with each other is still lacking. However, it has been hypothesized with some evidence [Lopez-Pamies and Kamarei, 2025; Ward and Kumar, 2025] that the strength surface acts as a constraint on the variational statement of Griffith fracture as put forward by Francfort and Marigo [1998]. Mathematically, this may be stated as follows: the deformation field $\varphi(X,t)$ and the crack set $\Gamma(t)$ minimize the functional

$$\mathcal{E}(\varphi, \Gamma) \coloneqq \int_{\Omega_0\setminus\Gamma} W(\mathbf{F}, \mathbf{G}, \mathbf{g})\,\mathrm{d}V + G_c\,\mathcal{A}(\Gamma)\,, \tag{3.1}$$

among all admissible pairs (φ, Γ) such that

$$\Gamma \subset \mathcal{B}_f(t)\,, \tag{3.2}$$

where

$$\mathcal{B}_f(t) = \{X \in \mathcal{B} : \mathsf{F}(\mathbf{P}, \mathbf{F}, \mathbf{g}, \mathbf{G}) \geq 0\}\,, \tag{3.3}$$

is the failed subset of the body, while $\mathsf{F}(\mathbf{P}, \mathbf{F}, \mathbf{g}, \mathbf{G}) = 0$ defines the strength surface. A phase field model corresponding to this formulation has been developed by Kumar et al. [2020] and Kumar and Lopez-Pamies [2020] and has proven successful in predicting crack nucleation and propagation in a wide variety of brittle materials under arbitrary loading conditions, including finite deformations [Kamarei et al., 2026]. The model is not developed as a direct regularization of the above constrained variational formulation; however, it contains the essential features.

As discussed in the Introduction, a critical question to evaluate in this definition is the phrase "spatially uniform stress". To see the difficulty, recall that for a homogeneous compressible hyperelastic solid a homogeneous deformation, or equivalently a uniform strain, induces a spatially uniform stress field. However, a spatially uniform stress field may also be supported by an inhomogeneous deformation, so that uniform stress need not correspond to a homogeneous strain field. This can already be seen for a compressible hyperelastic material with energy function $W = W(J)$, for which the Cauchy stress is purely hydrostatic and depends only on J (a hyperelastic fluid). Consider the inhomogeneous simple shear deformation $\varphi(X_1, X_2, X_3) = \big(X_1\,, X_2 + f(X_1)\,, X_3\big)$

[7] For Cauchy elastic solids, the elastic response is characterized by constitutive response functions rather than by a strain-energy function.

[8] The phrase "monotonically increasing stress" is informal and is intended only to exclude loading paths that involve unloading or cyclic variations of the applied stress state. In particular, earlier formulations such as Malmeister [1966] restrict attention to simple uniform (proportional) loading, in which the loading path in stress space is a straight ray emanating from the origin.

with a non-affine function f. This deformation is isochoric, i.e., $J = 1$, while the strain is inhomogeneous because deformation gradient is written as

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ f'(X_1) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.4}$$

and $f'(X_1)$ varies with X_1. Nonetheless, the Cauchy stress is spatially uniform, namely $\boldsymbol{\sigma} = W'(1)\,\mathbf{g}^\sharp$. It is known more generally that homogeneous Cauchy stress in isotropic hyperelasticity does not necessarily imply homogeneous deformations [Mihai and Neff, 2017; Mihai and Neff, 2018]. The examples constructed in these works are piecewise homogeneous deformations whose deformation gradients are rank-one connected across planar interfaces, in the spirit of the classical Ball–James theory of microstructure [Ball and James, 1987]. However, if the stored-energy function is strictly rank-one convex, then such non-homogeneous rank-one connected deformations cannot support a homogeneous Cauchy stress, since in that case the Cauchy stress is injective along rank-one connected lines [Neff and Mihai, 2017].

These examples show that, in finite elasticity, the notion of a spatially uniform stress state is more subtle than in the linear theory. This naturally leads to the question of what measure of stress should be used in defining material strength.

3.1 What measure of stress should be used in defining material strength?

If one wishes to define material strength as the set of critical spatially uniform stresses, one must first decide which stress measure is to be used and under what assumptions a uniform stress state is regarded as representing a homogeneous loading state. This issue is especially important in finite elasticity, where different stress measures live in different configurations and encode different geometric information.

Let us consider a homogeneous undeformed body made of some hyperelastic solid. Let us define the material strength of this solid as the set of critical spatially uniform stresses under which the body fractures. For an arbitrary material point $X \in \mathcal{B}$ and given any element of area dA with unit normal $\mathbf{N}(X)$, traction in the deformed configuration is $\mathbf{t} = \mathbf{P}\mathbf{N}^\flat$, where $\mathbf{P}$ is the first Piola–Kirchhoff stress. Traction $\mathbf{t}$ acts on the element of area da with unit normal $\mathbf{n} \circ \varphi(X)$ in the current configuration. By Nanson's formula one has $\mathbf{n}^\flat\, da = J\,\mathbf{F}^{-\star}\,\mathbf{N}^\flat\, dA$, or equivalently $\mathbf{n}\, da = J\mathbf{F}^{-\mathsf{T}}\mathbf{N}\, dA$. These considerations suggest that, if one wants to characterize strength in terms of boundary loading, the first Piola–Kirchhoff stress is a natural stress measure to use. Indeed, $\mathbf{P}$ directly determines the traction vector associated with a given oriented area element in the reference configuration, which is the materially homogeneous stress-free configuration of the body. In this sense, $\mathbf{P}$ is convenient for prescribing and characterizing spatially homogeneous loading by boundary tractions.

On the other hand, if one wishes to regard homogeneity in a purely spatial sense, then the Cauchy stress $\boldsymbol{\sigma}$ is a more natural stress measure. Thus, even at the level of definitions, one must distinguish between uniformity relative to the reference configuration and uniformity relative to the current configuration. However, uniformity of the Cauchy stress, in general, is not equivalent to uniformity of the first Piola–Kirchhoff stress. It should be emphasized that the various stress measures are mechanically equivalent in the sense that, when paired with the corresponding area element in the appropriate configuration, they represent the same physical traction vector. Nevertheless, the notion of strength depends, in general, on which stress measure is assumed to be homogeneous at the onset of fracture. Once strength is defined with respect to one stress measure, one may then ask how the corresponding strength function is represented in terms of any other stress measure. We examine each stress measure in turn—namely the first and second Piola–Kirchhoff stresses and the Cauchy stress—and compare the corresponding notions of uniformity in the reference and spatial configurations.

Uniform first Piola–Kirchhoff stress. Let us follow [Kumar and Lopez-Pamies, 2020; Kumar et al., 2020] and assume that the first Piola–Kirchhoff stress is spatially uniform at the onset of fracture. For the two-point tensor $\mathbf{P}$, spatial uniformity means that its spatial covariant derivative vanishes, i.e., in components $P^{aA}{}_{|b} = 0$. Recall that [Marsden and Hughes, 1983]

$$P^{aA}{}_{|B} = P^{aA}{}_{,B} + \gamma^a{}_{cd}\, F^d{}_B\, P^{cA} + \Gamma^A{}_{CB}\, P^{aC}\,, \tag{3.5}$$

where $\gamma^a{}_{bc}$ are the Christoffel symbols of the spatial metric $\mathbf{g}$ in a coordinate chart $\{x^a\}$ and $\Gamma^A{}_{BC}$ are the Christoffel symbols of the material metric $\mathbf{G}$ in a coordinate chart $\{X^A\}$. We also know that $P^{aA}{}_{|B} = F^b{}_B\, P^{aA}{}_{|b}$.

Since the deformation gradient is invertible, it follows that

$$P^{aA}{}_{|b} = 0 \quad \Leftrightarrow \quad P^{aA}{}_{|B} = 0\,. \tag{3.6}$$

In other words, material and spatial uniformity of the first Piola-Kirchhoff stress are equivalent.

The notion of spatial or material uniformity for a two-point tensor such as the first Piola–Kirchhoff stress requires some care. Unlike a purely spatial or purely material tensor, $\mathbf{P}$ maps material covectors to spatial vectors, i.e., $\mathbf{P} : T^*_X\mathcal{B} \to T_x\mathcal{C}$, and therefore depends simultaneously on material and spatial positions. For this reason, it does not make sense to define uniformity by saying that the components of $\mathbf{P}$ are constant in a given coordinate system. The appropriate notion of uniformity is covariant constancy. The condition $P^{aA}{}_{|b} = 0$ means that $\mathbf{P}$ does not vary from point to point in the spatial manifold once the geometry of the current configuration is taken into account, while $P^{aA}{}_{|B} = 0$ means that $\mathbf{P}$ does not vary from point to point in the material manifold once the geometry of the reference configuration is taken into account. Since these two conditions are related by $P^{aA}{}_{|B} = F^b{}_B\, P^{aA}{}_{|b}$ and the deformation gradient is invertible, spatial and material uniformity are equivalent. Thus, for a two-point tensor, homogeneity means that the mapping from material covectors to spatial forces is the same everywhere in the body, after accounting for the geometries of the reference and current configurations.

We next examine the implications of spatial uniformity of the first Piola–Kirchhoff stress for the second Piola–Kirchhoff and Cauchy stresses.

Uniform second Piola–Kirchhoff stress. The material covariant derivative of the second Piola–Kirchhoff stress with respect to the material metric $\mathbf{G}$, $S^{AB}{}_{|C}$, is computed as follows. Recall that $S^{AB} = F^{-A}{}_a\, P^{aB}$. Hence,

$$S^{AB}{}_{|C} = F^{-A}{}_{a|C}\, P^{aB} + F^{-A}{}_a\, P^{aB}{}_{|C} = F^{-A}{}_{a|C}\, P^{aB}\,. \tag{3.7}$$

Thus, uniformity of the first Piola–Kirchhoff stress does not, in general, imply material uniformity of the second Piola–Kirchhoff stress. The latter holds only under the additional condition $F^{-A}{}_{a|C} = 0$, i.e., for homogeneous deformations.

Remark 3.1. A clarification is needed here. When the balance of linear momentum is written in terms of the second Piola–Kirchhoff stress, the operator $\mathrm{Div}\,\mathbf{S}$ is the divergence with respect to the metric $\mathbf{C}^\flat$, not the material metric $\mathbf{G}$. Thus, the equilibrium equation for $\mathbf{S}$ involves the Levi–Civita connection of $\mathbf{C}^\flat$. This should not be confused with the notion of material uniformity. Uniformity of $\mathbf{S}$ is defined using the material covariant derivative associated with $\mathbf{G}$, since uniformity is a geometric notion on the material manifold $(\mathcal{B}, \mathbf{G})$. In elasticity, $\mathbf{G}$ is the induced metric on the reference configuration, namely $\mathbf{G} = \mathbf{g}\big|_{\mathcal{B}}$. Therefore, two different metrics appear in the discussion of $\mathbf{S}$. The metric $\mathbf{C}^\flat$ enters the equilibrium equation through the divergence operator, whereas the metric $\mathbf{G}$ enters the definition of material uniformity. These two notions are distinct and should not be conflated.

Uniform Cauchy stress. Recall that $\sigma^{ab} = J^{-1}P^{aA}F^b{}_A$. Taking the spatial covariant derivative, and assuming that the first Piola–Kirchhoff stress has vanishing covariant derivative, one obtains

$$\sigma^{ab}{}_{|c} = (J^{-1})_{,c}\, P^{aA}F^b{}_A + J^{-1}P^{aA}F^b{}_{A|c}\,. \tag{3.8}$$

Now, since $J_{,c} = F^{-C}{}_c\, J_{,C}$ and $J_{,C} = J\, F^{-A}{}_d\, F^d{}_{A|C}$, one finds $(J^{-1})_{,c} = -J^{-1}F^{-C}{}_c\, F^{-A}{}_d\, F^d{}_{A|C}$. Also, $F^b{}_{A|c} = F^b{}_{A|C}\, F^{-C}{}_c$. Therefore,

$$\begin{aligned}\sigma^{ab}{}_{|c} &= -J^{-1}F^{-C}{}_c\, F^{-A}{}_d\, F^d{}_{A|C}\, \sigma^{ab} + J^{-1}P^{aA}F^b{}_{A|C}\, F^{-C}{}_c \\ &= J^{-1}F^{-C}{}_c \left(P^{aA}F^b{}_{A|C} - F^{-A}{}_d\, F^d{}_{A|C}\, \sigma^{ab}\right)\,.\end{aligned} \tag{3.9}$$

Thus, vanishing covariant derivative of the first Piola–Kirchhoff stress does not, in general, imply vanishing covariant derivative of the Cauchy stress. For homogeneous deformations, however, $F^a{}_{A|B} = 0$, and hence $\sigma^{ab}{}_{|c} = 0$, i.e., the Cauchy stress is spatially uniform.

In summary, uniformity of one measure of stress does not, in general, imply uniformity of another. In particular, for the first Piola–Kirchhoff stress, spatial and material uniformity are equivalent. However, uniformity of the first Piola–Kirchhoff stress does not, in general, imply either spatial uniformity of the Cauchy stress or

material uniformity of the second Piola–Kirchhoff stress. For homogeneous deformations, on the other hand, spatial uniformity of the Cauchy stress, material uniformity of the second Piola–Kirchhoff stress, and uniformity of the first Piola–Kirchhoff stress are equivalent.

Remark 3.2. Local invertibility of the stress-strain relation is not sufficient to conclude that a spatially uniform stress field necessarily corresponds to a spatially uniform strain field. As a matter of fact, local invertibility only guarantees uniqueness of strain within a neighborhood of a given state, and therefore does not exclude the possibility that the same uniform stress may correspond to strain states lying on different branches of the constitutive equation. In contrast, if the constitutive equation is globally invertible, then a uniform stress field uniquely determines a uniform strain field. In particular, assume that the constitutive equation $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g})$ is invertible. Let $\boldsymbol{\sigma}_0$ be a homogeneous Cauchy stress. Then $\mathbf{b}^\sharp = \mathbf{b}_0^\sharp = \hat{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\sigma}_0, \mathbf{g})$ is uniform. This gives a unique uniform spatial stretch tensor that corresponds to a homogeneous deformation [Ciarlet, 1988; Mihai and Neff, 2018]. In this case, the first Piola–Kirchhoff, Cauchy, and second Piola–Kirchhoff stresses are uniform simultaneously.

Remark 3.3 (Invertibility of the linear isotropic stress-strain map)**.** The linear isotropic constitutive equatins $\boldsymbol{\sigma} = 2\,\mu\,\operatorname{dev}\boldsymbol{\varepsilon} + \kappa\,\operatorname{tr}(\boldsymbol{\varepsilon})\,\mathbf{g}$ (μ is shear modulus and κ is bulk modulus) is invertible if and only if $\mu \neq 0$ and $\kappa \neq 0$. Note that taking the trace, one finds $\operatorname{tr}(\boldsymbol{\sigma}) = 3\,\kappa\,\operatorname{tr}(\boldsymbol{\varepsilon})$, and hence $\operatorname{tr}(\boldsymbol{\varepsilon}) = \dfrac{1}{3\,\kappa}\operatorname{tr}(\boldsymbol{\sigma})$, provided $\kappa \neq 0$. Taking the deviatoric part, one finds $\operatorname{dev}\boldsymbol{\sigma} = 2\,\mu\operatorname{dev}\boldsymbol{\varepsilon}$, and hence $\operatorname{dev}\boldsymbol{\varepsilon} = \dfrac{1}{2\,\mu}\operatorname{dev}\boldsymbol{\sigma}$, provided $\mu \neq 0$. Therefore, both $\operatorname{tr}(\boldsymbol{\varepsilon})$ and $\operatorname{dev}\boldsymbol{\varepsilon}$ are uniquely determined by $\boldsymbol{\sigma}$. Using the decomposition $\boldsymbol{\varepsilon} = \operatorname{dev}\boldsymbol{\varepsilon} + \dfrac{1}{3}\operatorname{tr}(\boldsymbol{\varepsilon})\,\mathbf{g}$, one concludes that $\boldsymbol{\varepsilon}$ is uniquely determined by $\boldsymbol{\sigma}$.

Remark 3.4 (Non-invertibility of $\mathbf{P} = \hat{\mathbf{P}}(\mathbf{F}, \mathbf{G}, \mathbf{g})$ with respect to $\mathbf{F}$)**.** Let us consider a Cauchy elastic solid for which $\mathbf{P} = \hat{\mathbf{P}}(\mathbf{F}, \mathbf{G}, \mathbf{g})$. It is a classic result that constitutive equations cannot be invertible when written with respect to deformation gradient [Schweickert et al., 2018]. Objectivity of $\hat{\mathbf{P}}$ implies that for every $\mathbf{Q} \in SO(3)$, one has

$$\hat{\mathbf{P}}(\mathbf{Q}\mathbf{F}, \mathbf{G}, \mathbf{g}) = \mathbf{Q}\hat{\mathbf{P}}(\mathbf{F}, \mathbf{G}, \mathbf{g})\,. \tag{3.10}$$

Assume now, for the sake of contradiction, that $\hat{\mathbf{P}}$ is injective in $\mathbf{F}$ for fixed $\mathbf{G}$ and $\mathbf{g}$. Then

$$\hat{\mathbf{P}}(\mathbf{F}_1, \mathbf{G}, \mathbf{g}) = \hat{\mathbf{P}}(\mathbf{F}_2, \mathbf{G}, \mathbf{g}) \quad \Longrightarrow \quad \mathbf{F}_1 = \mathbf{F}_2\,. \tag{3.11}$$

Consider the stress-free reference configuration corresponding to the inclusion map $\iota : \mathcal{B} \hookrightarrow \mathcal{S}$, and denote its tangent map by $\mathbf{I} = T\iota$. Assume that $\hat{\mathbf{P}}(\mathbf{I}, \mathbf{G}, \mathbf{g}) = \mathbf{0}$. Using material frame indifference with $\mathbf{F} = \mathbf{I}$, for any $\mathbf{Q} \in SO(3)$, one obtains

$$\hat{\mathbf{P}}(\mathbf{Q}, \mathbf{G}, \mathbf{g}) = \hat{\mathbf{P}}(\mathbf{Q}\mathbf{I}, \mathbf{G}, \mathbf{g}) = \mathbf{Q}\hat{\mathbf{P}}(\mathbf{I}, \mathbf{G}, \mathbf{g}) = \mathbf{Q}\mathbf{0} = \mathbf{0} = \hat{\mathbf{P}}(\mathbf{I}, \mathbf{G}, \mathbf{g})\,. \tag{3.12}$$

Since $\hat{\mathbf{P}}$ is injective in $\mathbf{F}$, it follows that $\mathbf{Q} = \mathbf{I}$, $\forall\, \mathbf{Q} \in SO(3)$, which is absurd. Therefore, a constitutive equation satisfying material frame indifference cannot be injective in $\mathbf{F}$. In particular, $\mathbf{P} = \hat{\mathbf{P}}(\mathbf{F}, \mathbf{G}, \mathbf{g})$ cannot be inverted uniquely to recover $\mathbf{F}$.

When written in terms of the Cauchy stress, constitutive equations may be invertible with respect to the strain measure $\mathbf{b}^\sharp$, i.e., $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g})$ is invertible. In such cases, a uniform state of Cauchy stress uniquely determines $\mathbf{b}^\sharp$. Consequently, a uniform state of Cauchy stress corresponds to a unique homogeneous deformation up to isometries of the ambient space.

Example 3.1. Consider the standard compressible neo-Hookean constitutive equation written in terms of the Cauchy stress:

$$\boldsymbol{\sigma} = \frac{\mu}{J}\,(\mathbf{b}^\sharp - \mathbf{g}^\sharp) + \frac{\kappa \ln J}{J}\,\mathbf{g}^\sharp\,, \tag{3.13}$$

where $J > 0$ and $\det \mathbf{b}^\sharp = J^2(\det \mathbf{g})^{-1}$. Unlike linear elasticity, this is not an affine relation in $\mathbf{b}^\sharp$. Even if $\boldsymbol{\sigma}$ is prescribed, the unknown $\mathbf{b}^\sharp$ appears both explicitly and implicitly through J, since J itself depends on $\mathbf{b}^\sharp$. Rearranging the constitutive equation, one obtains

$$\mathbf{b}^\sharp = \frac{J}{\mu}\,\boldsymbol{\sigma} + \left(1 - \frac{\kappa}{\mu}\ln J\right)\mathbf{g}^\sharp\,. \tag{3.14}$$

This is not yet a solution for $\mathbf{b}^\sharp$, because $J = J(\mathbf{b}^\sharp, \mathbf{g})$. Substituting this expression into the determinant relation leads to a nonlinear scalar equation for J. In general, it is not clear a priori whether this equation has one solution, several solutions, or none. Thus, in the general case, global invertibility of the constitutive mapping $\mathbf{b}^\sharp \mapsto \boldsymbol{\sigma}$ is a genuinely nontrivial question. To see concretely what can happen, let us restrict attention to purely dilatational deformations and assume $\mathbf{b}^\sharp = \alpha\,\mathbf{g}^\sharp$, $\alpha > 0$. Then, $\det \mathbf{b}^\sharp = \alpha^3 \det \mathbf{g}^\sharp = \alpha^3 (\det \mathbf{g})^{-1}$. Comparing with $\det \mathbf{b}^\sharp = J^2 / \det \mathbf{g}$, one obtains $J = \alpha^{\frac{3}{2}}$. Substituting this into the constitutive equation gives us

$$\boldsymbol{\sigma} = \frac{\mu}{J}\,(\alpha\,\mathbf{g}^\sharp - \mathbf{g}^\sharp) + \frac{\kappa \ln J}{J}\,\mathbf{g}^\sharp = f(\alpha)\,\mathbf{g}^\sharp\,, \qquad f(\alpha) = \frac{\mu(\alpha - 1) + \frac{3\kappa}{2}\ln\alpha}{\alpha^{\frac{3}{2}}}\,. \tag{3.15}$$

Thus, along purely dilatational deformations, the question of global invertibility reduces to the injectivity of the scalar function $\alpha \mapsto f(\alpha)$. Now observe that

$$\lim_{\alpha \to 0^+} f(\alpha) = -\infty\,, \qquad f(1) = 0\,, \qquad \lim_{\alpha \to \infty} f(\alpha) = 0^+\,. \tag{3.16}$$

Moreover,

$$f'(1) = \frac{2\mu + 3\kappa}{2} > 0\,. \tag{3.17}$$

Hence $f(\alpha)$ is increasing near $\alpha = 1$, but tends to 0^+ as $\alpha \to \infty$. Therefore, it must attain a positive maximum at some $\alpha_* > 1$. Consequently, for any f satisfying $0 < f < f(\alpha_*)$, there exist two distinct values $\alpha_1 \neq \alpha_2$ such that $f(\alpha_1) = f(\alpha_2) = f$. Therefore,

$$\hat{\boldsymbol{\sigma}}(\alpha_1\,\mathbf{g}^\sharp, \mathbf{g}) = \hat{\boldsymbol{\sigma}}(\alpha_2\,\mathbf{g}^\sharp, \mathbf{g})\,, \qquad \alpha_1 \neq \alpha_2\,, \tag{3.18}$$

and hence the constitutive mapping $\mathbf{b}^\sharp \mapsto \boldsymbol{\sigma}$ is not globally injective.

3.2 Loading conditions for defining strength

As we have seen, the notion of strength is based on considering homogeneous states of stress and identifying those that cause a homogeneous body to fracture. One may then ask how such a stress state is realized physically. Are only boundary tractions applied, or are body forces also involved? If one uses the first Piola–Kirchhoff stress as the stress measure in defining strength, and assumes quasistatic loading, then the balance of linear momentum reads $\operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \mathbf{0}$, where ρ_0 and $\mathbf{B}$ are the material mass density and the body force per unit mass, respectively. Since $P^{aA}{}_{|B} = 0$, one immediately has $P^{aA}{}_{|A} = 0$, and hence $\mathbf{B} = \mathbf{0}$. Thus, in this setting, the definition of material strength corresponds to loading a homogeneous body only through boundary tractions.

Remark 3.5. It should be noted that a homogeneous state of stress need not be constitutively admissible. For compressible isotropic solids, one may therefore ask which stress states, in the absence of body forces, can be maintained for an arbitrary material in this class. This is precisely the question posed and answered by Carroll [1973a]. In analogy with Ericksen's universal deformations,[9] we call such stress states *universal stresses*. Carroll [1973a] showed that universal stresses must be homogeneous (he called them controllable stresses). The converse, however, does not hold: not every homogeneous stress state is constitutively admissible, as the corresponding strain field may be incompatible.

Remark 3.6. From this point of view, the notion of strength is not tied to the existence of an explicit stress–strain map, but rather to the geometry of admissible stress states. The strength surface is the subset of constitutively admissible homogeneous stress states at which fracture first occurs. This viewpoint separates two distinct questions: the constitutive question of admissibility, determined by the elastic response, and the fracture question of which admissible stress states are critical. This distinction becomes particularly important for implicit constitutive theories, for materials with internal constraints, and, more generally, whenever the stress-strain relation is given implicitly.

[9] Universal deformations are deformations that can be maintained, in the absence of body forces, for every material within a given class [Ericksen, 1954, 1955]. Generalizations of this notion to anisotropic solids, and in particular to fiber-reinforced solids, can be found in [Yavari and Goriely, 2021, 2023; Yavari, 2025].

Let us consider a hyperelastic brittle solid with energy function $W = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g})$. In defining the strength function, we restrict attention to uniform stress states produced solely by boundary tractions. This leads naturally to a connection with universal deformations. Stress and strain are related through the constitutive equation

$$\mathbf{P} = \mathbf{g}^{\sharp} \frac{\partial W}{\partial \mathbf{F}} \,. \tag{3.19}$$

For a general Cauchy elastic solid we have the constitutive equation $\mathbf{P} = \hat{\mathbf{P}}(X, \mathbf{F}, \mathbf{G}, \mathbf{g})$. One may then ask which uniform stress states can be maintained by applying only boundary tractions. A related question was posed by Ericksen [1955]: for a homogeneous body made of an arbitrary compressible isotropic hyperelastic material, which deformations can be maintained by boundary tractions alone in the absence of body forces? Such deformations are called *universal deformations*. Ericksen showed that, for this class of materials, the universal deformations are precisely the homogeneous deformations. However, the present problem is different. Here, we fix a single hyperelastic material and consider all constitutively admissible homogeneous stress states, and then identify those at which the brittle solid fractures. Thus, unlike the problem of universal deformations, the present problem is not to characterize all stress states or deformations sustainable for an entire class of materials, but to determine, for a fixed brittle hyperelastic material, the constitutively admissible homogeneous stress states that are critical for fracture. These critical states define the strength hypersurface.

3.3 Strength function and strength hypersurface, and covariance

Strength has traditionally been formulated as a stress-based criterion, where a particular stress measure is assumed to be uniform at fracture. However, once such a definition is adopted, expressing the same strength criterion in terms of a different stress measure generally requires knowledge of the deformation, and hence the strain. Therefore, strain dependence cannot be avoided. Purely stress-based criteria correspond to the special case where the strength function is independent of strain.

Let us assume that the strength hypersurface is written in terms of the first Piola stress as

$$\mathsf{F}(\mathbf{P}, \mathbf{g} \circ \varphi, \mathbf{G}) = 0 \,. \tag{3.20}$$

Note that the first Piola–Kirchhoff stress $\mathbf{P}$ is a two-point tensor with components P^{aA}. More precisely, it maps material covectors to spatial vectors, $\mathbf{P} : T^*_X\mathcal{B} \to T_{\varphi(X)}\mathcal{C}$. Since $\mathbf{P}$ connects objects in two different configurations, a scalar-valued function of $\mathbf{P}$ cannot be formed without the use of the material and spatial metrics to lower and contract indices. For this reason, the strength function must depend explicitly on both the material metric $\mathbf{G}$ and the spatial metric $\mathbf{g} \circ \varphi$ in addition to $\mathbf{P}$. As a generalization of material-frame-indifference, we assume that the strength function is spatially covariant, i.e., it is left invariant under the action of the group $\mathrm{Diff}(\mathcal{S})$: $L_\xi \mathsf{F} = \mathsf{F}$. More specifically, this implies that

$$\mathsf{F}(\xi_*\mathbf{P}, \xi_*\mathbf{g}, \mathbf{G}) = \mathsf{F}\left((T\xi) \cdot \mathbf{P}, (T\xi)^{-\star} \cdot \mathbf{g} \cdot (T\xi)^{-1}, \mathbf{G}\right) = \mathsf{F}(\mathbf{P}, \mathbf{g} \circ \varphi, \mathbf{G}) \,. \tag{3.21}$$

Similar to what was done for energy function in §2.6, in the above spatial covariance relation we may choose ξ to be any diffeomorphic extension of φ^{-1} to $\mathcal{S}$. Thus,[10]

$$\mathsf{F}(\mathbf{P}, \mathbf{g} \circ \varphi, \mathbf{G}) = \mathsf{F}\big(\mathbf{F}^{-1}\mathbf{P}, \mathbf{F}^{\star}\mathbf{g}\,\mathbf{F}, \mathbf{G}\big) = \hat{\mathsf{F}}(\mathbf{S}, \mathbf{C}^{\flat}, \mathbf{G}) \,. \tag{3.22}$$

We observe that when a stress-based strength function is formulated in terms of $\mathbf{P}$, its representation in terms of $\mathbf{S}$ necessarily depends on $\mathbf{C}$ as well.

More generally, when using the first Piola–Kirchhoff stress $\mathbf{P}$, strength function can depend on both $\mathbf{P}$ and deformation gradient:

$$\mathsf{F}(\mathbf{P}, \mathbf{F}, \mathbf{g}, \mathbf{G}) = 0 \,. \tag{3.23}$$

In this case, spatial covariance implies that

$$\mathsf{F}(\mathbf{P}, \mathbf{F}, \mathbf{g}, \mathbf{G}) = \mathsf{F}(\varphi^*\mathbf{P}, \varphi^*\mathbf{F}, \varphi^*\mathbf{g}, \mathbf{G}) = \mathsf{F}(\mathbf{S}, \mathrm{Id}_{T_X\mathcal{B}}, \mathbf{C}^{\flat}, \mathbf{G}) \,. \tag{3.24}$$

[10] Kumar and Lopez-Pamies [2020] explicitly state that the strength surface must be a function of an objective stress measure and accordingly use the eigenvalues of the Biot stress.

Now the strength hypersurface in terms of second Piola–Kirchhoff stress is defined as[11]

$$\hat{\mathsf{F}}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G}) \coloneqq \mathsf{F}(\mathbf{S}, \mathrm{Id}_{T_X\mathcal{B}}, \mathbf{C}^\flat, \mathbf{G}) = 0\,. \tag{3.25}$$

In what follows we assume that the strength hypersurface is written in terms of the second Piola–Kirchhoff stress as

$$\hat{\mathsf{F}}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G}) = 0\,. \tag{3.26}$$

Remark 3.7. The equation $\hat{\mathsf{F}}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G}) = 0$ defines a hypersurface in the 12-dimensional space of the pairs $(\mathbf{S}, \mathbf{C}^\flat)$, since both $\mathbf{S}$ and $\mathbf{C}^\flat$ are symmetric second-order tensors and hence each has six independent components. However, $\mathbf{S}$ and $\mathbf{C}^\flat$ are not independent, as they are related by constitutive equations. In implicit elasticity, one has constitutive equations of the form $\mathbf{f}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G}) = \mathbf{0}$, which represent six scalar constraints. Thus, the constitutively admissible states lie on a 6-dimensional submanifold of the original 12-dimensional space. The strength hypersurface should therefore be understood as a hypersurface in this reduced constitutive manifold. In particular, even in implicit elasticity, the notion of strength is attached not to the full space of stress-strain pairs, but to the subset of pairs that are constitutively admissible.

Recall that

$$\varphi_*\mathbf{S} = J\boldsymbol{\sigma}, \qquad \varphi_*\mathbf{C}^\flat = \mathbf{g}, \qquad \varphi_*\mathbf{G} = \mathbf{c}^\flat\,, \tag{3.27}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\mathbf{g}$ is the spatial metric, and $\mathbf{c}^\flat$ is the spatial metric induced on the body. Let us push forward the strength function to the current configuration (spatial covariance). Covariance of the strength function implies that

$$\varphi_*\hat{\mathsf{F}}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G}) = \hat{\mathsf{F}}(\varphi_*\mathbf{S}, \varphi_*\mathbf{C}^\flat, \varphi_*\mathbf{G}) = \hat{\mathsf{F}}(J\boldsymbol{\sigma}, \mathbf{g}, \mathbf{c}^\flat)\,. \tag{3.28}$$

Thus, the spatial representation of the strength function is given by

$$\mathsf{f}(\boldsymbol{\sigma}, \mathbf{c}^\flat, \mathbf{g}) \coloneqq \hat{\mathsf{F}}(J\boldsymbol{\sigma}, \mathbf{g}, \mathbf{c}^\flat)\,. \tag{3.29}$$

In much of the literature, material strength is assumed to be described by a stress-based criterion. However, this assumption is tied to the particular stress measure being used. For example, suppose that the strength hypersurface is described in terms of the Cauchy stress by $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g}) = 0$. Since the various stress measures are related through the deformation, or equivalently through the strain, any change of stress measure is necessarily accompanied by the corresponding strain measure. More precisely, assuming that f is independent of $\mathbf{c}^\flat$, from $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{c}^\flat, \mathbf{g}) = \hat{\mathsf{F}}(J\boldsymbol{\sigma}, \mathbf{g}, \mathbf{c}^\flat)$, one concludes that $\hat{\mathsf{F}} = \hat{\mathsf{F}}(J\boldsymbol{\sigma}, \mathbf{g})$. Pulling this back to the reference configuration and using covariance, one obtains $\varphi^*\hat{\mathsf{F}}(J\boldsymbol{\sigma}, \mathbf{g}) = \hat{\mathsf{F}}(\mathbf{S}, \mathbf{C}^\flat)$. Similarly, if one assumes that $\hat{\mathsf{F}} = \hat{\mathsf{F}}(\mathbf{S}, \mathbf{G})$, then one concludes that $\mathsf{f} = \mathsf{f}(\boldsymbol{\sigma}, \mathbf{c}^\flat)$. Thus, a criterion that is stress-based when written in terms of the Cauchy stress will, in general, depend on both stress and strain when written in terms of the second Piola–Kirchhoff stress. In this sense, once a strength function is fixed in terms of a given stress measure, rewriting it in terms of another stress measure generally introduces dependence on the corresponding strain.

Remark 3.8. Experimental evidence for viscoelastic elastomers shows that strength depends on both stress and strain, or equivalently, on stress and deformation [Smith, 1958, 1963, 1964a,b; Knauss, 1967]. In particular, the experimental results show that failure cannot, in general, be characterized by a unique critical stress, a unique critical strain, or a unique critical strain-energy density. Rather, the critical stress-strain pair along the loading path is what determines fracture. Consistent with this experimental evidence, Kamarei et al. [2025] used a strength function depending on both stress and strain in modeling fracture of viscoelastic elastomers. Here, we have shown that even for elastic solids, spatial covariance requires the strength function to depend on both stress and strain.

Example 3.2. The Drucker–Prager strength function reads [Kumar et al., 2020]

$$\mathcal{F}(\mathbf{S}) = \sqrt{\mathcal{J}_2} + \frac{s_{cs} - s_{ts}}{\sqrt{3}\,(s_{cs} + s_{ts})}\,\mathcal{I}_1 - \frac{2 s_{cs} s_{ts}}{\sqrt{3}\,(s_{cs} + s_{ts})} = 0\,, \tag{3.30}$$

[11] Note that $\mathbf{F}^{-1}\mathbf{F} = \mathrm{id}_{T_X\mathcal{B}}$ and $\mathbf{F}\mathbf{F}^{-1} = \mathrm{id}_{T_{\varphi(X)}\mathcal{C}}$.

where s_{ts} and s_{cs} denote the uniaxial tensile and compressive strengths, respectively. The invariants $\mathcal{I}_1$ and $\mathcal{J}_2$ are defined in terms of the principal values $\beta_1, \beta_2, \beta_3$ of the Biot stress by

$$\mathcal{I}_1 = \beta_1+\beta_2+\beta_3\,, \qquad \mathcal{J}_2 = \frac{1}{3}\left[(\beta_1+\beta_2+\beta_3)^2-\beta_1^2-\beta_2^2-\beta_3^2\right] = \frac{1}{6}\Big[(\beta_1-\beta_2)^2+(\beta_2-\beta_3)^2+(\beta_3-\beta_1)^2\Big]\,. \tag{3.31}$$

For isotropic solids, recall that the eigenvalues of the Biot stress are related to those of the Cauchy and second Piola–Kirchhoff stresses as given in (2.38). Thus, the invariants may be written in terms of the principal values of the Cauchy stress as

$$\begin{aligned}\mathcal{I}_1 &= \lambda_2\lambda_3\,\sigma_1+\lambda_1\lambda_3\,\sigma_2+\lambda_1\lambda_2\,\sigma_3\,,\\ \mathcal{J}_2 &= \frac{1}{3}\Big[\left(\lambda_2\lambda_3\,\sigma_1+\lambda_1\lambda_3\,\sigma_2+\lambda_1\lambda_2\,\sigma_3\right)^2-(\lambda_2\lambda_3\,\sigma_1)^2-(\lambda_1\lambda_3\,\sigma_2)^2-(\lambda_1\lambda_2\,\sigma_3)^2\Big]\,.\end{aligned} \tag{3.32}$$

Similarly, in terms of the principal values of the second Piola–Kirchhoff stress one has

$$\begin{aligned}\mathcal{I}_1 &= \lambda_1 S_1+\lambda_2 S_2+\lambda_3 S_3\,,\\ \mathcal{J}_2 &= \frac{1}{3}\Big[(\lambda_1 S_1+\lambda_2 S_2+\lambda_3 S_3)^2-(\lambda_1 S_1)^2-(\lambda_2 S_2)^2-(\lambda_3 S_3)^2\Big]\,.\end{aligned} \tag{3.33}$$

This example shows that a strength function expressed in terms of the Biot stress can be written in terms of the Cauchy stress or the second Piola–Kirchhoff stress only at the expense of introducing explicit dependence on the principal stretches. Thus, when the strength surface is taken to be a function of the Biot stress, it is independent of deformation only in that representation, while its representations in terms of $\boldsymbol{\sigma}$ or $\mathbf{S}$ are deformation dependent.

Remark 3.9. Spatial and material covariance[12] allows one to pass from a material representation of constitutive relations in terms of $(\mathbf{S}, \mathbf{C}^\flat)$ to a spatial representation in terms of $(\boldsymbol{\sigma}, \mathbf{c}^\flat)$ via push-forward, where $\mathbf{c}^\flat = \varphi_*\mathbf{G}$ is the push-forward of the material metric. One may equivalently use $(\boldsymbol{\sigma}, \mathbf{b}^\sharp)$, since $\mathbf{c} = \mathbf{b}^{-1}$. However, neither $\mathbf{c}$ nor $\mathbf{b}$ is work-conjugate to the Cauchy stress $\boldsymbol{\sigma}$.[13] This is in contrast with the pairs $(\mathbf{P}, \mathbf{F})$ and $(\mathbf{S}, \mathbf{C}^\flat)$, which are work-conjugate by construction. There is no inconsistency here. Covariance is a geometric requirement: it prescribes how tensor fields transform under changes of configuration via push-forward and pull-back operations. Work-conjugacy is an energetic requirement: it is dictated by the stress power identity and singles out pairs that preserve the bilinear pairing of power. In general, push-forward and pull-back operations do not preserve this bilinear pairing, and hence they do not preserve work-conjugacy. Consequently, while $(\boldsymbol{\sigma}, \mathbf{c}^\flat)$ is natural from the standpoint of covariance, it is not a work-conjugate pair.

Remark 3.10. There are strain-energy-based strength functions in the literature. Beltrami [1885] proposed a total strain-energy criterion, in which failure occurs when the stored elastic energy reaches a critical value. A similar strength function has been adopted for elastomers where the nucleation for a flaw-free material is defined through work of fracture [Chen et al., 2017]. But such strength functions face two issues: (i) they lead to surfaces that are centro-symmetric in the stress space, that is, invariant under stress inversion, $\mathsf{F}(\mathbf{S}, \mathbf{G}) = \mathsf{F}(-\mathbf{S}, \mathbf{G})$, which is inconsistent with experiments [Kumar et al., 2020], and (ii) they can not capture cavitation-like nucleation under dominant hydrostatic tensile stresses as mentioned in the Introduction and discussed explicitly below in Example 3.4. The first limitation has been widely recognized in literature. Huber [1904] (see the English translation [Huber, 2004]) formulated the distortion-energy criterion, proposing that failure is governed by the distortional (deviatoric) part of the strain energy rather than the total strain energy. The von Mises–Hencky theory [von Mises, 1913; Hencky, 1924] is a commonly utilized distortion-energy criterion. There are many other similar criteria that have been proposed in the literature, all based on the idea that only a part of the strain energy drives crack growth. These criteria have been widely utilized in the computational modeling of fracture with continuum damage models [Mazars, 1986; Mazars and Pijaudier-Cabot, 1989; Comi, 2001; Badel et al., 2007] and phase field models [Amor et al., 2009; Miehe et al., 2010]. Strain energy based criteria typically do not agree with the experimental observations on strength in the entire stress space [Lopez-Pamies et al.,

[12] Material covariance is a generalization of isotropy. For anisotopic solids elastic energy and strength functions become materially covariant when structural tensors are included.

[13] As a matter of fact, $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is work conjugate to Hencky's logarithmic strain [Gurtin and Spear, 1983; Hoger, 1986; Xiao et al., 1997].

2025]. Nevertheless, all such strength functions are a special case of the general framework discussed here. Let us denote the relevant energy measure by $\mathbb{E} = \mathbb{E}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G})$. Then the corresponding strength function has the form $\mathfrak{F} = \mathfrak{F}(\mathbb{E})$, and hence

$$\hat{\mathsf{F}}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G}) = \mathfrak{F}\big(\mathbb{E}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G})\big)\,. \tag{3.34}$$

Thus, energy-based strength criteria are a special case of stress–strain-based strength criteria.

Remark 3.11. The strength hypersurface represents a physical condition for fracture and therefore must be independent of the particular configuration used to describe the body. A deformation merely changes the representation of stresses and geometric tensors through push-forward and pull-back operations, but it does not alter the underlying physical state of the material. Consequently, if a given stress state causes fracture when expressed in the reference configuration, the same physical state expressed in the current configuration must also correspond to fracture. This requirement implies that the strength function must be spatially covariant under the change of configuration induced by the deformation map. In other words, the functional relation defining the strength hypersurface must be preserved when the arguments are transformed by the natural push-forward and pull-back associated with the motion. Spatial covariance therefore expresses the representation-independence of the fracture criterion: it guarantees that the same set of physical failure states is described consistently in both the reference and the spatial configurations.

3.3.1 The manifold of constitutively admissible homogeneous stresses

Let us assume that strength is defined in terms of uniform Cauchy stress $\boldsymbol{\sigma}$. It should be emphasized that not all homogeneous stress fields are constitutively admissible. First, note that the set of all symmetric 3×3 matrices, $\mathrm{Sym}(3)$, is a 6-dimensional real vector space and hence a smooth 6-dimensional manifold. In the present explicit isotropic compressible setting, the set of constitutively admissible homogeneous stress fields is a subset of $\mathrm{Sym}(3)$, i.e., the set of all homogeneous stresses $\boldsymbol{\sigma}$ such that there exists a deformation φ with associated $\mathbf{b}^\sharp$ satisfying the compatibility conditions and the constitutive relation $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g})$. Let us denote this set by $\mathcal{A}$. Note that the set $\mathcal{A}$ is non-empty. To show this, consider a homogeneous dilatational deformation $\varphi(X) = \alpha\, X$, $\alpha > 0$. Then the deformation gradient is uniform with components $F^a{}_A = \alpha\, \delta^a_A$, and hence the left Cauchy–Green tensor is written as $\mathbf{b}^\sharp = \mathbf{F}\mathbf{G}^\sharp\mathbf{F}^\star = \alpha^2\, \mathbf{g}^\sharp$. The constitutive relation $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g})$ then implies that the corresponding Cauchy stress is uniform. Therefore, $\boldsymbol{\sigma} \in \mathcal{A}$, and hence $\mathcal{A} \neq \varnothing$.

Proposition 3.1. *Let $\hat{\boldsymbol{\sigma}} : \mathrm{Sym}^+(3) \to \mathrm{Sym}(3)$ be the constitutive map of an isotropic compressible elastic solid, and let*

$$\mathcal{A} = \hat{\boldsymbol{\sigma}}\big(\mathrm{Sym}^+(3)\big) \subset \mathrm{Sym}(3)\,, \tag{3.35}$$

denote the set of constitutively admissible homogeneous Cauchy stresses. Suppose that, for every $\mathbf{b}^\sharp \in \mathrm{Sym}^+(3)$ under consideration, the derivative map

$$D\hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g}) : T_{\mathbf{b}^\sharp}\mathrm{Sym}^+(3) \to T_{\hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g})}\mathrm{Sym}(3)\,, \tag{3.36}$$

has full rank 6. Then $\mathcal{A}$ is a smooth 6-dimensional manifold.

Proof. Recall that $\mathrm{Sym}(3)$ is a 6-dimensional smooth manifold, and that $\mathrm{Sym}^+(3)$, being an open subset of $\mathrm{Sym}(3)$, is also a 6-dimensional smooth manifold. Since the derivative map

$$D\hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g}) : T_{\mathbf{b}^\sharp}\mathrm{Sym}^+(3) \to T_{\hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g})}\mathrm{Sym}(3)\,, \tag{3.37}$$

has full rank 6 for every $\mathbf{b}^\sharp \in \mathrm{Sym}^+(3)$ under consideration, it is an isomorphism. Equivalently, $\det D\hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g}) \neq 0$.[14] Therefore, by the inverse function theorem [Lee, 2013], for every $\mathbf{b}^\sharp \in \mathrm{Sym}^+(3)$ there exist open neighborhoods $\mathcal{U}_{\mathbf{b}^\sharp} \subset \mathrm{Sym}^+(3)$ and $\mathcal{V}_{\hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g})} \subset \mathrm{Sym}(3)$ such that the restriction

$$\hat{\boldsymbol{\sigma}}\big|_{\mathcal{U}_{\mathbf{b}^\sharp}} : \mathcal{U}_{\mathbf{b}^\sharp} \to \mathcal{V}_{\hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g})}\,, \tag{3.38}$$

[14] A closely related non-degeneracy condition appears in [Neff et al., 2025], where the authors use the condition $\det D_{\mathbf{B}}\boldsymbol{\sigma}(\mathbf{B}) \neq 0$ and relate it to the invertibility of the associated tangent operator; see in particular their equations (1.10) and (2.69).

is a diffeomorphism. In particular, for every $\mathbf{b}^\sharp \in \mathrm{Sym}^+(3)$, the image of a neighborhood of $\mathbf{b}^\sharp$ is an open subset of Sym(3). Hence, the image

$$\mathcal{A} = \hat{\boldsymbol{\sigma}}\big(\mathrm{Sym}^+(3)\big) \subset \mathrm{Sym}(3)\,, \tag{3.39}$$

is locally open in Sym(3). Equivalently, every point of $\mathcal{A}$ has a neighborhood, open in $\mathcal{A}$, that is also open in Sym(3). Since Sym(3) is a smooth 6-dimensional manifold, it follows that $\mathcal{A}$ inherits the structure of a smooth 6-dimensional manifold. □

This proposition shows that, under the non-degeneracy assumption $\det D\hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g}) \neq 0$, the set of constitutively admissible homogeneous Cauchy stresses is not an arbitrary subset of Sym(3), but a smooth 6-dimensional manifold. Thus, the strength hypersurface should be understood as a hypersurface in this constitutively admissible stress manifold. The non-degeneracy assumption is equivalent to local invertibility of the constitutive map $\mathbf{b}^\sharp \mapsto \hat{\boldsymbol{\sigma}}(\mathbf{b}^\sharp, \mathbf{g})$, and has a clear physical meaning: the elastic response is locally non-degenerate, in the sense that small changes in strain produce unique changes in stress, and there are no non-trivial nearby strain states that leave the stress unchanged. This may be viewed as a natural constitutive regularity condition for a reasonable elastic solid.

Assuming that the tangent map $D_{\mathbf{b}^\sharp}\boldsymbol{\sigma}$ is invertible is a strong local non-degeneracy condition, as it implies that no nontrivial perturbation of $\mathbf{b}^\sharp$ produces a vanishing increment of stress. In particular, it guarantees local uniqueness of the constitutive response. Let $\mathrm{Lin}(T_X\mathcal{B}, T_x\mathcal{C})$ denote the space of linear maps from $T_X\mathcal{B}$ to $T_x\mathcal{C}$, and $\mathrm{Lin}^+(T_X\mathcal{B}, T_x\mathcal{C})$ denote the subset of orientation-preserving linear isomorphisms. An elastic energy $W(\mathbf{F}, \mathbf{G}, \mathbf{g})$ is said to be rank-one convex if, for every $\mathbf{F} \in \mathrm{Lin}^+(T_X\mathcal{B}, T_x\mathcal{C})$ and every rank-one tensor $\mathbf{a} \otimes \mathbf{N}^\flat$, the function $t \mapsto W(\mathbf{F} + t\,\mathbf{a} \otimes \mathbf{N}^\flat, \mathbf{G}, \mathbf{g})$ is convex for all t for which $\mathbf{F} + t\,\mathbf{a} \otimes \mathbf{N}^\flat \in \mathrm{Lin}^+(T_X\mathcal{B}, T_x\mathcal{C})$ [Ball, 1976]. If W is of class C^2, this is equivalent to the Legendre–Hadamard condition, or strong ellipticity, namely

$$D^2W(\mathbf{F}, \mathbf{G}, \mathbf{g})\left[\mathbf{a} \otimes \mathbf{N}^\flat, \mathbf{a} \otimes \mathbf{N}^\flat\right] \geq 0\,, \qquad \forall\, \mathbf{a} \in T_x\mathcal{C} \setminus \{\mathbf{0}\}\,,\ \forall\, \mathbf{N}^\flat \in T_X^*\mathcal{B} \setminus \{\mathbf{0}\}\,, \tag{3.40}$$

and in components

$$\frac{\partial^2 W}{\partial F^a{}_A \partial F^b{}_B}\, a^a\, N_A\, a^b\, N_B \geq 0\,, \qquad \forall\, \mathbf{a} \in T_x\mathcal{C} \setminus \{\mathbf{0}\}\,,\ \forall\, \mathbf{N}^\flat \in T_X^*\mathcal{B} \setminus \{\mathbf{0}\}\,. \tag{3.41}$$

Thus, strong ellipticity requires positivity only along rank-one directions.

This condition is strictly weaker than invertibility of $D_{\mathbf{b}^\sharp}\boldsymbol{\sigma}$. Indeed, invertibility of $D_{\mathbf{b}^\sharp}\boldsymbol{\sigma}$ requires non-degeneracy in all directions in the space of admissible perturbations of $\mathbf{b}^\sharp$, whereas rank-one convexity only requires non-degeneracy along rank-one directions. The restriction to rank-one directions is not arbitrary. Rank-one perturbations correspond to deformation gradients that are compatible across surfaces, and hence describe the simplest admissible discontinuities through the Hadamard jump condition. Loss of strong ellipticity along such directions signals loss of ellipticity, the onset of material instability, and the possible formation of microstructure. For this reason, rank-one convexity, or equivalently strong ellipticity in the C^2 setting, is the minimal local stability requirement in elasticity, while invertibility of $D_{\mathbf{b}^\sharp}\boldsymbol{\sigma}$ is a stronger constitutive assumption.

3.3.2 Constitutively admissible homogeneous stresses in isotropic solids

For isotropic solids, all one needs for describing the strength properties are the principal stresses. For a hyperelastic solid the Cauchy principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ are related to principal stretches $(\lambda_1, \lambda_2, \lambda_3)$ as [Ogden, 1997]

$$\sigma_i = \frac{\lambda_i}{\lambda_1\lambda_2\lambda_3}\,\frac{\partial W}{\partial \lambda_i}\,, \qquad i = 1, 2, 3\,. \tag{3.42}$$

Thus, one has the principal stress map

$$\begin{aligned} \hat{\boldsymbol{\sigma}} : \mathbb{R}^3_+ &\to \mathbb{R}^3 \\ (\lambda_1, \lambda_2, \lambda_3) &\mapsto (\sigma_1, \sigma_2, \sigma_3)\,. \end{aligned} \tag{3.43}$$

Suppose that, for every $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+$ under consideration, the derivative map

$$D\hat{\boldsymbol{\sigma}}(\lambda_1, \lambda_2, \lambda_3) : T_{(\lambda_1,\lambda_2,\lambda_3)}\mathbb{R}^3_+ \to T_{\hat{\boldsymbol{\sigma}}(\lambda_1,\lambda_2,\lambda_3)}\mathbb{R}^3\,, \tag{3.44}$$

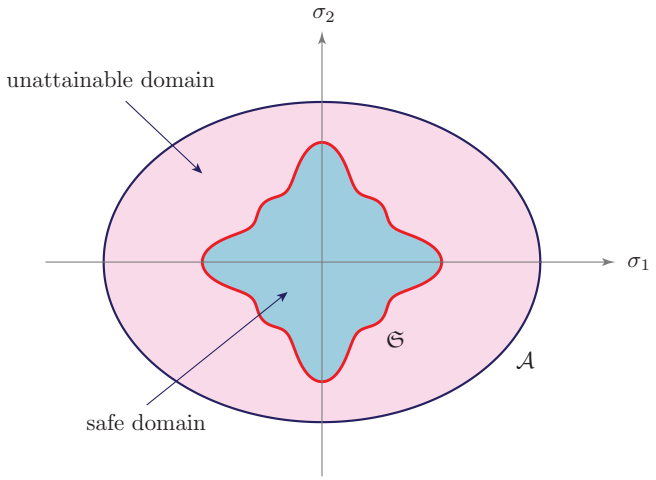

Figure 2: *Schematic illustration of the constitutively-admissible stress manifold $\mathcal{A}$ (bounded by the black ellipse) and the strength hypersurface $\mathfrak{S}$ (red curve) in principal stress space. The strength hypersurface is a hypersurface of the constitutively-admissible stress manifold and separates safe and unsafe stress states.*

has full rank 3. Equivalently, the Jacobian matrix

$$\left[\frac{\partial \sigma_i}{\partial \lambda_j}\right] , \tag{3.45}$$

is invertible. This is precisely the condition that the map $(\lambda_1, \lambda_2, \lambda_3) \mapsto (\sigma_1, \sigma_2, \sigma_3)$ is locally invertible. By the inverse function theorem [Lee, 2013], its image is locally open in $\mathbb{R}^3$. Therefore, the set of constitutively admissible triples of principal stresses is a smooth 3-dimensional manifold. Consequently, for isotropic solids, the strength hypersurface may be viewed equivalently as a hypersurface in this 3-dimensional manifold of admissible principal stresses.

3.3.3 The strength hypersurface

In the following we restrict attention to a subclass of strength functions that depend only on stress, since all existing examples in the literature belong to this class. Let us consider a homogeneous solid in a state of spatially uniform Cauchy stress $\boldsymbol{\sigma}$. We use the Cauchy stress without loss of generality; the same conclusions hold if one starts with any other stress measure. Suppose the set of all uniform stresses under which the body fractures is described by the equation $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g}) = 0$.[15] Assume that $\mathsf{f} : \mathbb{R}^6 \to \mathbb{R}$ is continuous and define the corresponding strength hypersurface:

$$\mathfrak{S} = \{\boldsymbol{\sigma} \in \mathcal{A} : \mathsf{f}(\boldsymbol{\sigma}, \mathbf{g}) = 0\} . \tag{3.46}$$

Any physical state of stress in the body is assumed to satisfy the constraint $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g})) \leq 0$, and the set $\{\mathsf{f} < 0\}$ is the set of *safe stresses*. The domain $\{\mathsf{f} > 0\}$ is assumed to be physically unattainable, see Fig. 2. Clearly, the domains $\{\mathsf{f} < 0\}$ and $\{\mathsf{f} > 0\}$ are disjoint. Consequently, the strength hypersurface $\mathfrak{S}$ partitions the stress space into two disjoint sets corresponding to safe and unattainable stresses. Note that every stress state on the strength hypersurface must be a threshold state separating safe stresses from failing stresses. In physical terms this means the following: Let $\boldsymbol{\sigma}_0 \in \mathfrak{S}$ be a critical stress state satisfying $\mathsf{f}(\boldsymbol{\sigma}_0, \mathbf{g}) = 0$. Then arbitrarily small perturbations of the stress must produce both safe and failing states. More precisely, for every $\varepsilon > 0$ there exist stresses $\boldsymbol{\sigma}_- \in \mathbb{R}^6$ and $\boldsymbol{\sigma}_+ \in \mathbb{R}^6$ such that[16]

$$\|\boldsymbol{\sigma}_- - \boldsymbol{\sigma}_0\| < \varepsilon, \qquad \|\boldsymbol{\sigma}_+ - \boldsymbol{\sigma}_0\| < \varepsilon , \tag{3.47}$$

with

$$\mathsf{f}(\boldsymbol{\sigma}_-, \mathbf{g}) < 0 , \qquad \mathsf{f}(\boldsymbol{\sigma}_+, \mathbf{g}) > 0 . \tag{3.48}$$

[15] Note that a scalar function of a spatial second-order tensor necessarily involves the spatial metric $\mathbf{g}$ through index contractions.

[16] We denote by $\|\cdot\|$ any norm on $\mathrm{Sym}(3) \cong \mathbb{R}^6$. Since $\mathrm{Sym}(3)$ is a finite-dimensional vector space, all norms are equivalent. Therefore, the particular choice of norm is immaterial, as it induces the same topology and hence the same notion of neighborhoods.

In other words, every neighborhood of $\boldsymbol{\sigma}_0$ contains stresses that do not fracture the material and stresses that do fracture the material. This expresses the physical idea that $\mathfrak{S}$ is a true threshold between safe and failing stresses.

Proposition 3.2. *Let $\mathsf{f} : \mathbb{R}^6 \to \mathbb{R}$ be continuous and let $\mathfrak{S}$ be defined by $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g})) = 0$. Then the sets $\{\mathsf{f} < 0\}$ and $\{\mathsf{f} > 0\}$ are open and disjoint, and*

$$\partial\{\mathsf{f} < 0\} \subset \mathfrak{S}\,, \qquad \partial\{\mathsf{f} > 0\} \subset \mathfrak{S}\,. \tag{3.49}$$

If, in addition, the threshold property stated above holds for every $\boldsymbol{\sigma}_0 \in \mathfrak{S}$, then

$$\partial\{\mathsf{f} < 0\} = \partial\{\mathsf{f} > 0\} = \mathfrak{S}\,. \tag{3.50}$$

Proof. Since f is continuous, the preimages $\{\mathsf{f} < 0\} = \mathsf{f}^{-1}((-\infty, 0))$ and $\{\mathsf{f} > 0\} = \mathsf{f}^{-1}((0, \infty))$ are open. They are disjoint because f cannot be simultaneously negative and positive.

To prove $\partial\{\mathsf{f} < 0\} \subset \mathfrak{S}$, let $\boldsymbol{\sigma}_0 \in \partial\{\mathsf{f} < 0\}$. Suppose, for contradiction, that $\mathsf{f}(\boldsymbol{\sigma}_0) > 0$. By continuity of f at $\boldsymbol{\sigma}_0$, there exists $\varepsilon > 0$ such that $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g}) > 0$ for all $\boldsymbol{\sigma}$ with $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0\| < \varepsilon$. Hence, $B_\varepsilon(\boldsymbol{\sigma}_0) \cap \{\mathsf{f} < 0\} = \emptyset$, where $B_\varepsilon(\boldsymbol{\sigma}_0) = \{\boldsymbol{\sigma} \in \mathbb{R}^n : \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0\| < \varepsilon\}$ denotes the open ball of radius ε centered at $\boldsymbol{\sigma}_0$. This implies that $\boldsymbol{\sigma}_0 \notin \partial\{\mathsf{f} < 0\}$, a contradiction. Therefore $\mathsf{f}(\boldsymbol{\sigma}_0) \le 0$. Next suppose, again for contradiction, that $\mathsf{f}(\boldsymbol{\sigma}_0) < 0$. By continuity, there exists $\varepsilon > 0$ such that $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g}) < 0$ whenever $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0\| < \varepsilon$. Hence $B_\varepsilon(\boldsymbol{\sigma}_0) \subset \{\mathsf{f} < 0\}$, so $\boldsymbol{\sigma}_0$ is an interior point of $\{\mathsf{f} < 0\}$ and cannot lie in its boundary, again a contradiction. Thus $\mathsf{f}(\boldsymbol{\sigma}_0) = 0$, i.e. $\boldsymbol{\sigma}_0 \in \mathfrak{S}$, and therefore $\partial\{\mathsf{f} < 0\} \subset \mathfrak{S}$. The inclusion $\partial\{\mathsf{f} > 0\} \subset \mathfrak{S}$ is proved identically, with the inequalities reversed.

Finally, assume the threshold property. Let $\boldsymbol{\sigma}_0 \in \mathfrak{S}$. Then every neighborhood of $\boldsymbol{\sigma}_0$ contains a point of $\{\mathsf{f} < 0\}$ and a point of $\{\mathsf{f} > 0\}$. Hence every neighborhood of $\boldsymbol{\sigma}_0$ intersects both $\{\mathsf{f} < 0\}$ and its complement, so $\boldsymbol{\sigma}_0 \in \partial\{\mathsf{f} < 0\}$. Similarly, $\boldsymbol{\sigma}_0 \in \partial\{\mathsf{f} > 0\}$. Therefore $\mathfrak{S} \subset \partial\{\mathsf{f} < 0\}$ and $\mathfrak{S} \subset \partial\{\mathsf{f} > 0\}$. Combined with the previously proved inclusions $\partial\{\mathsf{f} < 0\} \subset \mathfrak{S}$ and $\partial\{\mathsf{f} > 0\} \subset \mathfrak{S}$, we conclude that $\partial\{\mathsf{f} < 0\} = \partial\{\mathsf{f} > 0\} = \mathfrak{S}$. □

Example 3.3. This example illustrates that, even for spatially uniform stress fields, constitutive admissibility imposes nontrivial constraints. In particular, we show that not every uniform stress field can be realized by a deformation. The implicit constitutive equations for an isotropic solid are rewritten as [Rivlin and Ericksen, 1955; Morgan, 1966; Yavari and Goriely, 2024]

$$\begin{aligned}\mathbf{f}(\boldsymbol{\sigma}, \mathbf{b}) = \alpha_0\, \mathbf{g}^\sharp + \alpha_1 \boldsymbol{\sigma} + \alpha_2 \boldsymbol{\sigma}^2 + \alpha_3\, \mathbf{b}^\sharp + \alpha_4\, \mathbf{c}^\sharp + \alpha_5 \left(\boldsymbol{\sigma}\mathbf{b}^\sharp + \mathbf{b}^\sharp\boldsymbol{\sigma}\right) + \alpha_6 \left(\boldsymbol{\sigma}^2\mathbf{b}^\sharp + \mathbf{b}^\sharp\boldsymbol{\sigma}^2\right) \\ + \alpha_7 \left(\boldsymbol{\sigma}\mathbf{c}^\sharp + \mathbf{c}^\sharp\boldsymbol{\sigma}\right) + \alpha_8 \left(\boldsymbol{\sigma}^2\mathbf{c}^\sharp + \mathbf{c}^\sharp\boldsymbol{\sigma}^2\right) = \mathbf{0}\,,\end{aligned} \tag{3.51}$$

where α_i, $i = 0, \dots, 8$, are functions of the following ten invariants

$$\begin{aligned}&I_1 = \operatorname{tr}\boldsymbol{\sigma}\,, \quad I_2 = \operatorname{tr}\boldsymbol{\sigma}^2\,, \quad I_3 = \operatorname{tr}\boldsymbol{\sigma}^3\,, \quad I_4 = \operatorname{tr}\mathbf{b}^\sharp\,, \quad I_5 = \frac{1}{2}\left[I_4^2 - \operatorname{tr}\mathbf{b}^{2\sharp}\right]\,, \quad I_6 = \det\mathbf{b}^\sharp\,, \\ &I_7 = \operatorname{tr}\left(\boldsymbol{\sigma}\mathbf{b}^\sharp\right)\,, \quad I_8 = \operatorname{tr}\left(\boldsymbol{\sigma}\mathbf{b}^{2\sharp}\right)\,, \quad I_9 = \operatorname{tr}\left(\boldsymbol{\sigma}^2\mathbf{b}^\sharp\right)\,, \quad I_{10} = \operatorname{tr}\left(\boldsymbol{\sigma}^2\mathbf{b}^{2\sharp}\right)\,.\end{aligned} \tag{3.52}$$

Suppose that the Cauchy stress field is uniform, and hence, its principal values σ_i, $i = 1, 2, 3$, are constants throughout the body. Assume $\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0$, i.e.,

$$\alpha_2\boldsymbol{\sigma}^2 + \alpha_1\boldsymbol{\sigma} = -\alpha_0\, \mathbf{g}^\sharp - \alpha_3\, \mathbf{b}^\sharp - \alpha_4\, \mathbf{c}^\sharp\,. \tag{3.53}$$

We ask whether there exists a deformation corresponding to this uniform stress field. Let $\lambda_i = \lambda_i(x)$, $i = 1, 2, 3$, denote the principal stretches of the deformation at a point $x \in \mathcal{C}$. We denote the eigenvalues of $\mathbf{b}^\sharp(x)$ by $\lambda_i^2(x)$. For this class of materials, $\boldsymbol{\sigma}$ and $\mathbf{b}^\sharp$ have the same eigenvectors and can be diagonalized simultaneously.[17] Thus, at each point $x \in \mathcal{C}$, with respect to the common eigenbasis one has

$$\alpha_3\, \lambda_i^2(x) + \alpha_4\, \lambda_i^{-2}(x) = -\alpha_0 - \alpha_1\sigma_i - \alpha_2\sigma_i^2\,, \qquad i = 1, 2, 3\,. \tag{3.54}$$

Multiplying by $\lambda_i^2(x)$, we obtain

$$\alpha_3\, \lambda_i^4(x) + \left(\alpha_0 + \alpha_1\sigma_i + \alpha_2\sigma_i^2\right) \lambda_i^2(x) + \alpha_4 = 0\,, \qquad i = 1, 2, 3\,. \tag{3.55}$$

[17] It should be emphasized that for the more general class of elastic solids with constitutive equations (3.51), $\boldsymbol{\sigma}$ and $\mathbf{b}^\sharp$ do not have the same eigenvectors, in general.

Note that the coefficients $\alpha_j = \alpha_j(I_1, \dots, I_{10})$ are not constants in general, because the invariants $I_4, \dots, I_{10}$ depend on $\mathbf{b}^\sharp(x)$ and hence on the position-dependent stretches $\lambda_i(x)$. Therefore, the above equations are not quadratic equations with constant coefficients for $\lambda_i^2(x)$; rather, they form a coupled nonlinear algebraic system that must be satisfied pointwise. More precisely, for each point $x \in \mathcal{C}$ one must solve

$$\Psi_i\big(\lambda_1^2(x), \lambda_2^2(x), \lambda_3^2(x); \sigma_1, \sigma_2, \sigma_3\big) = 0\,, \qquad i = 1, 2, 3\,, \tag{3.56}$$

where

$$\Psi_i := \alpha_3\, \lambda_i^4 + \big(\alpha_0 + \alpha_1 \sigma_i + \alpha_2 \sigma_i^2\big)\, \lambda_i^2 + \alpha_4\,, \tag{3.57}$$

and each α_j is evaluated at the invariants (3.52). Thus, constitutive admissibility of a uniform stress field requires that, at every point $x \in \mathcal{C}$, this nonlinear algebraic system admit a real positive solution $(\lambda_1^2(x), \lambda_2^2(x), \lambda_3^2(x))$. If, for a given uniform stress field $\boldsymbol{\sigma}$, there exists a point $x \in \mathcal{C}$ such that the above system has no real positive solution, then no deformation can realize this stress field. Hence, not every uniform stress field is constitutively admissible. A concrete subclass is obtained by assuming that α_i, $i = 0, \dots, 4$, are constants and that $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = \alpha_4 = 1$, and $\alpha_0 = -c$, where c is a constant. In this case, the constitutive equations reduce to

$$\lambda_i^2(x) + \lambda_i^{-2}(x) = c\,, \qquad i = 1, 2, 3\,. \tag{3.58}$$

Multiplying by $\lambda_i^2(x)$, one obtains

$$\lambda_i^4(x) - c\, \lambda_i^2(x) + 1 = 0\,, \qquad i = 1, 2, 3\,. \tag{3.59}$$

This quadratic equation admits real positive solutions for $\lambda_i^2(x)$ if and only if $c^2 - 4 \geq 0$, i.e., if and only if $c \geq 2$ or $c \leq -2$. Since $\lambda_i^2(x) + \lambda_i^{-2}(x) \geq 2$ for every positive $\lambda_i^2(x)$, only the case $c \geq 2$ is physically admissible. Therefore, if $0 < c < 2$, there is no real positive solution for $\lambda_i^2(x)$ at any point $x \in \mathcal{C}$. Thus, for this subclass, no deformation can realize such a uniform stress field, and hence not every uniform stress field is constitutively admissible.

3.3.4 Star-shapedness of the safe domain

Next, we show that, under a natural assumption, the safe domain is star-shaped with respect to the zero-stress state (the origin in stress space).

Definition 3.1 (Star-shaped domain)**.** Let $\Omega \subset \mathbb{R}^n$ with $n \in \{2, 3\}$. We say that Ω is *star-shaped with respect to the origin* if $\mathbf{0} \in \Omega$ and if for every $\boldsymbol{x} \in \Omega$ and every $t \in [0, 1]$ one has $t\boldsymbol{x} \in \Omega$ [Rockafellar, 1970]. Equivalently, for every $\boldsymbol{x} \in \Omega$ the line segment $\{t\boldsymbol{x} : t \in [0, 1]\}$ is contained in Ω. In $n = 2$ we write $\boldsymbol{x} = (\sigma_1, \sigma_2)$, while in $n = 3$ we write $\boldsymbol{x} = (\sigma_1, \sigma_2, \sigma_3)$.

Definition 3.2 (Proportional reduction safety)**.** Let $\mathsf{f} : \mathbb{R}^n \to \mathbb{R}$ and let $\Omega = \{\boldsymbol{\sigma} \in \mathbb{R}^n : \mathsf{f}(\boldsymbol{\sigma}, \mathbf{g})) < 0\}$ denote the safe domain. We say that f satisfies the *proportional reduction safety property* if for every safe stress $\boldsymbol{\sigma} \in \Omega$ and every $t \in [0, 1)$, the proportionally reduced stress $t\boldsymbol{\sigma}$ is also safe, i.e.,

$$\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g})) < 0 \;\Rightarrow\; \mathsf{f}(t\boldsymbol{\sigma}, \mathbf{g}) < 0\,, \qquad \forall\, \boldsymbol{\sigma} \in \mathbb{R}^n\,,\ \forall\, t \in [0, 1)\,. \tag{3.60}$$

Proposition 3.3. *Let $\mathsf{f} : \mathbb{R}^n \to \mathbb{R}$ with $n \in \{2, 3\}$ be a (continuous) strength function, and let Ω be its safe domain. If f satisfies the proportional reduction safety property, then Ω is star-shaped with respect to the origin.*

Proof. Let $\boldsymbol{\sigma} \in \Omega$. Since $\boldsymbol{\sigma} \in \Omega$, one has $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g})) < 0$. Hence, by the proportional reduction safety property, $t\boldsymbol{\sigma} \in \Omega$ for every $t \in [0, 1)$. Since also $\boldsymbol{\sigma} \in \Omega$, it follows that $t\boldsymbol{\sigma} \in \Omega$ for every $t \in [0, 1]$. Thus, Ω is star-shaped with respect to the origin. □

One should note that this conclusion cannot be deduced from continuity of f alone. Star-shapedness encodes a constitutive requirement on strength: if a stress state is safe, then every proportionally reduced stress state along the segment joining that state to the origin must also be safe. The proportional reduction safety property is a mathematical expression of this requirement.

Remark 3.12. Every convex domain is star-shaped with respect to each of its points. The converse is false: a star-shaped domain need not be convex. It should be emphasized that star-shapedness is a much weaker condition than convexity. We have no reason to expect the safe domain to be convex. On the contrary, experimental evidence shows that non-convex safe domains do occur [Ashkenazi, 1965]. However, such domains are still star-shaped with respect to the origin. See also Fig. 3.

3.3.5 Topology of the strength hypersurface

Let $\mathsf{f} : \mathcal{A} \to \mathbb{R}$ be a smooth strength function. From a physical point of view, the strength hypersurface (3.46) represents the set of critical stress states separating safe and failing states. It is therefore natural to expect it be a smooth hypersurface and that it be bounded, since sufficiently large admissible stresses should not remain safe. We now make these requirements precise.

Proposition 3.4. *Let $\mathsf{f} : \mathcal{A} \to \mathbb{R}$ be a smooth function, where $\mathcal{A} \subset \mathbb{R}^6$ is the manifold of constitutively admissible homogeneous stresses. Suppose that 0 is a regular value of f and that either*

$$\mathsf{f}(\boldsymbol{\sigma}) > 0\,, \qquad \forall\, \boldsymbol{\sigma} \in \mathcal{A} \text{ with } \|\boldsymbol{\sigma}\| > R\,, \tag{3.61}$$

for some $R > 0$, or, more strongly,[18]

$$\mathsf{f}(\boldsymbol{\sigma}) \to +\infty \quad \text{as} \quad \|\boldsymbol{\sigma}\| \to \infty\,, \qquad \boldsymbol{\sigma} \in \mathcal{A}\,. \tag{3.62}$$

Then the strength hypersurface $\mathfrak{S} = \mathsf{f}^{-1}(0)$ is a smooth compact embedded hypersurface in $\mathcal{A}$, and hence also in $\mathbb{R}^6$.

Proof. A value $c \in \mathbb{R}$ is a regular value of f if, for every $\boldsymbol{\sigma} \in \mathsf{f}^{-1}(c)$, the derivative map

$$d\mathsf{f}_{\boldsymbol{\sigma}} : T_{\boldsymbol{\sigma}}\mathcal{A} \to T_c\mathbb{R} \simeq \mathbb{R}\,, \tag{3.63}$$

is surjective, which is equivalent to $d\mathsf{f}_{\boldsymbol{\sigma}} \neq \mathbf{0}$. By the preimage theorem [Lee, 2013], if 0 is a regular value of f, then $\mathfrak{S} = \mathsf{f}^{-1}(0)$ is a smooth embedded submanifold of $\mathcal{A}$ of codimension one. Hence, $\mathfrak{S}$ is a smooth embedded hypersurface in $\mathcal{A}$, and since $\mathcal{A}$ is an embedded submanifold of $\mathbb{R}^6$, it follows that $\mathfrak{S}$ is also an embedded hypersurface in $\mathbb{R}^6$.

Under either assumption, there exists $R > 0$ such that $\mathsf{f}(\boldsymbol{\sigma}) > 0$ for all $\boldsymbol{\sigma} \in \mathcal{A}$ with $\|\boldsymbol{\sigma}\| > R$. Therefore,

$$\mathfrak{S} \subset \overline{B_R(0)}\,, \tag{3.64}$$

and hence $\mathfrak{S}$ is bounded. Since f is continuous and $\{0\}$ is closed, the set $\mathfrak{S} = \mathsf{f}^{-1}(0)$ is closed in $\mathcal{A}$, and therefore closed in $\mathbb{R}^6$. Thus, $\mathfrak{S}$ is closed and bounded in $\mathbb{R}^6$, and hence compact by the Heine–Borel theorem [Rudin, 1976]. □

In summary, if f is smooth, if 0 is a regular value, and if sufficiently large admissible stresses are unsafe, then the strength hypersurface $\mathfrak{S}$ is a smooth compact embedded hypersurface that separates safe and failing stresses.

3.4 Material strength in the presence of internal constraints

In the presence of an internal constraint, the stress has a reactive part and a constitutive part. As an example, let us consider incompressible solids, for which $J = 1$. In this case, the second Piola–Kirchhoff stress admits the following standard additive decomposition[19]

$$\mathbf{S} = -\,p_0\,\mathbf{C}^{-\sharp} + \bar{\mathbf{S}}\,, \tag{3.66}$$

where p_0 is an undetermined scalar Lagrange multiplier field and $\bar{\mathbf{S}}$ is the constitutive stress. The scalar field p_0 is not determined by the constitutive equations; it is fixed only through equilibrium and boundary conditions together with the incompressibility constraint $\det \mathbf{C}^{\flat} = \det \mathbf{G}$. Consequently, in any constitutive equation, whether implicit or explicit, it is the constitutive part of the stress $\bar{\mathbf{S}}$ that enters directly, rather than the total stress $\mathbf{S}$.

[18] This assumption is the standard coercivity condition. It is stronger than the first, since $\mathsf{f}(\boldsymbol{\sigma}) \to +\infty$ as $\|\boldsymbol{\sigma}\| \to \infty$ implies that there exists $R > 0$ such that $\mathsf{f}(\boldsymbol{\sigma}) > 0$ whenever $\|\boldsymbol{\sigma}\| > R$.

[19] In terms of the first Piola–Kirchhoff and Cauchy stresses this is rewritten as

$$\mathbf{P} = -p_0\,\mathbf{g}^{\sharp}\,\mathbf{F}^{-\star} + \bar{\mathbf{P}}\,, \qquad \boldsymbol{\sigma} = -p\,\mathbf{g}^{\sharp} + \bar{\boldsymbol{\sigma}}\,, \tag{3.65}$$

where $p = J^{-1}p_0$.

If the strength function is assumed to depend only on stress, i.e., $\hat{\mathsf{F}}(\mathbf{S},\mathbf{G})=0$, then the internal constraint does not appear explicitly in the equation defining the strength hypersurface. However, the admissible strength states must satisfy

$$\begin{cases} \hat{\mathsf{F}}(\mathbf{S},\mathbf{G})=0\,, \\ \mathbf{S}=-\,p_0\,\mathbf{C}^{-\sharp}+\bar{\mathbf{S}}\,, \\ \mathfrak{f}(\bar{\mathbf{S}},\mathbf{C}^{\flat},\mathbf{G})=\mathbf{0}\,, \\ \det\mathbf{C}^{\flat}=\det\mathbf{G}\,, \end{cases} \tag{3.67}$$

where $(3.67)_3$ is the implicit constitutive equation. Thus, the internal constraint restricts the set of admissible stresses without modifying the form of the strength function. On the other hand, if the strength function depends on both stress and strain, i.e., $\hat{\mathsf{F}}(\mathbf{S},\mathbf{C}^{\flat},\mathbf{G})=0$, then the admissible strength states are determined by

$$\begin{cases} \hat{\mathsf{F}}(\mathbf{S},\mathbf{C}^{\flat},\mathbf{G})=0\,, \\ \mathbf{S}=-\,p_0\,\mathbf{C}^{-\sharp}+\bar{\mathbf{S}}\,, \\ \mathfrak{f}(\bar{\mathbf{S}},\mathbf{C}^{\flat},\mathbf{G})=\mathbf{0}\,, \\ \det\mathbf{C}^{\flat}=\det\mathbf{G}\,. \end{cases} \tag{3.68}$$

Thus, for constrained materials, a stress-based strength criterion is restricted only indirectly through constitutive admissibility, whereas a strength function depending on both stress and strain is restricted directly through the admissible stress-strain pairs.

Example 3.4 (Uniform hydrostatic tensile stress in an incompressible solid)**.** Consider a homogeneous incompressible solid for which the total first Piola–Kirchhoff stress is uniform. In particular, there are no body forces, and the loading is applied only through boundary tractions. Let the reference configuration be a ball of radius R_0. We use spherical coordinates (R,Θ,Φ) and (r,θ,ϕ) in the reference and current configurations, respectively. The material and spatial metrics have the classical representations $\mathbf{G}=\mathrm{diag}(1,\,R^2,\,R^2\sin^2\Theta)$ and $\mathbf{g}=\mathrm{diag}(1,\,r^2,\,r^2\sin^2\theta)$. We consider radial deformations:

$$r(R,\Theta,\Phi)=r(R)\,,\qquad \theta(R,\Theta,\Phi)=\Theta\,,\qquad \phi(R,\Theta,\Phi)=\Phi\,. \tag{3.69}$$

The deformation gradient has the representation $\mathbf{F}=\mathrm{diag}(r'(R),\,1,\,1)$. Incompressibility implies that

$$J=\sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\,\det\mathbf{F}=\frac{r^2(R)\,r'(R)}{R^2}=1\,. \tag{3.70}$$

Assuming that $r(0)=0$, one concludes that $r(R)=R$. Thus, under radial deformations the spherical ball does not deform, i.e., the deformation is the identity map, and the body behaves effectively as a rigid solid under this loading. The principal invariants are calculated as

$$I_1=\frac{R^6+2r^6(R)}{R^2r^4(R)}=3\,,\qquad I_2=\frac{2R^6+r^6(R)}{R^4r^2(R)}=3\,. \tag{3.71}$$

The non-zero Cauchy stress components are

$$\begin{aligned} \sigma^{rr}(R)&=-p+\frac{2R^4W_1}{r^4(R)}-\frac{2W_2\,r^4(R)}{R^4}=-p+2(W_1-W_2)\,, \\ \sigma^{\theta\theta}(R)&=-\frac{p}{r^2(R)}+\frac{2W_1}{R^2}-\frac{2R^2W_2}{r^4(R)}=-\frac{p}{R^2}+\frac{2(W_1-W_2)}{R^2}\,, \\ \sigma^{\phi\phi}(R)&=\left[-\frac{p}{r^2(R)}+\frac{2W_1}{R^2}-\frac{2R^2W_2}{r^4(R)}\right]\csc^2\Theta=\left[-\frac{p}{R^2}+\frac{2(W_1-W_2)}{R^2}\right]\csc^2\Theta\,. \end{aligned} \tag{3.72}$$

The only non-trivial equilibrium equation is

$$\frac{\partial\sigma^{rr}}{\partial r}+\frac{2}{r}\sigma^{rr}-r\,\sigma^{\theta\theta}-(r\sin^2\theta)\,\sigma^{\phi\phi}=0\,, \tag{3.73}$$

which simplifies to $\frac{\partial \sigma^{rr}(R)}{\partial R} = 0$. Therefore, if $\sigma^{rr}(R_0) = \sigma_0$, then $\sigma^{rr}(R) = \sigma_0$ throughout the body. It follows that $p(R) = -\sigma_0 + 2(W_1 - W_2)$, and hence the corresponding physical components of the other two normal stresses are also equal to σ_0. Thus, the deformation map is the identity and the stress field is a uniform hydrostatic tension, i.e., $\boldsymbol{\sigma} = \sigma_0 \, \mathbf{g}^\sharp$. This example demonstrates that strength functions formulated solely in terms of strain, strain energy, or its distortional component are not, in general, adequate. It was shown in Proposition 3.3 that the safe domain is star-shaped with respect to the origin. Thus, every ray emanating from the origin intersects the strength hypersurface exactly once. In particular, along the hydrostatic tensile ray $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_0$ with $\sigma_0 > 0$, there exists a unique intersection with the strength surface.

4 Material Strength of Isotropic Solids

In this section, we specialize the general framework to isotropic solids and examine several classical strength criteria as examples, with particular attention to the geometry of the associated strength surfaces and safe domains.

For an isotropic material three principal values $(\sigma_1, \sigma_2, \sigma_3)$ of the Cauchy stress control fracture. In this case the strength hypersurface $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g}) = \hat{\mathsf{f}}(\sigma_1, \sigma_2, \sigma_3) = 0$ is a surface in a 3-manifold. Isotropy implies that we must have the following permutation symmetries

$$\hat{\mathsf{f}}(\sigma_1, \sigma_2, \sigma_3) = \hat{\mathsf{f}}(\sigma_1, \sigma_3, \sigma_2) = \hat{\mathsf{f}}(\sigma_2, \sigma_1, \sigma_3) = \hat{\mathsf{f}}(\sigma_2, \sigma_3, \sigma_1) = \hat{\mathsf{f}}(\sigma_3, \sigma_1, \sigma_2) = \hat{\mathsf{f}}(\sigma_3, \sigma_2, \sigma_1) \, . \tag{4.1}$$

Obviously, $\hat{\mathsf{f}}(0, 0, 0) < 0$. Each transposition of two principal stresses corresponds to a reflection across one of the planes $\sigma_1 = \sigma_2$, $\sigma_1 = \sigma_3$, or $\sigma_2 = \sigma_3$. Therefore, the strength hypersurface is mirror-symmetric with respect to each of these three planes. In particular, if $(\sigma_1, \sigma_2, \sigma_3)$ lies on the strength hypersurface, then its mirror image with respect to any of these planes also lies on the strength hypersurface. Consequently, the surface is completely determined by its restriction to any one of the six regions corresponding to the six possible orderings of the principal stresses, namely $\sigma_1 \geq \sigma_2 \geq \sigma_3$, $\sigma_1 \geq \sigma_3 \geq \sigma_2$, $\sigma_2 \geq \sigma_1 \geq \sigma_3$, $\sigma_2 \geq \sigma_3 \geq \sigma_1$, $\sigma_3 \geq \sigma_1 \geq \sigma_2$, and $\sigma_3 \geq \sigma_2 \geq \sigma_1$. For example, one may restrict attention to the region $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and the remaining parts of the surface then follow by symmetry.

In the case of states of plane stress, the strength hypersurface reduces to a curve defined by $\mathsf{f}(\boldsymbol{\sigma}, \mathbf{g}) = \hat{\mathsf{f}}(\sigma_1, \sigma_2) = 0$ with the symmetry $\hat{\mathsf{f}}(\sigma_1, \sigma_2) = \hat{\mathsf{f}}(\sigma_2, \sigma_1)$. Let us define the reflection map $S(\sigma_1, \sigma_2) = (\sigma_2, \sigma_1)$. The symmetry condition can be written as $\mathsf{f} = \mathsf{f} \circ S$. Therefore, if (σ_1, σ_2) belongs to the strength curve, i.e., if $\hat{\mathsf{f}}(\sigma_1, \sigma_2) = 0$, then $\mathsf{f}(S(\sigma_1, \sigma_2)) = \hat{\mathsf{f}}(\sigma_2, \sigma_1) = 0$ as well. Hence, the set $\{(\sigma_1, \sigma_2) : \hat{\mathsf{f}}(\sigma_1, \sigma_2) = 0\}$ is invariant under the reflection $(\sigma_1, \sigma_2) \mapsto (\sigma_2, \sigma_1)$, and the strength curve is mirror-symmetric with respect to the line $\sigma_1 = \sigma_2$. Consequently, the strength curve is completely determined by its restriction to either of the two regions $\sigma_1 \geq \sigma_2$ or $\sigma_2 \geq \sigma_1$. For example, one may restrict attention to the region $\sigma_1 \geq \sigma_2$, and the remaining part of the curve then follows by symmetry.

4.1 Examples of Isotropic Strength Surfaces

In this section, several classical isotropic strength criteria [Chockalingam et al., 2026] are written explicitly in terms of the principal Cauchy stresses and analyzed within the framework introduced earlier. For each criterion we first rewrite the strength surface in a form compatible with our stress convention and parameter notation. We then examine whether the corresponding safe domain satisfies the proportional reduction safety property, and hence whether the admissible stress region is star-shaped with respect to the origin.

4.1.1 Mohr–Coulomb Strength Surface

The Mohr–Coulomb strength surface depends only on the maximum and minimum principal stresses. Let us define

$$\sigma_{\max} = \max\{\sigma_1, \sigma_2, \sigma_3\} \, , \qquad \sigma_{\min} = \min\{\sigma_1, \sigma_2, \sigma_3\} \, . \tag{4.2}$$

Then the Mohr–Coulomb strength surface can be written as

$$\mathsf{f}_{\mathrm{MC}}(\sigma_1, \sigma_2, \sigma_3) = \beta_1 \, \sigma_{\max} + \beta_2 \, \sigma_{\min} - 1 = 0 \, . \tag{4.3}$$

Here β_1 and β_2 are material constants. They are determined by requiring that the strength surface pass through the uniaxial tensile and uniaxial compressive strength states. Under uniaxial tension with tensile strength $\sigma_{\mathrm{ts}}^{\mathrm{MC}}$, the principal stresses are $(\sigma_{\mathrm{ts}}^{\mathrm{MC}}, 0, 0)$, and hence $\sigma_{\max} = \sigma_{\mathrm{ts}}^{\mathrm{MC}}$ and $\sigma_{\min} = 0$. Substituting into the Mohr–Coulomb strength surface gives $\beta_1\, \sigma_{\mathrm{ts}}^{\mathrm{MC}} - 1 = 0$, and therefore

$$\beta_1 = \frac{1}{\sigma_{\mathrm{ts}}^{\mathrm{MC}}}\,. \tag{4.4}$$

Under uniaxial compression with compressive strength $\sigma_{\mathrm{cs}}^{\mathrm{MC}}$, the principal stresses are $(-\sigma_{\mathrm{cs}}^{\mathrm{MC}}, 0, 0)$, and hence $\sigma_{\max} = 0$ and $\sigma_{\min} = -\sigma_{\mathrm{cs}}^{\mathrm{MC}}$. Substituting into the Mohr–Coulomb strength surface gives $-\beta_2\, \sigma_{\mathrm{cs}}^{\mathrm{MC}} - 1 = 0$, and therefore

$$\beta_2 = -\frac{1}{\sigma_{\mathrm{cs}}^{\mathrm{MC}}}\,. \tag{4.5}$$

Thus the Mohr–Coulomb strength surface may be written as

$$\mathsf{f}_{\mathrm{MC}}(\sigma_1, \sigma_2, \sigma_3) = \frac{\sigma_{\max}}{\sigma_{\mathrm{ts}}^{\mathrm{MC}}} - \frac{\sigma_{\min}}{\sigma_{\mathrm{cs}}^{\mathrm{MC}}} - 1 = 0\,. \tag{4.6}$$

The surface is plotted in Fig. 3(a). This criterion does not depend on the intermediate principal stress.

We next show that the safe domain of the Mohr–Coulomb criterion is star-shaped with respect to the origin. To this end, let $(\sigma_1, \sigma_2, \sigma_3)$ be a safe stress state, so that $\mathsf{f}_{\mathrm{MC}}(\sigma_1, \sigma_2, \sigma_3) < 0$, and let $t \in [0, 1)$. Since $t \geq 0$, one has $(t\boldsymbol{\sigma})_{\max} = t\, \sigma_{\max}$ and $(t\boldsymbol{\sigma})_{\min} = t\, \sigma_{\min}$, and therefore

$$\mathsf{f}_{\mathrm{MC}}(t\sigma_1, t\sigma_2, t\sigma_3) = t \left(\frac{\sigma_{\max}}{\sigma_{\mathrm{ts}}^{\mathrm{MC}}} - \frac{\sigma_{\min}}{\sigma_{\mathrm{cs}}^{\mathrm{MC}}} \right) - 1\,. \tag{4.7}$$

Since $\mathsf{f}_{\mathrm{MC}}(\sigma_1, \sigma_2, \sigma_3) < 0$, one has $\frac{\sigma_{\max}}{\sigma_{\mathrm{ts}}^{\mathrm{MC}}} - \frac{\sigma_{\min}}{\sigma_{\mathrm{cs}}^{\mathrm{MC}}} < 1$. Multiplying by $t \in [0, 1)$ gives $t \left(\frac{\sigma_{\max}}{\sigma_{\mathrm{ts}}^{\mathrm{MC}}} - \frac{\sigma_{\min}}{\sigma_{\mathrm{cs}}^{\mathrm{MC}}} \right) < t < 1$, and hence $\mathsf{f}_{\mathrm{MC}}(t\sigma_1, t\sigma_2, t\sigma_3) < 0$. Thus the Mohr–Coulomb criterion satisfies the proportional reduction safety property, and consequently its safe domain is star-shaped with respect to the origin.

4.1.2 Hoek–Brown Strength Surface

The Hoek–Brown strength surface is usually written in the rock-mechanics convention in which compressive stress is taken to be positive. Thus, if $(\sigma_1, \sigma_2, \sigma_3)$ denote the principal Cauchy stresses in our convention, then one sets $s_i = -\sigma_i$. It follows that $s_{\max} = -\sigma_{\min}$ and $s_{\min} = -\sigma_{\max}$. In the notation of the paper, the generalized Hoek–Brown criterion is written as

$$s_{\max} = s_{\min} + \sigma_c \left(m_b \frac{s_{\min}}{\sigma_c} + s_{\mathrm{HB}} \right)^a\,. \tag{4.8}$$

Here σ_c, m_b, s_{HB}, and a are material parameters. In the usual Hoek–Brown model one has $0 < a \leq 1$, and therefore the expression is well defined only when $m_b \frac{s_{\min}}{\sigma_c} + s_{\mathrm{HB}} \geq 0$. For intact rock one has $m_b = m_i$, $s_{\mathrm{HB}} = 1$, and $a = 0.5$. Rewriting the criterion in terms of $(\sigma_1, \sigma_2, \sigma_3)$ gives

$$-\sigma_{\min} = -\sigma_{\max} + \sigma_c \left(-m_b \frac{\sigma_{\max}}{\sigma_c} + s_{\mathrm{HB}} \right)^a\,. \tag{4.9}$$

Hence

$$\sigma_{\max} - \sigma_{\min} = \sigma_c \left(s_{\mathrm{HB}} - m_b \frac{\sigma_{\max}}{\sigma_c} \right)^a\,. \tag{4.10}$$

Therefore, in our notation, the Hoek–Brown strength surface may be written as

$$\mathsf{f}_{\mathrm{HB}}(\sigma_1, \sigma_2, \sigma_3) = \sigma_{\max} - \sigma_{\min} - \sigma_c \left(s_{\mathrm{HB}} - m_b \frac{\sigma_{\max}}{\sigma_c} \right)^a = 0\,. \tag{4.11}$$

The surface is plotted in Fig. 3(b). Unlike the Mohr–Coulomb criterion, this relation is nonlinear in the principal stresses. It is calibrated through the parameters σ_c, m_b, s_{HB}, and a, and the corresponding uniaxial tensile and

compressive strengths are obtained by substituting the uniaxial states into the criterion, which yields nonlinear relations among these parameters. Nevertheless, like the Mohr–Coulomb criterion, the Hoek–Brown surface depends only on $\sigma_{\max}$ and $\sigma_{\min}$ and therefore ignores the intermediate principal stress.

The safe domain associated with the Hoek–Brown strength surface is also star-shaped with respect to the origin, under the usual assumption $0 < a \le 1$ and provided the fractional power remains well-defined along the proportional reduction path. To see this, let $(\sigma_1, \sigma_2, \sigma_3)$ be a safe stress state, so that $\mathsf{f}_{\mathrm{HB}}(\sigma_1, \sigma_2, \sigma_3) < 0$, and let $t \in [0, 1)$. Since $t \ge 0$, one has $(t\boldsymbol{\sigma})_{\max} = t\,\sigma_{\max}$ and $(t\boldsymbol{\sigma})_{\min} = t\,\sigma_{\min}$. Therefore

$$\mathsf{f}_{\mathrm{HB}}(t\sigma_1, t\sigma_2, t\sigma_3) = t(\sigma_{\max} - \sigma_{\min}) - \sigma_c \left(s_{\mathrm{HB}} - m_b \frac{t\,\sigma_{\max}}{\sigma_c} \right)^a . \tag{4.12}$$

Next observe that $s_{\mathrm{HB}} - m_b \frac{t\,\sigma_{\max}}{\sigma_c} = (1-t)s_{\mathrm{HB}} + t\left(s_{\mathrm{HB}} - m_b \frac{\sigma_{\max}}{\sigma_c}\right)$. Since $0 < a \le 1$, the function $f(x) = x^a$ is concave on $[0, \infty)$, and therefore for any $x_0, x_1 \ge 0$ and $t \in [0, 1]$ one has

$$f((1-t)x_0 + tx_1) \ge (1-t)f(x_0) + tf(x_1). \tag{4.13}$$

Applying this with $x_0 = s_{\mathrm{HB}}$ and $x_1 = s_{\mathrm{HB}} - m_b \frac{\sigma_{\max}}{\sigma_c}$ yields

$$\left(s_{\mathrm{HB}} - m_b \frac{t\,\sigma_{\max}}{\sigma_c} \right)^a \ge (1-t)s_{\mathrm{HB}}^a + t\left(s_{\mathrm{HB}} - m_b \frac{\sigma_{\max}}{\sigma_c} \right)^a . \tag{4.14}$$

Since $(1-t)s_{\mathrm{HB}}^a \ge 0$, it follows that

$$\left(s_{\mathrm{HB}} - m_b \frac{t\,\sigma_{\max}}{\sigma_c} \right)^a \ge t\left(s_{\mathrm{HB}} - m_b \frac{\sigma_{\max}}{\sigma_c} \right)^a . \tag{4.15}$$

Substituting this estimate into $\mathsf{f}_{\mathrm{HB}}(t\sigma_1, t\sigma_2, t\sigma_3)$ gives

$$\begin{aligned} \mathsf{f}_{\mathrm{HB}}(t\sigma_1, t\sigma_2, t\sigma_3) &= t(\sigma_{\max} - \sigma_{\min}) - \sigma_c \left(s_{\mathrm{HB}} - m_b \frac{t\,\sigma_{\max}}{\sigma_c} \right)^a \\ &\le t(\sigma_{\max} - \sigma_{\min}) - t\,\sigma_c \left(s_{\mathrm{HB}} - m_b \frac{\sigma_{\max}}{\sigma_c} \right)^a \\ &= t\left[\sigma_{\max} - \sigma_{\min} - \sigma_c \left(s_{\mathrm{HB}} - m_b \frac{\sigma_{\max}}{\sigma_c} \right)^a \right] = t\,\mathsf{f}_{\mathrm{HB}}(\sigma_1, \sigma_2, \sigma_3) < 0 . \end{aligned} \tag{4.16}$$

Thus, the Hoek–Brown criterion satisfies the proportional reduction safety property, and consequently its safe domain is star-shaped with respect to the origin.

4.1.3 Generalized Three-Dimensional Hoek–Brown Strength Surface

The generalized three-dimensional Hoek–Brown surface is written in terms of τ_{oct} and $s_{m,2}$, where

$$\tau_{\mathrm{oct}} = \sqrt{\frac{2}{3} J_2}\,, \qquad J_2 = \frac{1}{6}\Big[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \Big], \tag{4.17}$$

and

$$s_{m,2} = \frac{1}{2}(s_{\max} + s_{\min}) = -\frac{1}{2}(\sigma_{\max} + \sigma_{\min}) . \tag{4.18}$$

Thus

$$\tau_{\mathrm{oct}} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} . \tag{4.19}$$

The generalized three-dimensional Hoek–Brown surface is described as

$$\mathsf{f}_{\mathrm{GHB}}(\sigma_1, \sigma_2, \sigma_3) = \beta_1 \big(3\sqrt{2}\,\tau_{\mathrm{oct}}\big)^{\frac{1}{a}} + \beta_2 \left(\frac{3}{2}\sqrt{2}\,\tau_{\mathrm{oct}} - s_{m,2} \right) - 1 = 0 . \tag{4.20}$$

The surface is plotted in Fig. 3(c). Using $s_{m,2} = -(\sigma_{\max} + \sigma_{\min})/2$, this becomes

$$\mathsf{f}_{\mathrm{GHB}}(\sigma_1, \sigma_2, \sigma_3) = \beta_1 \big(3\sqrt{2}\,\tau_{\mathrm{oct}}\big)^{\frac{1}{a}} + \beta_2 \left(\frac{3}{2}\sqrt{2}\,\tau_{\mathrm{oct}} + \frac{1}{2}(\sigma_{\max} + \sigma_{\min})\right) - 1 = 0\,. \tag{4.21}$$

The material constants are written as

$$\beta_1 = \frac{1}{\big(\sigma_{\mathrm{cs}}^{\mathrm{HB}}\big)^{\frac{1}{a}}}\,, \qquad \beta_2 = \frac{1}{\sigma_{\mathrm{ts}}^{\mathrm{HB}}}\left[1 - \left(\frac{\sigma_{\mathrm{ts}}^{\mathrm{HB}}}{\sigma_{\mathrm{cs}}^{\mathrm{HB}}}\right)^{\frac{1}{a}}\right]. \tag{4.22}$$

Unlike the preceding two criteria, this one depends on all three principal stresses through J_2, and therefore includes the effect of the intermediate principal stress.

The safe domain associated with the generalized three-dimensional Hoek–Brown surface is also star-shaped with respect to the origin, provided $\beta_1 \geq 0$, $\beta_2 \geq 0$, and $0 < a \leq 1$. To see this, let $(\sigma_1, \sigma_2, \sigma_3)$ be a safe stress state, so that $\mathsf{f}_{\mathrm{GHB}}(\sigma_1, \sigma_2, \sigma_3) < 0$, and let $t \in [0, 1)$. Since τ_{oct} is homogeneous of degree one in the principal stresses, one has $\tau_{\mathrm{oct}}(t\sigma_1, t\sigma_2, t\sigma_3) = t\,\tau_{\mathrm{oct}}$, while $(t\boldsymbol{\sigma})_{\max} = t\,\sigma_{\max}$ and $(t\boldsymbol{\sigma})_{\min} = t\,\sigma_{\min}$. Therefore

$$\mathsf{f}_{\mathrm{GHB}}(t\sigma_1, t\sigma_2, t\sigma_3) = t^{\frac{1}{a}}\beta_1 \left(3\sqrt{2}\,\tau_{\mathrm{oct}}\right)^{\frac{1}{a}} + t\beta_2 \left[\frac{3}{2}\sqrt{2}\,\tau_{\mathrm{oct}} + \frac{1}{2}(\sigma_{\max} + \sigma_{\min})\right] - 1\,. \tag{4.23}$$

Since $0 < a \leq 1$, one has $\frac{1}{a} \geq 1$, and hence $t^{\frac{1}{a}} \leq t \leq 1$ for every $t \in [0, 1)$. Using also $\beta_1 \geq 0$ and $\beta_2 \geq 0$, it follows that

$$\begin{aligned}
\mathsf{f}_{\mathrm{GHB}}(t\sigma_1, t\sigma_2, t\sigma_3) &\leq t\beta_1 \left(3\sqrt{2}\,\tau_{\mathrm{oct}}\right)^{\frac{1}{a}} + t\beta_2 \left[\frac{3}{2}\sqrt{2}\,\tau_{\mathrm{oct}} + \frac{1}{2}(\sigma_{\max} + \sigma_{\min})\right] - 1 \\
&= t\left\{\beta_1 \left(3\sqrt{2}\,\tau_{\mathrm{oct}}\right)^{\frac{1}{a}} + \beta_2 \left[\frac{3}{2}\sqrt{2}\,\tau_{\mathrm{oct}} + \frac{1}{2}(\sigma_{\max} + \sigma_{\min})\right]\right\} - 1 \\
&< \beta_1 \left(3\sqrt{2}\,\tau_{\mathrm{oct}}\right)^{\frac{1}{a}} + \beta_2 \left[\frac{3}{2}\sqrt{2}\,\tau_{\mathrm{oct}} + \frac{1}{2}(\sigma_{\max} + \sigma_{\min})\right] - 1 = \mathsf{f}_{\mathrm{GHB}}(\sigma_1, \sigma_2, \sigma_3) < 0\,.
\end{aligned} \tag{4.24}$$

Thus, the generalized three-dimensional Hoek–Brown criterion satisfies the proportional reduction safety property, and consequently its safe domain is star-shaped with respect to the origin.

4.1.4 Mogi–Coulomb Strength Surface

The Mogi–Coulomb surface is written in the paper as

$$\mathsf{f}_{\mathrm{MgC}}(\sigma_1, \sigma_2, \sigma_3) = \beta_1\, s_{m,2} + \beta_2\, \tau_{\mathrm{oct}} - 1 = 0\,. \tag{4.25}$$

Using $s_{m,2} = -\frac{1}{2}(\sigma_{\max} + \sigma_{\min})$, this becomes

$$\mathsf{f}_{\mathrm{MgC}}(\sigma_1, \sigma_2, \sigma_3) = -\frac{\beta_1}{2}(\sigma_{\max} + \sigma_{\min}) + \beta_2\, \tau_{\mathrm{oct}} - 1 = 0\,. \tag{4.26}$$

The parameters are given by

$$\beta_1 = \frac{1}{\sigma_{\mathrm{cs}}^{\mathrm{MC}}} - \frac{1}{\sigma_{\mathrm{ts}}^{\mathrm{MC}}}\,, \qquad \beta_2 = \frac{3\sqrt{2}}{4}\left(\frac{1}{\sigma_{\mathrm{ts}}^{\mathrm{MC}}} + \frac{1}{\sigma_{\mathrm{cs}}^{\mathrm{MC}}}\right). \tag{4.27}$$

Therefore

$$\mathsf{f}_{\mathrm{MgC}}(\sigma_1, \sigma_2, \sigma_3) = -\frac{1}{2}\left(\frac{1}{\sigma_{\mathrm{cs}}^{\mathrm{MC}}} - \frac{1}{\sigma_{\mathrm{ts}}^{\mathrm{MC}}}\right)(\sigma_{\max} + \sigma_{\min}) + \frac{3\sqrt{2}}{4}\left(\frac{1}{\sigma_{\mathrm{ts}}^{\mathrm{MC}}} + \frac{1}{\sigma_{\mathrm{cs}}^{\mathrm{MC}}}\right)\tau_{\mathrm{oct}} - 1 = 0\,. \tag{4.28}$$

The surface is plotted in Fig. 3(d). This criterion depends on all three principal stresses through τ_{oct} and hence accounts for the intermediate principal stress.

The safe domain associated with the Mogi–Coulomb strength surface is star-shaped with respect to the origin. To see this, let $(\sigma_1, \sigma_2, \sigma_3)$ be a safe stress state, so that $\mathsf{f}_{\mathrm{MgC}}(\sigma_1, \sigma_2, \sigma_3) < 0$, and let $t \in [0, 1)$. Since τ_{oct}

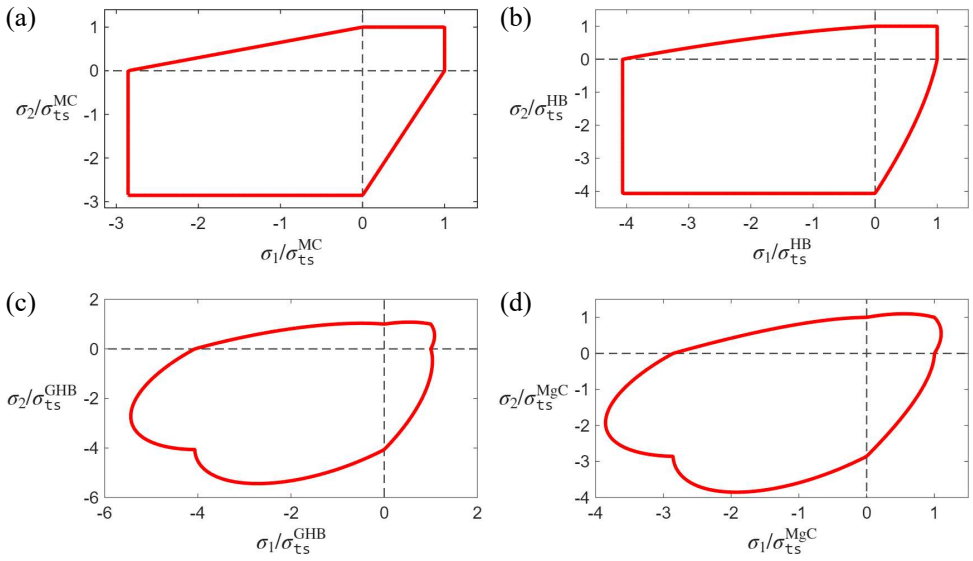

Figure 3: *Plot of various isotropic strength surfaces in principal stress space under plane stress conditions. (a) Mohr-Coulomb surface, (b) Hoek-Brown surface, (c) 3D Hoek-Brown or GZZ surface, and (d) Mogi-Coulomb surface.*

is homogeneous of degree one in the principal stresses, one has $\tau_{\rm oct}(t\sigma_1, t\sigma_2, t\sigma_3) = t\,\tau_{\rm oct}$, while $(t\boldsymbol{\sigma})_{\max} = t\,\sigma_{\max}$ and $(t\boldsymbol{\sigma})_{\min} = t\,\sigma_{\min}$. Therefore

$$\mathsf{f}_{\rm MgC}(t\sigma_1, t\sigma_2, t\sigma_3) = t\left[-\frac{\beta_1}{2}(\sigma_{\max}+\sigma_{\min}) + \beta_2\,\tau_{\rm oct}\right] - 1\,. \tag{4.29}$$

Since $\mathsf{f}_{\rm MgC}(\sigma_1,\sigma_2,\sigma_3) < 0$, one has $-\frac{\beta_1}{2}(\sigma_{\max}+\sigma_{\min})+\beta_2\,\tau_{\rm oct} < 1$. Multiplying by $t \in [0,1)$ gives $t\big[-\frac{\beta_1}{2}(\sigma_{\max}+\sigma_{\min}) + \beta_2\,\tau_{\rm oct}\big] < t < 1$, and hence, $\mathsf{f}_{\rm MgC}(t\sigma_1, t\sigma_2, t\sigma_3) < 0$. Thus, the Mogi–Coulomb criterion satisfies the proportional reduction safety property, and consequently its safe domain is star-shaped with respect to the origin.

Fig. 3 compares the four isotropic strength surfaces discussed in this section in principal stress space under plane stress conditions. Although all four criteria are isotropic, their geometric structures are markedly different, reflecting different assumptions about the dependence of strength on the principal stresses. All four surfaces are star-shaped with respect to the origin. In addition, the surfaces in panels (a) and (b) are convex, whereas those in panels (c) and (d) are not.

5 Material Strength in the Presence of Residual Stresses

We next consider solids with residual stresses and anelastic distortions, and examine how the definition and representation of material strength are modified in anelasticity. Anelasticity, in the sense of Eckart [1948], is characterized by the presence of internal distortions that define a local relaxed configuration, so that the elastic response is measured relative to this evolving stress-free state rather than a global Euclidean reference configuration. One may ask whether material strength can be affected by residual stresses. It should be emphasized that, in the present framework, material strength is a local material property associated with a homogeneous body under boundary tractions and in a spatially uniform state of stress. In a traction-free body with no body forces, a spatially uniform residual stress must be trivial, i.e., it must vanish [Hoger, 1985]. In this sense, residual stress does not enter the definition of material strength at a given point. Rather, material strength is defined with respect to a stress-free state, namely for a homogeneous body in its relaxed state before a homogeneous state of stress is imposed that induces fracture.

5.1 Material metric

Suppose a body $\mathcal{B}$ has a distribution of eigenstrains.[20] Let us consider a material point $X \in \mathcal{B}$ that is mapped to $x = \varphi(X)$ in the current configuration $\mathcal{C}$. An infinitesimal line element $\mathbf{U} \in T_X\mathcal{B}$ in the reference configuration is mapped to $\mathbf{FU}$ after deformation. Now imagine that independently of the rest of the body the deformed line element is allowed to elastically relax and denote it by $\overset{e}{\mathbf{F}}{}^{-1}\mathbf{FU}$, where $\overset{e}{\mathbf{F}}{}^{-1} : T_x\mathcal{C} \to T_X\mathcal{B}$ is a local elastic unloading map. The resulting line element is different from that in the reference configuration. The two are related as $\overset{e}{\mathbf{F}}{}^{-1}\mathbf{FU} = \overset{a}{\mathbf{F}}\mathbf{U}$, where $\overset{a}{\mathbf{F}}(X) : T_X\mathcal{B} \to T_X\mathcal{B}$ is the anelastic distortion. As $\mathbf{U}$ is an arbitrary vector, one obtains $\mathbf{F} = \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}$—the Biblby-Kröner-Lee multiplicative decomposition of deformation gradient [Bilby et al., 1957; Kröner, 1959; Lee and Liu, 1967; Lee, 1969], see also [Sadik and Yavari, 2017; Yavari and Sozio, 2023]. Let us denote the induced flat metric of the reference configuration in the absence of eigenstrains by $\mathring{\mathbf{G}}$. The square relaxed line element length is calculated as follows

$$\langle\!\langle \overset{a}{\mathbf{F}}\mathbf{U}, \overset{a}{\mathbf{F}}\mathbf{U} \rangle\!\rangle_{\mathring{\mathbf{G}}} = \langle\!\langle \mathbf{U}, \mathbf{U} \rangle\!\rangle_{\overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{G}}} = \langle\!\langle \mathbf{U}, \mathbf{U} \rangle\!\rangle_{\mathbf{G}}\,, \tag{5.1}$$

where $\mathbf{G} = \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{G}} = \overset{a}{\mathbf{F}}{}^{\star}\mathring{\mathbf{G}}\overset{a}{\mathbf{F}}$ is the material metric [Yavari and Goriely, 2013; Yavari, 2021].

5.2 Elastic energy function in anelasticity

Let us consider an isotropic hyperelastic solid with energy function W, which explicitly depends on the elastic distortion: $W = W(X, \overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathbf{g})$. For an isotropic solid the energy function is materially covariant [Marsden and Hughes, 1983], and hence,

$$W(X, \overset{e}{\mathbf{F}}, \mathring{\mathbf{G}}, \mathbf{g}) = W(X, \overset{a}{\mathbf{F}}{}^{*}\overset{e}{\mathbf{F}}, \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{G}}, \mathbf{g}) = W(X, \overset{e}{\mathbf{F}}\overset{a}{\mathbf{F}}, \mathbf{G}, \mathbf{g}) = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g})\,. \tag{5.2}$$

Therefore, in the presence of eigenstrains, one can work with the total strain as long as the material metric is used instead of the induced flat Euclidean metric. This can be generalized to anisotropic solids as well [Yavari and Sozio, 2023].

5.3 Material strength function of an anelastic brittle solid

In defining material strength, one considers a thought experiment in which a small material neighborhood is isolated and allowed to relax to a stress-free configuration. The specimen is then loaded starting from this relaxed state until fracture occurs. Thus, the notion of strength is tied to the relaxed (or intermediate) configuration rather than to the current configuration viewed through the total deformation. In particular, the relevant kinematic quantity is the elastic distortion $\overset{e}{\mathbf{F}}$, and the corresponding energetically conjugate stress is

$$\mathring{\mathbf{P}} = \mathbf{g}^{\sharp}\,\frac{\partial W}{\partial \overset{e}{\mathbf{F}}}\,. \tag{5.3}$$

Consequently, the stress entering the strength function is not the first Piola–Kirchhoff stress $\mathbf{P}$ associated with the total deformation $\mathbf{F}$, but rather $\mathring{\mathbf{P}}$, which is defined with respect to the intermediate configuration. Therefore, the strength function is written in the following form

$$\mathsf{F}(X, \mathring{\mathbf{P}}, \overset{e}{\mathbf{F}}, \mathbf{g}, \mathring{\mathbf{G}}) = 0\,. \tag{5.4}$$

The explicit dependence on X emphasizes that, in general, the material may be inhomogeneous.

For an isotropic material the strength function is materially covariant, i.e.,

$$\mathsf{F}(X, \mathring{\mathbf{P}}, \overset{e}{\mathbf{F}}, \mathbf{g}, \mathring{\mathbf{G}}) = \mathsf{F}(X, \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{P}}, \overset{a}{\mathbf{F}}{}^{*}\overset{e}{\mathbf{F}}, \mathbf{g}, \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{G}}) = \mathsf{F}(X, \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{P}}, \mathbf{F}, \mathbf{g}, \mathbf{G})\,. \tag{5.5}$$

[20] Reissner [1931] formulated a theory of *Eigenspannungen* (eigenstresses) generated by incompatible initial distortions acting as internal stress sources. In modern terminology, these initial distortions may be interpreted as eigenstrains. The term *eigenstrain* was later popularized by Mura [Kinoshita and Mura, 1971; Mura, 1982]. Closely related notions appear in the literature under a variety of names, including *initial strain* [Kondo, 1949], *nuclei of strain* [Mindlin and Cheng, 1950], *transformation strain* [Eshelby, 1957], *inherent strain* [Ueda et al., 1975], and *residual strain* [Ambrosi et al., 2019]; see also [Jun and Korsunsky, 2010; Zhou et al., 2013].

From (5.2), one can write

$$\frac{\partial W}{\partial \overset{e}{\mathbf{F}}} = \frac{\partial W}{\partial \mathbf{F}} \frac{\partial \mathbf{F}}{\partial \overset{e}{\mathbf{F}}} = \frac{\partial W}{\partial \mathbf{F}} \overset{a}{\mathbf{F}}{}^{\star} . \tag{5.6}$$

We know that

$$\mathring{\mathbf{P}} = \mathbf{g}^{\sharp} \frac{\partial W}{\partial \overset{e}{\mathbf{F}}} , \qquad \mathbf{P} = \mathbf{g}^{\sharp} \frac{\partial W}{\partial \mathbf{F}} . \tag{5.7}$$

Hence, $\mathring{\mathbf{P}} = \mathbf{P}\, \overset{a}{\mathbf{F}}{}^{\star}$. Note that $\mathring{\mathbf{P}}(X) \in T_{\varphi(X)}\mathcal{S} \otimes T_X\mathcal{B}$, and therefore, under the pull-back by $\overset{a}{\mathbf{F}} : T_X\mathcal{B} \to T_X\mathcal{B}$, the material slot transforms with the inverse map. In coordinate-independent form, one writes $\overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{P}} = (\mathrm{id}_{T_{\varphi(X)}\mathcal{S}} \otimes \overset{a}{\mathbf{F}}{}^{-1})\, \mathring{\mathbf{P}} = \mathring{\mathbf{P}} \overset{a}{\mathbf{F}}{}^{-\star}$. In components, this reads $(\overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{P}})^{aA} = \mathring{P}^{aB}\, \overset{a}{F}{}^{-A}{}_{B}$. Thus, we have

$$\overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{P}} = \mathbf{P} . \tag{5.8}$$

Therefore,

$$\mathsf{F}(X, \mathring{\mathbf{P}}, \overset{e}{\mathbf{F}}, \mathbf{g}, \mathring{\mathbf{G}}) = \mathsf{F}(X, \mathbf{P}, \mathbf{F}, \mathbf{g}, \mathbf{G}) . \tag{5.9}$$

Thus, similar to the elastic energy function, in the presence of eigenstrains the strength function may be expressed in terms of the total deformation and the corresponding first Piola–Kirchhoff stress, provided that the material metric $\mathbf{G}$ is used in place of the Euclidean reference metric $\mathring{\mathbf{G}}$.

Example 5.1 (Drucker–Prager strength function for a residually-stressed solid)**.** The Drucker–Prager strength function has the form given in (3.30). More specifically, we have $\mathcal{F}(X, \overset{(1)}{\mathbf{S}}, \mathbf{G}) = 0$, where $\mathbf{G}$ is the material metric. This strength function is written in terms of the first invariant of the Biot stress and the second invariant of its deviatoric part. More precisely, $\mathcal{I}_1 = \mathrm{tr}_{\mathbf{G}}\, \overset{(1)}{\mathbf{S}}$ is the trace of the Biot stress, while $\mathcal{J}_2$ is the second invariant of the deviatoric Biot stress. Thus,

$$\mathcal{I}_1 = \mathrm{tr}_{\mathbf{G}}\, \overset{(1)}{\mathbf{S}} = \overset{(1)}{S}{}^{AB} G_{AB} . \tag{5.10}$$

The deviatoric Biot stress is defined by

$$\mathrm{dev}_{\mathbf{G}}\, \overset{(1)}{\mathbf{S}} = \overset{(1)}{\mathbf{S}} - \frac{1}{3}\mathcal{I}_1\, \mathbf{G}^{\sharp} , \qquad (\mathrm{dev}_{\mathbf{G}}\, \overset{(1)}{\mathbf{S}})^{AB} = \overset{(1)}{S}{}^{AB} - \frac{1}{3}\mathcal{I}_1 G^{AB} . \tag{5.11}$$

Therefore,

$$\mathcal{J}_2 = \frac{1}{2}\, \mathrm{dev}_{\mathbf{G}}\, \overset{(1)}{\mathbf{S}} : \mathrm{dev}_{\mathbf{G}}\, \overset{(1)}{\mathbf{S}} = \frac{1}{2}\, (\mathrm{dev}_{\mathbf{G}}\, \overset{(1)}{\mathbf{S}})^{AB} (\mathrm{dev}_{\mathbf{G}}\, \overset{(1)}{\mathbf{S}})^{MN} G_{AM} G_{BN} . \tag{5.12}$$

It is seen that the two invariants explicitly depend on the material metric, and hence, eigenstrains. As an explicit example of a residually-stressed solid, consider a spherical ball with a radially-symmetric distribution of pure dilatational eigenstrain for which in spherical coordinates the material metric has the following representation [Yavari, 2021]

$$\mathbf{G} = \mathbf{G}(R) = e^{2\omega(R)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \sin^2\Theta \end{bmatrix} , \tag{5.13}$$

where $\omega(R)$ is some function that quantifies the eigenstrain. Thus, both $\mathcal{I}_1$ and $\mathcal{J}_2$ depend explicitly on the function $\omega(R)$ through the components of the material metric and its inverse. Therefore, the Drucker–Prager strength function of a residually-stressed solid depends explicitly on the underlying eigenstrain distribution.

6 Material Strength of Anisotropic Solids

In this section, we develop the invariant-theoretic representation of strength functions for anisotropic solids and analyze the role of material symmetry in determining their admissible forms.

6.1 A brief history of strength surfaces for anisotropic solids

The modern foundations of failure and similarly motivated yield theories were laid in the classical works of von Mises [1913, 1928], in which invariant-based criteria for isotropic materials were introduced. Hill [1948] proposed a macroscopic yield criterion for anisotropic metals with three mutually orthogonal planes of symmetry, i.e., orthotropic solids, by introducing a homogeneous quadratic plastic potential with six anisotropy parameters, together with associated stress–strain-increment relations. He applied the theory to problems such as rolled sheets, torsion of thin-walled cylinders, and deep drawing, and showed that the model captured experimentally observed directional effects. This work is historically relevant as an early formulation of anisotropic yield surfaces using quadratic forms, although it pertained to plastic yielding rather than brittle material strength.

Building on these earlier works on anisotropic yielding, a number of phenomenological strength and failure criteria were proposed for brittle solids and composites. Marin [1957] proposed an early phenomenological extension of strength theories to anisotropic materials, allowing for different tensile and compressive strengths and formulating a criterion for combined stress states. His work already pointed to the empirical character of anisotropic strength criteria and to the need for additional information beyond simple uniaxial tests. Gol'Denblat and Kopnov [1965] proposed a general tensorial strength criterion for anisotropic materials, especially glass-reinforced plastics, in which the failure function is written in terms of tensorial strength coefficients and stress components. The formulation is constructed so that the strength function is a scalar and therefore respects material frame indifference under changes of coordinates. They also showed that several earlier criteria arise as special cases of their general theory and compared its predictions with experiments under combined loading. Azzi and Tsai [1965] adapted Hill's quadratic anisotropic yield criterion to composites and showed how it could be used to predict failure when the loading is not aligned with the principal material directions, while also allowing for different tensile and compressive strengths in different quadrants. Hoffman [1967] proposed a phenomenological failure criterion for orthotropic brittle materials, motivated by analogy with anisotropic yield criteria, especially Hill's quadratic form, while emphasizing that brittle fracture is fundamentally different from yielding. His criterion introduced distinct tensile, compressive, and shear strength parameters in the material symmetry directions and, for plane stress in a unidirectional composite, led to a fracture surface that he compared with available experimental data.

Tsai and Wu [1971] proposed a general polynomial failure criterion for anisotropic materials in which the strength surface is expressed as a quadratic function of the stress components, combining linear and quadratic terms to capture different failure modes. They imposed restrictions on the coefficients of this quadratic form so that the resulting strength surface is bounded and physically admissible. Their formulation unifies and generalizes several existing failure criteria and provides a systematic framework for incorporating material anisotropy through experimentally determined strength parameters. Cowin [1986] incorporated anisotropy into a strength criterion by expressing the strength coefficients in terms of a second-order fabric tensor that characterizes the underlying material microstructure. For historical surveys of macroscopic failure criteria for anisotropic and composite materials, including maximum-stress, tensor-polynomial, and quadratic criteria, see Franklin [1968], Tsai [1984], and Fan [1987]; for a broader review of strength theories for materials under complex stress states, see also Yu [2002].

6.2 Invariant representation of strength function

Constitutive equations of nonlinear elasticity and anelasticity have been developed systematically using invariant theory. Invariant theory studies scalar and tensorial quantities that remain unchanged under the action of a symmetry group. In continuum mechanics, it provides the mathematical foundation for writing constitutive equations in a form consistent with material symmetry. For isotropic solids, this explains why constitutive equations may be written in terms of scalar invariants of tensors such as $\mathbf{C}$ or $\mathbf{b}$ rather than their individual components. The algebraic foundations of the subject go back to Hilbert [1890], whose finiteness results showed that invariant algebras admit finite generating sets. Weyl [1939] later reformulated classical invariant theory in representation-theoretic terms, while Spencer [1972, 1982] systematically adapted these ideas to continuum mechanics and nonlinear elasticity, where structural tensors and integrity bases became standard tools.

In elasticity, the practical consequence of invariant theory is the representation theorem viewpoint: once the material symmetry group is specified, invariant theory identifies the scalar invariants that may appear in the elastic energy function or in constitutive response functions, and it also determines the corresponding tensorial representations. This was briefly discussed in §2.4.

The fracture properties of a solid are ultimately determined by the distribution of defects and microstructural inhomogeneities in the material. In many situations, particularly in brittle solids, these defects are distributed in a manner that reflects the same material symmetries that govern the elastic response. Consequently, the macroscopic strength of the material is expected to inherit the same symmetry group as the underlying elastic constitutive equations.[21] Similar to elastic properties and their symmetries, the symmetry of material strength is therefore inherently a material notion and must be defined on the material manifold. In particular, the action of the material symmetry group is naturally expressed in terms of material stress and strain measures. Hence, in order to characterize the effect of symmetry on strength, one must start with a material description in terms of $(\mathbf{S}, \mathbf{C}^\flat)$ or $(\mathbf{P}, \mathbf{F})$. In view of (2.39), it is then natural to require that the strength function be invariant under the same material symmetry group. Thus, if $\mathcal{G}_X$ is the material symmetry group at $X \in \mathcal{B}$, then one must have

$$\hat{\mathsf{F}}(\mathbf{K}^*\mathbf{S}, \mathbf{K}^*\mathbf{C}^\flat, \mathbf{G}) = \hat{\mathsf{F}}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G}) \,, \qquad \forall\, \mathbf{K} \in \mathcal{G}_X \leqslant \mathrm{Orth}(\mathbf{G}) \,. \tag{6.1}$$

Equation (6.1) is the statement that the strength function has the same material symmetry group as the elastic energy function. In other words, the fracture criterion inherits the same material symmetries as the elastic response.

The material symmetry group can be characterized by a finite collection of structural tensors $\boldsymbol{\zeta}_i$ of order μ_i, $i = 1, \dots, N$ [Liu, 1982; Boehler, 1987; Zheng and Spencer, 1993; Zheng, 1994; Lu and Papadopoulos, 2000; Mazzucato and Rachele, 2006]. More precisely,

$$\mathbf{Q} \in \mathcal{G}_X \leqslant \mathrm{Orth}(\mathbf{G}) \iff \langle \mathbf{Q} \rangle_{\mu_1} \boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_1 \,, \dots \,, \langle \mathbf{Q} \rangle_{\mu_N} \boldsymbol{\zeta}_N = \boldsymbol{\zeta}_N \,. \tag{6.2}$$

Thus, the material symmetry group $\mathcal{G}_X$ is precisely the invariance group of the structural tensors $\boldsymbol{\zeta}_i$, $i = 1, \dots, N$. In this sense, the structural tensors encode the material symmetry relevant to strength.

For a $\mathbf{G}$-orthogonal transformation $\mathbf{Q}$ and a tensor $\boldsymbol{\zeta}$ of order μ, the μ-th Kronecker power $\langle \mathbf{Q} \rangle_\mu$ is defined by

$$(\langle \mathbf{Q} \rangle_\mu \boldsymbol{\zeta})^{\bar{A}_1 \dots \bar{A}_\mu} = Q^{\bar{A}_1}{}_{A_1} \dots Q^{\bar{A}_\mu}{}_{A_\mu} \, \zeta^{A_1 \dots A_\mu} \,. \tag{6.3}$$

In particular, for arbitrary vectors $\mathbf{V}_i \in T_X\mathcal{B}$, $i = 1, \dots, m$, one has

$$\langle \mathbf{Q} \rangle_m \, (\mathbf{V}_1 \otimes \dots \otimes \mathbf{V}_m) = \mathbf{Q}\mathbf{V}_1 \otimes \dots \otimes \mathbf{Q}\mathbf{V}_m \,. \tag{6.4}$$

Accordingly, one may write the strength function in the form

$$\mathsf{F} = \widehat{\mathsf{F}}(X, \mathbf{S}, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_N) \,. \tag{6.5}$$

Denoting the set of structural tensors by $\boldsymbol{\Lambda} = \{\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_N\}$, one simply writes $\mathsf{F} = \widehat{\mathsf{F}}(X, \mathbf{S}, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\Lambda})$. When the structural tensors are included among its arguments, the strength function becomes an isotropic scalar-valued function of its arguments, in the same sense as in the classical principle of isotropy of space [Boehler, 1979].

Instead of working directly with the tensors $\{\mathbf{S}, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_N\}$, one may equivalently use a corresponding set of isotropic invariants. By Hilbert's theorem, any finite collection of tensors admits a finite integrity basis for the algebra of isotropic invariants [Spencer, 1971]. Denoting such an integrity basis by I_j, $j = 1, \dots, m$, one may therefore write

$$\mathsf{F} = \mathsf{F}(X, I_1, \dots, I_m) \,. \tag{6.6}$$

Example 6.1. For a transversely isotropic solid, the material symmetry group is characterized by a single material preferred direction at each material point. Let $\mathbf{N}(X) \in T_X\mathcal{B}$ be a unit vector normal to the plane of isotropy, i.e., $\langle\!\langle \mathbf{N}, \mathbf{N} \rangle\!\rangle_{\mathbf{G}} = 1$. In this case, a single structural tensor is sufficient to characterize the symmetry. It should be emphasized, however, that the choice of structural tensor is not unique. One convenient choice is

$$\boldsymbol{\zeta} = \mathbf{N} \otimes \mathbf{N} \,. \tag{6.7}$$

Accordingly, the strength function may be written as $\mathsf{F} = \widehat{\mathsf{F}}(X, \mathbf{S}, \mathbf{C}^\flat, \mathbf{G}, \mathbf{N} \otimes \mathbf{N})$. In this case, the integrity basis consists of five invariants, namely the three principal invariants of $\mathbf{C}$ together with the additional invariants $I_4 = \mathbf{C}^\flat(\mathbf{N}, \mathbf{N})$ and $I_5 = (\mathbf{C}^\flat \mathbf{G}^\sharp \mathbf{C}^\flat)(\mathbf{N}, \mathbf{N})$.

[21] Hoffman [1967], in his discussion of orthotropic brittle materials, observed that it is plausible to assume that the planes of elastic symmetry and the planes of strength symmetry coincide, but he also emphasized that this assumption should be made with caution. He noted that elastic properties are associated with an averaging of the underlying microstructure, whereas fracture strength is generally more sensitive to microstructural inhomogeneities and defects. This suggests that the symmetry group of strength may, in general, be a proper subgroup of the symmetry group of elasticity. In the present work, however, we assume for simplicity that the two symmetry groups coincide.

6.2.1 Isotropic solids

First, let us consider isotropic solids. An isotropic function of two symmetric second-order tensors can be expressed in terms of the following ten invariants [Rivlin and Ericksen, 1955]:

$$\operatorname{tr}\mathbf{S}\,,\quad \operatorname{tr}\mathbf{S}^2\,,\quad \operatorname{tr}\mathbf{S}^3\,,\quad \operatorname{tr}\mathbf{C}\,,\quad \operatorname{tr}\mathbf{C}^2\,,\quad \operatorname{tr}\mathbf{C}^3\,,\quad \operatorname{tr}(\mathbf{SC})\,,\quad \operatorname{tr}\left(\mathbf{SC}^2\right)\,,\quad \operatorname{tr}\left(\mathbf{S}^2\mathbf{C}\right)\,,\quad \operatorname{tr}\left(\mathbf{S}^2\mathbf{C}^2\right)\,. \tag{6.8}$$

Equivalently, we can use the following invariants[22]

$$\begin{aligned}&\mathcal{I}_1=\operatorname{tr}\mathbf{S}\,,\quad \mathcal{I}_2=\operatorname{tr}\mathbf{S}^2\,,\quad \mathcal{I}_3=\operatorname{tr}\mathbf{S}^3\,,\quad \mathcal{I}_4=\operatorname{tr}\mathbf{C}\,,\quad \mathcal{I}_5=\frac{1}{2}\left[\mathcal{I}_4^2-\operatorname{tr}\mathbf{C}^2\right]\,,\quad \mathcal{I}_6=\det\mathbf{C}\,,\\ &\mathcal{I}_7=\operatorname{tr}(\mathbf{SC})\,,\quad \mathcal{I}_8=\operatorname{tr}\left(\mathbf{SC}^2\right)\,,\quad \mathcal{I}_9=\operatorname{tr}\left(\mathbf{S}^2\mathbf{C}\right)\,,\quad \mathcal{I}_{10}=\operatorname{tr}\left(\mathbf{S}^2\mathbf{C}^2\right)\,.\end{aligned} \tag{6.9}$$

Therefore, the material strength function has the following functional form: $\hat{\mathsf{F}}(\mathbf{S},\mathbf{C}^\flat,\mathbf{G})=\bar{\mathsf{F}}(\mathcal{I}_1,\cdots,\mathcal{I}_{10})$.

Even for isotropic solids, once the strength function is allowed to depend on both stress and strain, the corresponding integrity basis becomes rather large. For this reason, and also in preparation for the anisotropic case where the number of invariants increases further, we restrict attention here to the special case in which the strength function depends only on stress. Thus, for isotropic solids the strength function admits the representation $\hat{\mathsf{F}}(\mathbf{S},\mathbf{G})=\bar{\mathsf{F}}(\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3)$.

6.2.2 Transversely isotropic solids

Recall that a transversely isotropic solid has, at each material point, a single preferred direction orthogonal to the plane of isotropy. This preferred direction at $X\in\mathcal{B}$ is denoted as $\mathbf{N}(X)$. In addition to $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$, we have the following two extra invariants

$$\mathcal{I}_4=\mathbf{N}\cdot\mathbf{S}\cdot\mathbf{N}\,,\qquad \mathcal{I}_5=\mathbf{N}\cdot\mathbf{S}^2\cdot\mathbf{N}\,. \tag{6.10}$$

Therefore, for a transversely isotropic solids the strength function has the following invariant representation

$$\hat{\mathsf{F}}=\bar{\mathsf{F}}(\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3,\mathcal{I}_4,\mathcal{I}_5)\,. \tag{6.11}$$

6.2.3 Orthotropic solids

An orthotropic solid is characterized by symmetry with respect to three mutually orthogonal reflection planes. Let $\mathbf{N}_1(X)$, $\mathbf{N}_2(X)$, and $\mathbf{N}_3(X)$ denote a $\mathbf{G}$-orthonormal triad that defines the corresponding principal material directions at X. Beyond $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$, four additional invariants arise:

$$\mathcal{I}_4=\mathbf{N}_1\cdot\mathbf{S}\cdot\mathbf{N}_1\,,\qquad \mathcal{I}_5=\mathbf{N}_1\cdot\mathbf{S}^2\cdot\mathbf{N}_1\,,\qquad \mathcal{I}_6=\mathbf{N}_2\cdot\mathbf{S}\cdot\mathbf{N}_2\,,\qquad \mathcal{I}_7=\mathbf{N}_2\cdot\mathbf{S}^2\cdot\mathbf{N}_2\,. \tag{6.12}$$

Accordingly, for an orthotropic solid the strength function admits the following invariant representation:

$$\hat{\mathsf{F}}=\bar{\mathsf{F}}(\mathcal{I}_1,\cdots,\mathcal{I}_7)\,. \tag{6.13}$$

6.2.4 Monoclinic solids

A monoclinic solid is characterized by a set of three preferred material directions described by unit vectors $\{\mathbf{N}_1,\mathbf{N}_2,\mathbf{N}_3\}$, where $\mathbf{N}_1$ and $\mathbf{N}_2$ are not orthogonal and $\mathbf{N}_3$ is orthogonal to the plane they span. In addition to the seven invariants associated with orthotropic solids, monoclinic solids admit two further invariants:

$$\mathcal{I}_8=\mathfrak{g}\,\mathbf{N}_1\cdot\mathbf{S}\cdot\mathbf{N}_2,\qquad \mathcal{I}_9=\mathfrak{g}^2\,, \tag{6.14}$$

where $\mathfrak{g}=\mathbf{N}_1\cdot\mathbf{N}_2$. Thus, for a monoclinic solid the strength function can be represented as

$$\hat{\mathsf{F}}=\bar{\mathsf{F}}(\mathcal{I}_1,\cdots,\mathcal{I}_9)\,. \tag{6.15}$$

[22] Clearly, one can use the pair I_4 and I_5 instead of I_4 and $\operatorname{tr}\mathbf{C}^2$. Using the Calyley-Hamilton theorem, one can show that $\operatorname{tr}\mathbf{C}^3=I_4^3-3I_4I_5+3I_6$ [Spencer, 1971].

6.2.5 Strength function for anisotropic anelastic solids

Let us denote the set of structural tensors with respect to the intermediate configuration by $\mathring{\mathbf{\Lambda}} = \{\mathring{\boldsymbol{\zeta}}_1, \dots, \mathring{\boldsymbol{\zeta}}_N\}$. With respect to this local stress-free configuration the strength function has the representation

$$\mathsf{F}(X, \mathring{\mathbf{P}}, \overset{e}{\mathbf{F}}, \mathbf{g}, \mathring{\mathbf{G}}, \mathring{\mathbf{\Lambda}}) = 0\,. \tag{6.16}$$

Using covariance under the anelastic distortion $\overset{a}{\mathbf{F}}$, one obtains

$$\mathsf{F}(X, \mathring{\mathbf{P}}, \overset{e}{\mathbf{F}}, \mathbf{g}, \mathring{\mathbf{G}}, \mathring{\mathbf{\Lambda}}) = \mathsf{F}(X, \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{P}}, \overset{a}{\mathbf{F}}{}^{*}\overset{e}{\mathbf{F}}, \mathbf{g}, \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{G}}, \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{\Lambda}}) = \mathsf{F}(X, \mathbf{P}, \mathbf{F}, \mathbf{g}, \mathbf{G}, \mathbf{\Lambda})\,, \tag{6.17}$$

where $\mathbf{\Lambda} = \overset{a}{\mathbf{F}}{}^{*}\mathring{\mathbf{\Lambda}}$ is the set of anelastic structural tensors. The treatment of anisotropy follows exactly as in the elastic case, with the only modification that all invariants are computed using the material metric $\mathbf{G}$ instead of the flat metric $\mathring{\mathbf{G}}$.

7 Material Strength in Linear Elastic and Anelastic Brittle Solids

We now consider the linearized (small-strain) limit and show how classical stress-based strength criteria are recovered.

The purpose of this section is to clarify the relation between the general covariant formulation of material strength developed in the preceding sections and the classical stress-based strength criteria of linear elasticity. In the small-strain regime, all stress measures coincide to leading order, and the covariance requirement that places stress and strain on the same footing is therefore automatically satisfied. This explains why classical strength criteria are formulated solely in terms of stress invariants. At the same time, we emphasize that such criteria are not obtained by linearizing a general nonlinear strength function, but are instead postulated models that arise after neglecting higher-order strain dependence. This discussion establishes the precise sense in which classical linear theories of strength are consistent with, yet more restrictive than, the general nonlinear framework.

7.1 Linearized kinematics

Let us consider a one-parameter family of motions and anelastic distortions φ_ϵ and $\overset{a}{\mathbf{F}}_\epsilon$ such that $\varphi_{\epsilon=0} = \mathring{\varphi}$ and $\overset{a}{\mathbf{F}}_{\epsilon=0} = \mathbf{I}$, where $\mathbf{I} = T\iota$ is the tangent map of the inclusion $\iota : \mathcal{B} \hookrightarrow \mathcal{S}$. In general, $\mathring{\varphi}$ can be any initial deformation map but for our purposes of discussing classical linear elasticity we choose $\mathring{\varphi} = \iota$, i.e., linearization with respect to a stress-free initial configuration. The corresponding variation fields are defined by

$$\delta\varphi = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\varphi_\epsilon\,, \qquad \delta\overset{a}{\mathbf{F}} = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\overset{a}{\mathbf{F}}_\epsilon\,. \tag{7.1}$$

Recall that $\overset{a}{\mathbf{F}}_\epsilon : T_X\mathcal{B} \to T_X\mathcal{B}$ for all ϵ, and hence $\delta\overset{a}{\mathbf{F}}$ is a well-defined material $\binom{1}{1}$-tensor. Let $\mathbf{U} = \delta\varphi$. Its spatial representation $\mathbf{u} = \delta\varphi \circ \mathring{\varphi}^{-1}$ is the displacement field of the classical theory of linear elasticity. We call $\overset{a}{\mathbf{U}} = \delta\overset{a}{\mathbf{F}}$ the *anelastic displacement*. Its spatial counterpart is denoted by $\overset{a}{\mathbf{u}} = \mathring{\varphi}_*\overset{a}{\mathbf{U}} = \overset{a}{\mathbf{U}} \circ \mathring{\varphi}^{-1}$.

Using the identity $\mathbf{C}^\flat = \varphi^*\mathbf{g}$, the linearization of the right Cauchy–Green deformation tensor is written as

$$\delta\mathbf{C}^\flat = \varphi^*_{\epsilon=0}\mathbf{L}_{\mathbf{u}}\mathbf{g} = \mathring{\varphi}^*\left(\nabla^{\mathbf{g}}\mathbf{u}^\flat + \left[\nabla^{\mathbf{g}}\mathbf{u}^\flat\right]^\star\right) = 2\mathring{\varphi}^*\boldsymbol{\epsilon} = 2\boldsymbol{\varepsilon}\,, \tag{7.2}$$

where

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla^{\mathbf{g}}\mathbf{u}^\flat + \left[\nabla^{\mathbf{g}}\mathbf{u}^\flat\right]^\star\right)\,, \tag{7.3}$$

is the linearized total strain, $\mathbf{L}$ is the Lie derivative operator, and $\boldsymbol{\varepsilon} = \mathring{\varphi}^*\boldsymbol{\epsilon}$. The linearization of the elastic right Cauchy–Green tensor is calculated as

$$\delta\overset{e}{\mathbf{C}}{}^\flat = \left(2\overset{e}{\mathbf{F}}{}^\star\boldsymbol{\epsilon}\overset{e}{\mathbf{F}} - \overset{e}{\mathbf{C}}{}^\flat(\delta\overset{a}{\mathbf{F}})\overset{a}{\mathbf{F}}{}^{-1} - \overset{a}{\mathbf{F}}{}^{-\star}(\delta\overset{a}{\mathbf{F}})^\star\overset{e}{\mathbf{C}}{}^\flat\right)\Big|_{\epsilon=0} = 2\mathring{\varphi}^*\boldsymbol{\epsilon} - \mathring{\mathbf{G}}\overset{a}{\mathbf{U}} - \overset{a}{\mathbf{U}}{}^\star\mathring{\mathbf{G}} = 2\mathring{\varphi}^*\left(\boldsymbol{\epsilon} - \overset{a}{\boldsymbol{\epsilon}}\right) = 2\mathring{\varphi}^*\overset{e}{\boldsymbol{\epsilon}} = 2\overset{e}{\boldsymbol{\varepsilon}}\,, \tag{7.4}$$

where

$$\overset{a}{\boldsymbol{\varepsilon}} = \mathring{\varphi}^*\overset{a}{\boldsymbol{\epsilon}} = \frac{1}{2}\left(\mathring{\mathbf{G}}\overset{a}{\mathbf{U}} + \overset{a}{\mathbf{U}}{}^\star\mathring{\mathbf{G}}\right)\,, \tag{7.5}$$

is the linearized anelastic strain, and $\overset{e}{\boldsymbol{\varepsilon}} = \mathring{\varphi}^* \overset{e}{\boldsymbol{\epsilon}}$ is the linearized elastic strain. Thus, the linearized strain admits the additive decomposition

$$\boldsymbol{\varepsilon} = \overset{e}{\boldsymbol{\varepsilon}} + \overset{a}{\boldsymbol{\varepsilon}} \,. \tag{7.6}$$

Finally, the material metric $\mathbf{G} = \overset{a}{\mathbf{F}}{}^* \mathring{\mathbf{G}} = \overset{a}{\mathbf{F}}{}^\star \mathring{\mathbf{G}} \overset{a}{\mathbf{F}}$ is linearized as

$$\delta\mathbf{G} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathbf{G}_\epsilon = \delta\overset{a}{\mathbf{F}}{}^\star \, \mathring{\mathbf{G}} \, \overset{a}{\mathbf{F}}_\epsilon \Big|_{\epsilon=0} + \overset{a}{\mathbf{F}}{}^\star_\epsilon \Big|_{\epsilon=0} \mathring{\mathbf{G}} \, \delta\overset{a}{\mathbf{F}} = \delta\overset{a}{\mathbf{F}}{}^\star \, \mathring{\mathbf{G}} \, \mathbf{I} + \mathbf{I} \, \mathring{\mathbf{G}} \, \delta\overset{a}{\mathbf{F}} = \overset{a}{\mathbf{U}}{}^\star \mathring{\mathbf{G}} + \mathring{\mathbf{G}} \overset{a}{\mathbf{U}} = 2\overset{a}{\boldsymbol{\varepsilon}} \,. \tag{7.7}$$

7.2 Strength function in linear elasticity

In linear elasticity, strains are assumed to be small, whereas stresses are of the same order. When linearizing with respect to a stress-free configuration, the deformation gradient is written as $\mathbf{F} = \mathbf{I} + \mathcal{O}(\nabla\mathbf{U})$, where $\mathbf{I} = T\iota$ is the tangent map of the inclusion $\iota : \mathcal{B} \hookrightarrow \mathcal{S}$ and $\mathbf{U}$ is the displacement field. It follows that all strain measures differ from the reference metric only by higher-order terms, and all stress measures coincide to leading order. This is consistent with the covariance requirement of the strength function, which places stress and strain on the same footing; in the small-strain regime, this requirement is automatically satisfied. Consequently, the strain-dependent corrections entering the full nonlinear strength function are higher-order, and one may work, to leading order, with a strength criterion depending only on stress. This explains why, in linear elasticity, strength is commonly described by a hypersurface in stress space alone, which in the isotropic case depends only on the invariants of the stress tensor.

It is then natural to ask whether the classical strength criteria of linear elasticity can be obtained by linearizing a general nonlinear strength function, for instance a strength function of the form $\mathsf{F}(\mathbf{P}, \mathbf{F}, \mathbf{g}, \mathbf{G}) = 0$, or equivalently, by covariance, $\hat{\mathsf{F}}(\mathbf{S}, \mathbf{C}^\flat, \mathbf{G}) = 0$. The answer is negative. Linearizing such a strength function about the undeformed configuration leads to only a local Taylor approximation in the stress and strain increments. At first order, one obtains the tangent hyperplane to the strength hypersurface at the chosen base state, while at second order one obtains only a local quadratic approximation. In either case, the result is merely a local description near one state, whereas a strength criterion is meant to describe a finite threshold for failure over a substantial region of stress space. Classical criteria such as Tresca, von Mises, and Drucker–Prager are therefore not obtained by linearizing a general nonlinear strength function. Rather, they are postulated directly in the linearized setting as hypersurfaces defined in terms of the principal stresses or stress invariants and motivated by symmetry and experiments. In this sense, the classical stress-based criteria of linear elasticity should be understood as leading-order approximations in the small-strain regime, and not as literal linearizations of a general nonlinear strength function.

However, it is important to emphasize that this conclusion does not imply any inconsistency between nonlinear and linear theories of strength. A general nonlinear strength function of the form $\mathsf{F}(\mathbf{P}, \mathbf{F}, \mathbf{g}, \mathbf{G}) = 0$ remains well-defined in the small-strain regime. Substituting the approximation $\mathbf{F} = \mathbf{I} + \mathcal{O}(\nabla\mathbf{U})$ gives a reduced strength function that depends on the pair $(\mathbf{P}, \nabla\mathbf{U})$. In this reduced setting, all stress measures coincide to leading order, so that $\mathbf{P}$ may be identified with any of the classical stress tensors. The resulting strength condition is therefore a well-defined relation coupling stress and strain, and it retains information about the underlying constitutive structure of the material. The key point is that, even in the small-strain regime, the dependence on $\nabla\mathbf{U}$ does not vanish identically; it is merely of higher order and is neglected only under an additional modeling assumption. Consequently, the passage from a general strength function $\mathsf{F}(\mathbf{P}, \mathbf{F}, \mathbf{g}, \mathbf{G}) = 0$ to a purely stress-based criterion is not a direct consequence of linearization, but rather reflects a deliberate simplification in which all strain dependence is discarded. This observation clarifies the status of classical strength criteria: they are not derived from a general nonlinear theory by a systematic limiting procedure, but are instead postulated models that are consistent with, but are not uniquely implied by the small-strain approximation. Conversely, if one starts with a stress-based strength function, it remains applicable in the linear regime without modification, since the identification of stress measures ensures that no ambiguity arises at leading order. In this sense, stress-based strength criteria are compatible with small-strain elasticity, but they do not encode the full structure of a general nonlinear strength function.

7.3 Strength function in linear anelasticity

The three stress invariants, to linear order in the eigenstrain, are given by

$$\begin{aligned}
\mathcal{I}_1 &= S^{AB}G_{AB} = \mathring{\mathcal{I}}_1 + 2\ \mathrm{tr}_{\overset{a}{\boldsymbol{\varepsilon}}}\,\mathbf{S} + o(\|\overset{a}{\boldsymbol{\varepsilon}}\|)\,,\\
\mathcal{I}_2 &= S^{AM}S^{BN}G_{AB}G_{MN} = \mathring{\mathcal{I}}_2 + 4\,S^{AM}S^{BN}\mathring{G}_{AB}\,\overset{a}{\varepsilon}_{MN} = \mathring{\mathcal{I}}_2 + 4\ \mathrm{tr}_{\overset{a}{\boldsymbol{\varepsilon}}}\,\mathring{\mathbf{S}}^2 + o(\|\overset{a}{\boldsymbol{\varepsilon}}\|)\,,\\
\mathcal{I}_3 &= S^{AM}S^{NK}S^{LB}\,G_{MN}G_{KL}G_{AB} = \mathring{\mathcal{I}}_3 + 6\,S^{AM}S^{NK}S^{LB}\mathring{G}_{KL}\mathring{G}_{AB}\,\overset{a}{\varepsilon}_{MN} = \mathring{\mathcal{I}}_3 + 6\ \mathrm{tr}_{\overset{a}{\boldsymbol{\varepsilon}}}\,\mathring{\mathbf{S}}^3 + o(\|\overset{a}{\boldsymbol{\varepsilon}}\|)\,,
\end{aligned} \tag{7.8}$$

where $\mathring{\mathbf{S}}^2$ and $\mathring{\mathbf{S}}^3$ denote $\mathbf{S}^2$ and $\mathbf{S}^3$ computed using the reference metric $\mathring{\mathbf{G}}$.

Using the linearized invariants and neglecting higher-order terms in $\overset{a}{\boldsymbol{\varepsilon}}$, the strength function is given by

$$\hat{\mathsf{F}}(\mathbf{S},\mathbf{G}) = \bar{\mathsf{F}}\big(\mathring{\mathcal{I}}_1 + 2\ \mathrm{tr}_{\overset{a}{\boldsymbol{\varepsilon}}}\,\mathbf{S},\ \mathring{\mathcal{I}}_2 + 4\ \mathrm{tr}_{\overset{a}{\boldsymbol{\varepsilon}}}\,\mathring{\mathbf{S}}^2,\ \mathring{\mathcal{I}}_3 + 6\ \mathrm{tr}_{\overset{a}{\boldsymbol{\varepsilon}}}\,\mathring{\mathbf{S}}^3\big)\,. \tag{7.9}$$

Therefore, even if the strength function is written in terms of stress invariants, eigenstrains enter through the linearization of the material metric and modify all invariants. In particular, the corresponding strength hypersurface in stress space depends on the internal anelastic state through $\overset{a}{\boldsymbol{\varepsilon}}$, and hence is no longer determined by stress alone.

Recall that the elastic strain is defined through the additive decomposition (7.6). Thus, the elastic strain is given by $\overset{e}{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} - \overset{a}{\boldsymbol{\varepsilon}}$. Since the constitutive response is formulated in terms of the elastic strain, the stress tensor depends on $\overset{e}{\boldsymbol{\varepsilon}}$ and hence depends explicitly on the eigenstrain $\overset{a}{\boldsymbol{\varepsilon}}$. Consequently, in linear anelasticity, eigenstrains influence the strength function both through their explicit contribution to the invariants and through their effect on the manifold of constitutively admissible stresses.

8 Conclusions

In this paper, we formulated material strength for brittle solids in the setting of finite elasticity. The central result is that a covariant strength function must depend on both stress and the corresponding strain measure. Only in that case can a strength function written in terms of one stress measure be rewritten in terms of another. Purely stress-based strength criteria were then identified as a special subclass obtained by suppressing the strain dependence.

We showed that the representation of a strength function depends on the choice of stress measure. Although the various stress measures are mechanically equivalent in the sense that they represent the same physical traction vector when paired with the appropriate area element, the notion of strength depends, in general, on which stress measure is assumed to be homogeneous at the onset of fracture. This leads naturally to the problem of relating strength functions written in terms of the first Piola, second Piola–Kirchhoff, and Cauchy stresses, and we showed that spatial covariance is the fundamental principle governing the relation among these different representations.

Restricting attention to the stress-based criteria that appear in the existing literature, we defined the corresponding strength hypersurface and discussed the associated safe domain. We distinguished constitutive admissibility from fracture and clarified that the strength hypersurface should be understood as a hypersurface in the constitutively admissible stress manifold. We show that, for stress-based strength functions satisfying standard regularity conditions and the requirement that sufficiently large stresses are inadmissible, the strength surface is a smooth compact hypersurface of the constitutively admissible stress manifold, defined as the zero level set of the strength function separating admissible from inadmissible stress states. For isotropic solids, we examined the symmetry of strength surfaces in principal stress space and showed that, under a natural proportional-reduction hypothesis, the safe domain is star-shaped with respect to the origin. We verified this property for several classical isotropic strength criteria, including the Mohr–Coulomb, Hoek–Brown, generalized three-dimensional Hoek–Brown, and Mogi–Coulomb surfaces.

We also discussed the role of material symmetry in strength. Motivated by the fact that fracture properties are determined by the same underlying microstructure that governs elastic response, we argued that the strength function should inherit the same material symmetry group as the constitutive equations. This provides a natural framework for discussing anisotropic strength hypersurfaces and their invariant-theoretic representations.

Finally, we showed that this framework extends naturally to materials with internal constraints and to solids with residual stress or distributed eigenstrains, for which the appropriate strength function must be referred to

the local relaxed configuration. In this setting, residual stresses do not enter merely as external parameters, but are encoded through the material metric, which modifies the invariants entering the strength function. Consequently, even when the strength function is expressed in terms of stress invariants, the corresponding strength hypersurface in stress space depends on the internal anelastic state. In particular, eigenstrains alter both the admissible set of stresses and the geometry of the strength surface, so that strength can no longer be characterized by a universal stress-based criterion independent of the material state. This highlights the essential role of the relaxed configuration and shows that, in the presence of residual stress, strength must be understood as a covariant relation involving both stress and the underlying anelastic structure. Taken together, these results provide a systematic and covariant continuum-mechanical formulation of material strength for brittle solids in both isotropic and anisotropic settings.

The present work provides a rational formulation of material strength for brittle solids within the framework of finite elasticity. Several natural directions for future work arise. A first is the extension to dissipative and rate-dependent solids, especially viscoelastic materials, where the strength function is expected to depend on additional internal variables and on the loading history. A second important direction is the study of fracture in residually-stressed solids, where the interaction of strength, incompatibility, and the material metric is expected to play a fundamental role. Another natural direction is the extension of the present framework to more general classes of anisotropic solids and to evolving material symmetries.

Acknowledgement

AY benefited from discussions with Patrizio Neff. AK acknowledges support from the National Science Foundation, United States, through the grant CMMI-2404808.

References

D. Ambrosi, M. Ben Amar, C. J. Cyron, A. DeSimone, A. Goriely, J. D. Humphrey, and E. Kuhl. Growth and remodelling of living tissues: perspectives, challenges and opportunities. *Journal of the Royal Society Interface*, 16(157):20190233, 2019.

H. Amor, J.-J. Marigo, and C. Maurini. Regularized formulation of the variational brittle fracture with unilateral contact: Numerical experiments. *Journal of the Mechanics and Physics of Solids*, 57(8):1209–1229, 2009.

T. L. Anderson. *Fracture Mechanics: Fundamentals and Applications*. CRC press, 2005.

E. K. Ashkenazi. Problems of the anisotropy of strength. *Polymer Mechanics*, 1(2):60–70, 1965.

V. D. Azzi and S. W. Tsai. Anisotropic strength of composites: Investigation aimed at developing a theory applicable to laminated as well as unidirectional composites, employing simple material properties derived from unidirectional specimens alone. *Experimental Mechanics*, 5(9):283–288, 1965.

P. Badel, V. Godard, and J.-B. Leblond. Application of some anisotropic damage model to the prediction of the failure of some complex industrial concrete structure. *International Journal of Solids and Structures*, 44 (18-19):5848–5874, 2007.

J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Archive for Rational Mechanics and Analysis*, 63(4):337–403, 1976.

J. M. Ball and R. D. James. Fine phase mixtures as minimizers of energy. *Archive for Rational Mechanics and Analysis*, 100(1):13–52, 1987.

G. I. Barenblatt. The formation of equilibrium cracks during brittle fracture. general ideas and hypotheses. axially-symmetric cracks. *Journal of Applied Mathematics and Mechanics*, 23(3):622–636, 1959.

E. Beltrami. Sull' interpretazione meccanica delle formule di Maxwell. *Rendiconti del Reale Istituto Lombardo di Scienze e Lettere*, 18:141–148, 1885.

B. A. Bilby, L. R. T. Lardner, and A. N. Stroh. Continuous distributions of dislocations and the theory of plasticity. In *Actes du IXe congrès international de mécanique appliquée, (Bruxelles, 1956)*, volume 8, pages 35–44, 1957.

J.-P. Boehler. A simple derivation of representations for non-polynomial constitutive equations in some cases of anisotropy. *Zeitschrift für Angewandte Mathematik und Mechanik*, 59(4):157–167, 1979.

J.-P. Boehler. *Applications of Tensor Functions in Solid Mechanics*, volume 292. Springer, 1987.

A. Bonacci, J. Dolbow, and J. Guilleminot. Stochastic modeling of anisotropic strength surfaces from atomistic simulations. *Theoretical and Applied Fracture Mechanics*, page 105594, 2026.

B. Bourdin, G. A. Francfort, and J.-J. Marigo. Numerical experiments in revisited brittle fracture. *Journal of the Mechanics and Physics of Solids*, 48(4):797–826, 2000.

M. M. Carroll. Controllable states of stress for compressible elastic solids. *Journal of Elasticity*, 3:57–61, 1973a.

M. M. Carroll. Controllable states of stress for incompressible elastic solids. *Journal of Elasticity*, 3(2):147–153, 1973b.

A.-L. Cauchy. Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou non élastique. *Exercises de Mathématiques*, 3:160–187, 1828.

C. Chen, Z. Wang, and Z. Suo. Flaw sensitivity of highly stretchable materials. *Extreme Mechanics Letters*, 10: 50–57, 2017.

S. Chockalingam, A. B. Tepole, and A. Kumar. The phase-field model of fracture incorporating Mohr–Coulomb, Mogi–Coulomb, and Hoek–Brown strength surfaces. *Engineering Fracture Mechanics*, 340:112108, 2026.

P. G. Ciarlet. *Mathematical Elasticity, Volume I: Three-Dimensional Elasticity*. North-Holland, Amsterdam, 1988.

C. Comi. A non-local model with tension and compression damage mechanisms. *European Journal of Mechanics-A/Solids*, 20(1):1–22, 2001.

S. C. Cowin. Fabric dependence of an anisotropic strength criterion. *Mechanics of Materials*, 5(3):251–260, 1986.

T. C. Doyle and J. L. Ericksen. Nonlinear elasticity. *Advances in Applied Mechanics*, 4:53–115, 1956.

D. S. Dugdale. Yielding of steel sheets containing slits. *Journal of the Mechanics and Physics of Solids*, 8(2): 100–104, 1960.

D. G. Ebin and J. Marsden. Groups of Diffeomorphisms and the motion of an incompressible fluid. *Annals of Mathematics*, 92(1):102–163, 1970.

C. Eckart. The thermodynamics of irreversible processes. IV. The theory of elasticity and anelasticity. *Physical Review*, 73(4):373, 1948.

J. L. Ericksen. Deformations possible in every isotropic, incompressible, perfectly elastic body. *Zeitschrift für Angewandte Mathematik und Physik*, 5(6):466–489, 1954.

J. L. Ericksen. Deformations possible in every compressible, isotropic, perfectly elastic material. *Journal of Mathematics and Physics*, 34(1-4):126–128, 1955.

J. D. Eshelby. The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proceedings of the Royal Society of London A*, 241(1226):376–396, 1957.

W.-x. Fan. On phenomenological anisotropic failure criteria. *Composites Science and Technology*, 28:269–278, 1987.

G. A. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *Journal of the Mechanics and Physics of Solids*, 46(8):1319–1342, 1998.

H. G. Franklin. Classic theories of failure of anisotropic materials. *Fibre Science and Technology*, 1(2):137–150, 1968.

D. Giulini. Some remarks on the notions of general covariance and background independence. *Lecture Notes in Physics*, 721:105–120, 2007.

I. I. Gol'Denblat and V. A. Kopnov. Strength of glass-reinforced plastics in the complex stress state. *Polymer Mechanics*, 1(2):54–59, 1965.

A. E. Green and P. M. Naghdi. A general theory of an elastic-plastic continuum. *Archive for Rational Mechanics and Analysis*, 18:251–281, 1965.

A. E. Green and R. S. Rivlin. On Cauchy's equations of motion. *Zeitschrift für Angewandte Mathematik und Physik*, 15(3):290–292, 1964.

G. Green. On the laws of the reflexion and refraction of light at the common surface of two non-crystallized media. *Transactions of the Cambridge Philosophical Society*, 7:1, 1838.

G. Green. On the propagation of light in crystallized media. *Transactions of the Cambridge Philosophical Society*, 7:121, 1839.

A. A. Griffith. VI. The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society of London. Series A*, 221(582-593):163–198, 1921.

M. E. Gurtin and K. Spear. On the relationship between the logarithmic strain rate and the stretching tensor. *International Journal of Solids and Structures*, 19(5):437–444, 1983.

A. Hamdi, M. N. Abdelaziz, N. Aït Hocine, P. Heuillet, and N. Benseddiq. A fracture criterion of rubber-like materials under plane stress conditions. *Polymer Testing*, 25(8):994–1005, 2006.

H. Hencky. Zur theorie plastischer deformationen und der hierdurch im material hervorgerufenen nachspannungen. *Zeitschrift für Angewandte Mathematik und Mechanik*, 4:323–334, 1924.

D. Hilbert. Ueber die theorie der algebraischen formen. *Mathematische Annalen*, 36:473–534, 1890.

R. Hill. A theory of the yielding and plastic flow of anisotropic metals. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 193(1033):281–297, 1948.

R. Hill. On constitutive inequalities for simple materials–I. *Journal of the Mechanics and Physics of Solids*, 16 (4):229–242, 1968.

R. Hill. Constitutive inequalities for isotropic elastic solids under finite strain. *Proceedings of the Royal Society of London A*, 314(1519):457–472, 1970.

R. Hill. Aspects of invariance in solid mechanics. *Advances in Applied Mechanics*, 18:1–75, 1978.

M. W. Hirsch. *Differential Topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1976.

O. Hoffman. The brittle strength of orthotropic materials. *Journal of Composite Materials*, 1(2):200–206, 1967.

A. Hoger. On the residual stress possible in an elastic body with material symmetry. *Archive for Rational Mechanics and Analysis*, 88(3):271–289, 1985.

A. Hoger. The material time derivative of logarithmic strain. *International Journal of Solids and Structures*, 22(9):1019–1032, 1986.

A. Hoger. The stress conjugate to logarithmic strain. *International Journal of Solids and Structures*, 23(12): 1645–1656, 1987.

M. T. Huber. O podstawach wytrzymałości materjałów. *Czasopismo Techniczne*, 1904.

M. T. Huber. Specific work of strain as a measure of material effort. *Archives of Mechanics*, 56(3):173–190, 2004. English translation of the 1904 Polish original.

T. J. R. Hughes and J. E. Marsden. Some applications of geometry is continuum mechanics. *Reports on Mathematical Physics*, 12(1):35–44, 1977.

T.-S. Jun and A. M. Korsunsky. Evaluation of residual stresses and strains using the eigenstrain reconstruction method. *International Journal of Solids and Structures*, 47(13):1678–1686, 2010.

F. Kamarei, E. Breedlove, and O. Lopez-Pamies. Nucleation and propagation of fracture in viscoelastic elastomers: A complete phase-field theory. *Computer Methods in Applied Mechanics and Engineering*, 446:118337, 2025.

F. Kamarei, B. Zeng, J. E. Dolbow, and O. Lopez-Pamies. Nine circles of elastic brittle fracture. *Computer Methods in Applied Mechanics and Engineering*, 448:118449, 2026.

N. Kinoshita and T. Mura. Elastic fields of inclusions in anisotropic media. *Physica Status Solidi (a)*, 5(3): 759–768, 1971.

W. G. Knauss. An upper bound of failure in viscoelastic materials subjected to multiaxial stress states. *International Journal of Fracture*, 3:267–277, 1967.

K. Kondo. A proposal of a new theory concerning the yielding of materials based on Riemannian geometry. *The Journal of the Japan Society of Aeronautical Engineering*, 2(8):29–31, 1949.

E. Kröner. Allgemeine kontinuumstheorie der versetzungen und eigenspannungen. *Archive for Rational Mechanics and Analysis*, 4(1):273–334, 1959.

A. Kumar and O. Lopez-Pamies. The phase-field approach to self-healable fracture of elastomers: A model accounting for fracture nucleation at large, with application to a class of conspicuous experiments. *Theoretical and Applied Fracture Mechanics*, 107:102550, 2020.

A. Kumar, G. A. Francfort, and O. Lopez-Pamies. Fracture and healing of elastomers: A phase-transition theory and numerical implementation. *Journal of the Mechanics and Physics of Solids*, 112:523–551, 2018.

A. Kumar, B. Bourdin, G. A. Francfort, and O. Lopez-Pamies. Revisiting nucleation in the phase-field approach to brittle fracture. *Journal of the Mechanics and Physics of Solids*, 142:104027, 2020.

A. Kumar, Y. Liu, J. E. Dolbow, and O. Lopez-Pamies. The strength of the brazilian fracture test. *Journal of the Mechanics and Physics of Solids*, 182:105473, 2024.

G. Lamé and B. P. É. Clapeyron. *Memoir on the internal equilibrium of homogeneous solid bodies*. Verlag nicht hermittelbar, 1833.

S. C. Lamont, N. Bouklas, and F. J. Vernerey. Cohesive instability in elastomers: Insights from a crosslinked Van der Waals fluid model. *International Journal of Fracture*, 249(1):20, 2025.

E. Lee and D. Liu. Finite-strain elastic-plastic theory with application to plane-wave analysis. *Journal of Applied Physics*, 38(1):19–27, 1967.

E. H. Lee. Elastic-plastic deformation at finite strains. *Journal of Applied Mechanics*, 36(1):1–6, 1969.

J. M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, 2 edition, 2013.

D. Li and L. N. Y. Wong. The brazilian disc test for rock mechanics applications: Review and new insights. *Rock Mechanics and Rock Engineering*, 46(2):269–287, 2013.

I. Liu. On representations of anisotropic invariants. *International Journal of Engineering Science*, 20(10): 1099–1109, 1982.

O. Lopez-Pamies and F. Kamarei. When and where do large cracks grow? Griffith energy competition constrained by material strength. *Extreme Mechanics Letters*, 81:102417, 2025.

O. Lopez-Pamies, J. E. Dolbow, G. A. Francfort, and C. J. Larsen. Classical variational models cannot predict fracture nucleation. *Computer Methods in Applied Mechanics and Engineering*, 433:117520, 2025.

J. Lu. A covariant constitutive theory for anisotropic hyperelastic solids with initial strains. *Mathematics and Mechanics of Solids*, 17(2):104–119, 2012.

J. Lu and P. Papadopoulos. A covariant constitutive description of anisotropic non-linear elasticity. *Zeitschrift für Angewandte Mathematik und Physik*, 51(2):204–217, 2000.

A. K. Malmeister. Geometry of theories of strength. *Polymer Mechanics*, 2(4):324–331, 1966.

J. Marin. Theories of strength for combined stresses and nonisotropic materials. *Journal of the Aeronautical Sciences*, 24(4):265–268, 1957.

J. E. Marsden and T. J. R. Hughes. *Mathematical Foundations of Elasticity*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1983.

J. Mazars. A model of a unilateral elastic damageable material and its application to concrete. *Fracture toughness and fracture energy of concrete*, pages 61–71, 1986.

J. Mazars and G. Pijaudier-Cabot. Continuum damage theory—application to concrete. *Journal of engineering mechanics*, 115(2):345–365, 1989.

A. L. Mazzucato and L. V. Rachele. Partial uniqueness and obstruction to uniqueness in inverse problems for anisotropic elastic media. *Journal of Elasticity*, 83(3):205–245, 2006.

J. Merodio and R. W. Ogden. Finite deformation elasticity theory. In *Constitutive Modelling of Solid Continua*, pages 17–52. Springer, 2020.

C. Miehe, F. Welschinger, and M. Hofacker. Thermodynamically consistent phase-field models of fracture: Variational principles and multi-field fe implementations. *International journal for numerical methods in engineering*, 83(10):1273–1311, 2010.

L. A. Mihai and P. Neff. Hyperelastic bodies under homogeneous Cauchy stress induced by non-homogeneous finite deformations. *International Journal of Non-Linear Mechanics*, 89:93–100, 2017.

L. A. Mihai and P. Neff. Hyperelastic bodies under homogeneous Cauchy stress induced by three-dimensional non-homogeneous deformations. *Mathematics and Mechanics of Solids*, 23(4):606–616, 2018.

R. D. Mindlin and D. H. Cheng. Nuclei of strain in the semi-infinite solid. *Journal of Applied Physics*, 21(9): 926–930, 1950.

A. J. A. Morgan. Some properties of media defined by constitutive equations in implicit form. *International Journal of Engineering Science*, 4(2):155–178, 1966.

T. Mura. *Micromechanics of Defects in Solids*. Martinus Nijhoff, 1982.

P. Neff and L. A. Mihai. Injectivity of the Cauchy-stress tensor along rank-one connected lines under strict rank-one convexity condition. *Journal of Elasticity*, 127(2):309–315, 2017.

P. Neff, S. Holthausen, M. V. d'Agostino, D. Bernardini, A. Sky, I.-D. Ghiba, and R. J. Martin. Hypo-elasticity, Cauchy-elasticity, corotational stability and monotonicity in the logarithmic strain. *Journal of the Mechanics and Physics of Solids*, 202:106074, 2025.

W. Noll. La mecanique classique, basee sur un axiome d'objectivite. In *La Methode Axiomatique dans les Mecaniques Classiques et NouveIies*, pages 47–56, Paris, 1963.

J. D. Norton. General covariance and the foundations of general relativity: Eight decades of dispute. *Reports on Progress in Physics*, 56(7):791–858, 1993.

R. W. Ogden. *Non-Linear Elastic Deformations*. Dover, 1997.

K. R. Rajagopal. On implicit constitutive theories. *Applications of Mathematics*, 48:279–319, 2003.

K. R. Rajagopal. The elasticity of elasticity. *Zeitschrift für angewandte Mathematik und Physik*, 58:309–317, 2007.

H. Reissner. Eigenspannungen und eigenspannungsquellen. *Zeitschrift für Angewandte Mathematik und Mechanik*, 11(1):1–8, 1931.

R. Rivlin and A. G. Thomas. Rupture of rubber. I. Characteristic energy for tearing. *Journal of Polymer Science*, 10(3):291–318, 1953.

R. S. Rivlin and J. L. Ericksen. Stress-deformation relations for isotropic materials. *Journal of Rational Mechanics and Analysis*, 4(6):323–425, 1955.

R. T. Rockafellar. *Convex Analysis*, volume 28 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1970.

W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 3 edition, 1976.

S. Sadik and A. Yavari. On the origins of the idea of the multiplicative decomposition of the deformation gradient. *Mathematics and Mechanics of Solids*, 22(4):771–772, 2017.

E. Schweickert, L. A. Mihai, and P. Neff. Homogeneous Cauchy stress induced by non-homogeneous deformations. *Proceedings in Applied Mathematics and Mechanics*, 18(1):e201800185, 2018.

J. C. Simo and J. E. Marsden. On the rotated stress tensor and the material version of the Doyle-Ericksen formula. *Archive for Rational Mechanics and Analysis*, 86(3):213–231, 1984.

J. C. Simo, J. E. Marsden, and P. S. Krishnaprasad. The Hamiltonian structure of nonlinear elasticity: the material and convective representations of solids, rods, and plates. *Archive for Rational Mechanics and Analysis*, 104(2):125–183, 1988.

T. L. Smith. Dependence of the ultimate properties of a GR-S rubber on strain rate and temperature. *Journal of Polymer Science*, 32:99–113, 1958.

T. L. Smith. Ultimate tensile properties of elastomers. I. Characterization by a time and temperature independent failure envelope. *Journal of Polymer Science Part A*, 1:3597–3613, 1963.

T. L. Smith. Relations between ultimate tensile properties of elastomers and their structure. *Proceedings of the Royal Society of London A*, 282:102–113, 1964a.

T. L. Smith. Ultimate tensile properties of elastomers. II. Comparison of failure envelopes for unfilled vulcanizates. *Journal of Applied Physics*, 35:27–36, 1964b.

A. Spencer. George Green and the foundations of the theory of elasticity. *Journal of Engineering Mathematics*, 95(1):5–6, 2015.

A. J. M. Spencer. Part III. Theory of invariants. *Continuum Physics*, 1:239–353, 1971.

A. J. M. Spencer. *Deformations of Fibre-Reinforced Materials*. Clarendon Press, Oxford, 1972.

A. J. M. Spencer. The formulation of constitutive equation for anisotropic solids. In *Mechanical Behavior of Anisotropic Solids/Comportment Méchanique des Solides Anisotropes*, pages 3–26. Springer, 1982.

A. J. M. Spencer. Modelling of finite deformations of anisotropic materials. In *Large Deformations of Solids: Physical Basis and Mathematical Modelling*, pages 41–52. Springer, 1986.

C. Truesdell. The mechanical foundations of elasticity and fluid dynamics. *Journal of Rational Mechanics and Analysis*, 1(1):125–300, 1952.

C. Truesdell and W. Noll. *The Nonlinear Field Theories of Mechanics*. Springer, Berlin, 2 edition, 2004.

S. W. Tsai. A survey of macroscopic failure criteria for composite materials. *Journal of Reinforced Plastics and Composites*, 3(1):40–62, 1984.

S. W. Tsai and E. M. Wu. A general theory of strength for anisotropic materials. *Journal of Composite Materials*, 5(1):58–80, 1971.

Y. Ueda, K. Fukuda, K. Nakacho, and S. Endo. A new measuring method of residual stresses with the aid of finite element method and reliability of estimated values. *Transactions of JWRI*, 4(2):123–131, 1975.

R. von Mises. Mechanik der festen Körper im plastisch-deformablen zustand. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1913:582–592, 1913.

R. von Mises. Mechanik der plastischen formänderung von kristallen. *Zeitschrift für Angewandte Mathematik und Mechanik*, 8(3):161–185, 1928.

O. Ward and A. Kumar. Why planar cracks fragment into echelon cracks. *arXiv preprint arXiv:2512.16053*, 2025.

H. Weyl. *The Classical Groups: Their Invariants and Representations*, volume 1 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1939.

H. Xiao, O. Bruhns, and A. Meyers. Logarithmic strain, logarithmic spin and logarithmic rate. *Acta Mechanica*, 124:89–105, 1997.

A. Yavari. On Eshelby's inclusion problem in nonlinear anisotropic elasticity. *Journal of Micromechanics and Molecular Physics*, 6(1):2150002, 2021.

A. Yavari. On universal deformations of compressible Cauchy elastic solids reinforced by inextensible fibers. *Journal of the Mechanics and Physics of Solids*, page 106340, 2025.

A. Yavari and A. Goriely. Nonlinear elastic inclusions in isotropic solids. *Proceedings of the Royal Society A*, 469(2160):20130415, 2013.

A. Yavari and A. Goriely. Universal deformations in anisotropic nonlinear elastic solids. *Journal of the Mechanics and Physics of Solids*, 156:104598, 2021.

A. Yavari and A. Goriely. The universal program of nonlinear hyperelasticity. *Journal of Elasticity*, 154(1): 91–146, 2023.

A. Yavari and A. Goriely. Controllable deformations in compressible isotropic implicit elasticity. *Zeitschrift für angewandte Mathematik und Physik*, 75(5):169, 2024.

A. Yavari and A. Goriely. Nonlinear Cauchy elasticity. *Archive for Rational Mechanics and Analysis*, 249(5): 57, 2025.

A. Yavari and J. E. Marsden. Covariantization of nonlinear elasticity. *Zeitschrift für Angewandte Mathematik und Physik*, 63(5):921–927, 2012.

A. Yavari and F. Sozio. On the direct and reverse multiplicative decompositions of deformation gradient in nonlinear anisotropic anelasticity. *Journal of the Mechanics and Physics of Solids*, 170:105101, 2023.

A. Yavari, J. E. Marsden, and M. Ortiz. On the spatial and material covariant balance laws in elasticity. *Journal of Mathematical Physics*, 47:85–112, 2006.

M.-H. Yu. Advances in strength theories for materials under complex stress state in the 20th century. *Applied Mechanics Reviews*, 55(3):169–218, 2002.

M.-H. Yu. *Unified Strength Theory and Its Applications*. Springer, Berlin, 2004.

B. Zeng, J. Guilleminot, and J. E. Dolbow. Examining crack nucleation under spatially uniform stress states with a complete phase-field model for fracture. *Theoretical and Applied Fracture Mechanics*, page 105170, 2025.

Q. S. Zheng. Theory of representations for tensor functions. *Applied Mechanics Reviews*, 47(11):545–587, 1994.

Q.-S. Zheng and A. J. M. Spencer. Tensors which characterize anisotropies. *International Journal of Engineering Science*, 31(5):679–693, 1993.

K. Zhou, H. J. Hoh, X. Wang, L. M. Keer, J. H. Pang, B. Song, and Q. J. Wang. A review of recent works on inclusions. *Mechanics of Materials*, 60:144–158, 2013.